Limits and Continuity

$$\lim_{x \to c} a = a$$

$$\lim_{x \to c} a f(x) = a \lim_{x \to c} f(x)$$

$$\lim_{x \to c} f(x) \pm g(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

$$\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} f(x) \Big/ \lim_{x \to c} g(x)$$

$$\lim_{x \to c} f(x)^n = \left(\lim_{x \to c} f(x) \right)^n$$

For c being a number or $+\infty$ or $-\infty$ or a one sided limit

Whenever the right hand side makes sense

Sandwich theorem: $g(x) \le f(x) \le h(x)$ on an interval around c
Then if $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, we have $\lim_{x \to c} f(x) = L$

L'Hospital rule: If $\lim_{x \to c} \frac{f(x)}{g(x)}$ is $\frac{\pm\infty}{\pm\infty}$ or $\frac{0}{0}$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

Continuity of $f(x)$ at $x = c$ amounts to being able to calculate $\lim_{x \to c} f(x)$ by plugging in $f(x)$, $x = c$,
that is, $f(x)$ is continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$

Some continuous functions: $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\cot x$, $\csc x$ (whenever defined), polynomials, rational functions (whenever defined), $\ln x$, e^x, x^n, n rational (whenever defined).

If f and g are continuous at x, so is $f \pm g$, $f \cdot g$, $a \cdot f$ (when a is a constant), f/g (whenever defined) and $f \circ g$ (if g is continuous at x and f is continuous at $g(x)$).

Some intuitive limit behavior:

$+\infty \cdot +\infty = +\infty$, $-\infty \cdot +\infty = -\infty$, $-\infty \cdot -\infty = +\infty$, $\frac{c}{0} = \pm\infty$, $\frac{c}{\pm\infty} = 0$ (when $c \neq 0$ and is a number)

Some useful limits:

$\lim_{x \to +\infty} e^x = \infty$, $\lim_{x \to -\infty} e^x = 0$, $\lim_{x \to +\infty} \ln x = +\infty$, $\lim_{x \to 0+} \ln x = -\infty$

Derivatives

$$\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$$

$$\left(af(x)\right)' = af'(x)$$

$$\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$ **Product Rule**

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$$ **Quotient Rule**

$$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$$ **Chain Rule**

$(a)' = 0$, when a is a constant

$\left(x^n\right)' = nx^{n-1}$, n rational

$$(\sin x)' = \cos x \qquad (\tan x)' = \sec^2 x \qquad (\sec x)' = \sec x \cdot \tan x$$

$$(\cos x)' = -\sin x \qquad (\cot x)' = -\csc^2 x \qquad (\csc x)' = -\csc x \cdot \cot x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arctan x)' = \frac{1}{1+x^2}$$

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}} \qquad (\mathrm{arcsec}\, x)' = \frac{1}{x\sqrt{x^2-1}}$$

$$\left(e^x\right)' = e^x \qquad (\ln x)' = \frac{1}{x}$$

Particular Cases of the Chain Rule

$$\left(f(x)^n\right)' = nf(x)^{n-1} \cdot f'(x)$$

$$\left(e^{f(x)}\right)' = f'(x) \cdot e^{f(x)}$$

$$\left(\ln f(x)\right)' = \frac{f'(x)}{f(x)}$$

Exponential, Logarithmic and Power Rules

$y = e^x$ is the exponential function

$e^{a+b} = e^a \cdot e^b$

$(e^a)^b = e^{ab}$

$\frac{e^a}{e^b} = e^{a-b}$

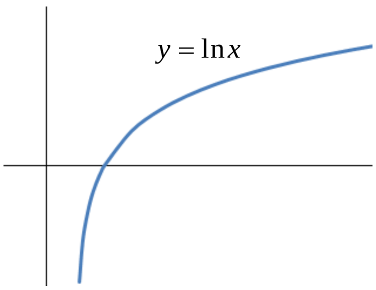
$e^{-a} = \frac{1}{e^a}$

$e^0 = 1$

$e^{\ln y} = y$

$\ln e^x = x$

$\int \frac{1}{x} dx = \ln|x| + C$



$\lim_{x \to +\infty} \ln x = +\infty \quad (= \ln(+\infty))$

$\lim_{x \to 0+} \ln x = -\infty \quad (= \ln(0))$

natural logarithm, $y = \ln x$ is the inverse of $y = e^x$

$\ln(ab) = \ln a + \ln b$

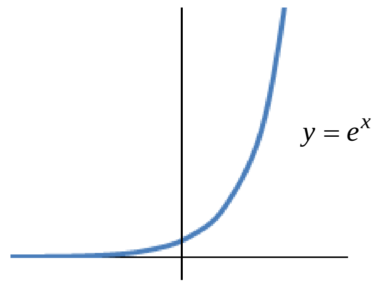
$\ln \frac{a}{b} = \ln a - \ln b$

$\ln a^p = p \ln a$

$\ln 1 = 0$

$\ln e = 1$

$\int e^x dx = e^x + C$



$\lim_{x \to +\infty} e^x = +\infty \quad (= e^{+\infty})$

$\lim_{x \to -\infty} e^x = 0 \quad (= e^{-\infty})$

Power Rules

$a^0 = 1$ $\quad a^n a^m = a^{n+m}$ $\quad (a^n)^m = a^{nm}$ $\quad \frac{a^n}{a^m} = a^{n-m}$

$a^{-n} = \frac{1}{a^n}$ $\quad \sqrt[n]{a} = a^{\frac{1}{n}}$ $\quad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$ $\quad a^n \cdot b^n = (a \cdot b)^n$

Roots of quadratic formula

$ax^2 + bx + c = 0$ the roots are $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Factorization

1) $x^2 + bx + c = 0$ if x_1, x_2 roots, then $x^2 + bx + c = (x - x_1)(x - x_2)$
2) $(a^2 - b^2) = (a+b)(a-b)$
3) $(a \pm b)^2 = a^2 \pm 2ab + b^2$

Divison of fractions

$$\frac{a/b}{c/d} = \frac{a \cdot d}{b \cdot c}$$

Integration Techniques

1) **Simple Substitution :** $\int f(g(x))\cdot g'(x)dx$

Set $u = g(x)$, $du = g'(x)dx$ $\Rightarrow$ $\int f(g(x))\cdot g'(x)dx = \int f(u)du$

2) **Integrals of the kind:** $\int \sin^n x \cos^m x dx$

a) When at least one of n or m is odd: use $\sin^2 t + \cos^2 t = 1$

b) When both n and m are even or zero : use $\cos^2 t = \dfrac{1+\cos 2t}{2}$ or $\sin^2 t = \dfrac{1-\cos 2t}{2}$

3) **Integration by parts :** Used (mostly) for integrals involving multipication of two functions $u = u(x),\ v = v(x)$.

$$\int u \cdot v' dx = u \cdot v - \int u' \cdot v dx$$

4) **Trig. Substitution :** ***See Trig. Substitutions and Some Useful Trig. Identities page***

5) **Integration of Rational Functions :** $\int \dfrac{p(x)}{q(x)}dx$, degree $p(x)$ < degree $q(x)$, and $q(x)$ monic.

a) Factor $q(x)$ into a multiplication of factors of the form $(x-a)^n$ and $(x^2+bx+c)^m$ where x^2+bx+c is irreducible.

b) For each factor $(x-a)^n$, with n maximal, corresponds a sum:

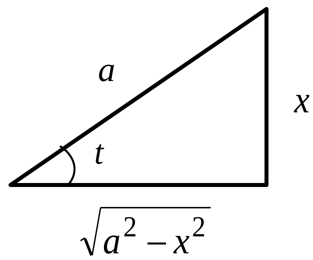
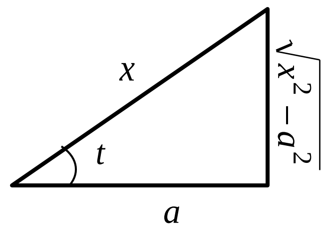
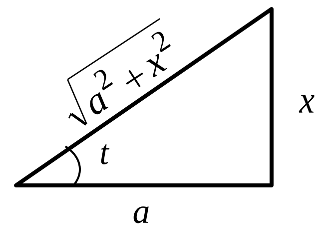
$$\frac{A_1}{(x-a)^n} + \frac{A_2}{(x-a)^{n-1}} + \ .\ .\ .\ . + \frac{A_n}{(x-a)},$$ where $A_1, . . . , A_n$ are numbers.

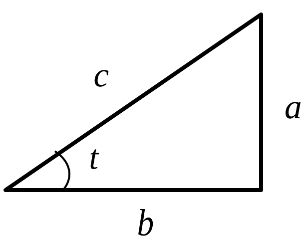
c) For each factor $(x^2+bx+c)^m$, with m maximal, corresponds a sum:

$$\frac{B_1 x + C_1}{(x^2+bx+c)^m} + \frac{B_2 x + C_2}{(x^2+bx+c)^{m-1}} + \ .\ .\ .\ . + \frac{B_m x + C_m}{(x^2+bx+c)},$$ where $B_1, . . , B_n$ and $C_1, . . , C_n$ are numbers.

d) $\dfrac{p(x)}{q(x)}$ = sum of all the sums found in b) and c).

Trig. Substitutions and Some Useful Trig. Identities

Integral Involves:	$\sqrt{a^2 - x^2}$	$\sqrt{x^2 - a^2}$	$\sqrt{a^2 + x^2}$
Triangle			
Substitution	$x = a\sin t$ $dx = a\cos t\,dt$	$x = a\sec t$ $dx = a\sec t\tan t\,dt$	$x = a\tan t$ $dx = a\sec^2 t\,dt$
Use	$\sqrt{a^2 - x^2} = a\cos t$ $\sin^2 t + \cos^2 t = 1$ $t = \arcsin\frac{x}{a}$	$\sqrt{x^2 - a^2} = a\tan t$ $\sec^2 t - 1 = \tan^2 t$ $t = \operatorname{arcsec}\frac{x}{a}$	$\sqrt{a^2 + x^2} = a\sec t$ $1 + \tan^2 t = \sec^2 t$ $t = \arctan\frac{x}{a}$



$\sin t = \frac{a}{c}$ $\quad \tan t = \frac{a}{b}$ $\quad \sec t = \frac{c}{b}$

$\cos t = \frac{b}{c}$ $\quad \cot t = \frac{b}{a}$ $\quad \csc t = \frac{c}{a}$

$\sin(-t) = -\sin t$
$\cos(-t) = \cos t$

$\sin(t + 2n\pi) = \sin t$
$\cos(t + 2n\pi) = \cos t$

$1 + \tan^2 t = \sec^2 t$
$1 + \cot^2 t = \csc^2 t$

$\cos(t + u) = \cos t\cos u - \sin t\sin u$
$\sin(t + u) = \sin t\cos u + \cos t\sin u$

$\sin 2t = 2\sin t\cos t$
$\cos 2t = \cos^2 t - \sin^2 t$

t(degrees)	t(radians)	$\sin t$	$\cos t$
0°	0	0	1
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$
90°	$\pi/2$	1	0
120°	$2\pi/3$	$\sqrt{3}/2$	$-1/2$
135°	$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$
150°	$5\pi/6$	$1/2$	$-\sqrt{3}/2$
180°	π	0	-1

$\sin^2 t + \cos^2 t = 1$

$\cos^2 t = \frac{1 + \cos 2t}{2}$

$\sin^2 t = \frac{1 - \cos 2t}{2}$

Basic Integration Formulas

$a, k,$ and b are constants

1) $\int u^r\,du = \frac{u^{r+1}}{r+1} + C, \quad \text{if } r \neq -1$

2) $\int (u-a)^r\,du = \frac{(u-a)^{r+1}}{r+1} + C, \quad \text{if } r \neq -1$

3) $\int \frac{1}{u}\,du = \ln|u| + C$

4) $\int \frac{1}{(u-a)}\,du = \ln|u-a| + C$

5) $\int \frac{1}{(ku-a)}\,du = \frac{1}{k}\ln|ku-a| + C$

6) $\int e^u\,du = e^u + C$

7) $\int e^{ku}\,du = \frac{1}{k}e^{ku} + C$

8) $\int b^u\,du = \frac{b^u}{\ln b} + C$

9) $\int \sin u\,du = -\cos u + C$

10) $\int \sin ku\,du = -\frac{1}{k}\cos ku + C$

11) $\int \cos u\,du = \sin u + C$

12) $\int \cos ku\,du = \frac{1}{k}\sin ku + C$

13) $\int \sec^2 u\,du = \tan u + C$

14) $\int \sec^3 u\,du = \frac{\sec u \tan u + \ln|\sec u + \tan u|}{2} + C$

15) $\int \csc^2 u\,du = -\cot u + C$

16) $\int \sec u \tan u\,du = \sec u + C$

17) $\int \csc u \cot u\,du = -\csc u + C$

18) $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$

19) $\int \cot u\,du = \ln|\sin u| + C$

20) $\int \sec u\,du = \ln|\sec u + \tan u| + C$

21) $\int \csc u\,du = \ln|\csc u - \cot u| + C$

21) $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = -\cos^{-1} u + C$

23) $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C, \text{ where } a > 0$

24) $\int \frac{du}{1+u^2} = \tan^{-1} u + C$

25) $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

26) $\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C$

27) $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C, \text{ where } a > 0$

Practice Exercise on Limits

1) $\lim\limits_{x \to 4} \dfrac{3\sqrt{x} - x^{3/2}}{x^{5/2} - 3x^{3/2}}$

2) $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x - 4}$

3) $\lim\limits_{x \to 0} x^2 \cdot \sin \dfrac{1}{x}$

4) $\lim\limits_{x \to 0^+} \dfrac{|x|}{x}$

5) $\lim\limits_{x \to 0^+} f(x)$ where $f(x) = \begin{cases} x^2 - 3x + 1 & \text{for } x \le 0 \\ 1 + 5x - x^2 & \text{for } x > 0 \end{cases}$

6) $\lim\limits_{x \to 0^-} f(x)$ where $f(x) = \begin{cases} x^2 - 3x + 1 & \text{for } x \le 0 \\ 1 + 5x - x^2 & \text{for } x > 0 \end{cases}$

7) $\lim\limits_{x \to 0} f(x)$ where $f(x) = \begin{cases} x^2 - 3x + 1 & \text{for } x \le 0 \\ 1 + 5x - x^2 & \text{for } x > 0 \end{cases}$

8) $\lim\limits_{x \to +\infty} \dfrac{x^2 - 4x + 1}{2x^2 + 5x - 8}$

9) $\lim\limits_{x \to +\infty} \dfrac{-2x^3 + 8x}{x^2 - 4}$

10) $\lim\limits_{x \to +\infty} x \cdot \sin \dfrac{1}{x}$

11) $\lim\limits_{x \to 1^-} \dfrac{x^2 - 2x + 5}{x^2 - 4x + 3}$

12) $\lim\limits_{x \to 0} \dfrac{e^x - x - 1}{x^2}$

13) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$

Practice Exercise on Derivatives

1) $\left[\left(7x^3-3x^2+4\right)^5\right]'$

2) $\left(\frac{3+2x}{3\text{-}2x}\right)'$

3) $\left[\left(\frac{x^2+1}{x^2-1}\right)^2\right]'$

4) $\left[\left(1+\sec^2 x\right)^3\right]'$

5) $\left(\frac{\cos^2 x}{1+\sin x}\right)'$

6) $\left[\ln\left(\sin^2 x\cdot\cos^2 x\right)\right]'$

7) $\left(\frac{\ln x}{x^4}\right)'$

8) $\left(e^{x^2+1}\right)'$

9) $\left(e^x\cdot\sin 2x\right)'$

10) $\left(\frac{e^{x^2}}{x^2+1}\right)'$

11) $\left(\arcsin\sqrt{x+1}\right)'$

12) $\left[\arctan(\ln x)\right]'$

13) $\left(\frac{\arccos x}{\sqrt{1\text{-}x^2}}\right)'$

Practice Exercise on Integration (page 1)

Simple substitution :

1) $\int e^{2x}dx$

2) $\int \sin(\ln x)\cdot\frac{1}{x}dx$

3) $\int \frac{\sin x}{\cos^3 x}dx$

4) $\int 6x(x^2-3)^{10}dx$

5) $\int \frac{\sin x}{1+\cos x}dx$

6) $\int \frac{x}{e^{x^2+1}}dx$

Integration by parts :

7) $\int x\sin x dx$

8) $\int x^2\ln x dx$

9) $\int \sin(\ln x)dx$

10) $\int xe^x dx$

11) $\int x\cos x dx$

12) $\int x^2\sin x dx$

13) $\int e^x\sin x dx$

Practice Exercise on Integration (page 2)

Integrals of power of $\sin$ and $\cos$:

14) $\int \sin^3 x\,dx$

15) $\int \cos^2 x\,dx$

16) $\int \sin^3 x \cos^2 x\,dx$

17) $\int \sin^2 x \cos^2 x\,dx$

Trig. substitutions :

18) $\int \frac{dx}{x^2\sqrt{4-x^2}}$

19) $\int x^2\sqrt{16-x^2}\,dx$

20) $\int \frac{dx}{x\sqrt{x^2+4}}$

21) $\int \frac{x^2}{\sqrt{x^2+6}}dx$

22) $\int \frac{dx}{x^3\sqrt{x^2-9}}$

23) $\int \frac{dx}{x^2\sqrt{x^2-7}}$

Rational Functions :

24) $\int \frac{dx}{x^2-4}$

25) $\int \frac{6x^2-2x+1}{4x^3-x}dx$

26) $\int \frac{dx}{(x+2)^2(x+1)}$