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Regular Symmetric Algebras

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INTRODUCTION

Let R be a commutative ring. R is called *a regular ring* if every finitely generated ideal of R has finite projective dimension. This notion, which has been extensively studied for Noetherian rings, was extended to coherent rings with a considerable degree of success [1, 12, 14]. Let R be a (coherent or Noetherian) regular ring and let S be an R algebra. The type of question considered here is under what conditions will S be a coherent regular ring. The algebras that are of special interest to us are: symmetric algebras of flat modules, polynomial rings and group rings of abelian groups.

The first difficulty encountered is to decide when is S a coherent ring. In that direction, an account of the results known in case S is a polynomial ring can be found in [15], and the case when S is a group ring of an abelian group has been solved in [5]. In case $S = S_R(M)$ is the symmetric algebra of a flat R module M, very little is known. Carrig [3] proved that if R is a Dedekind domain and M is a rank one flat R module then $S_R(M)$ is a coherent ring. He also provided an example that this may not be the case if rank $M > 1$ even if $R = Z$, the ring of integers. In Section 1 of this paper we prove that if R is any Noetherian ring of finite Krull dimension and M is a rank one flat R module than $S_R(M)$ is a coherent ring. The finite dimensionality condition may be dropped in certain important cases but the Noetherian hypothesis seems to be intrinsic. We conclude that symmetric algebras of rank one flat modules over Noetherian rings of finite Krull dimension are stably coherent rings.

We next turn our attention to the homological property of the regularity condition. In Section 2 of this paper we prove that if R is a Noetherian

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regular ring and M is a rank one flat R module then $S_R(M)$ is a regular ring. In the process we keep track of the exact relation between the weak dimensions of R and $S_R(M)$. Next we prove that the polynomial ring in any number of variables over a stably coherent ring is a regular ring. Finally, we make use of this result to solve a conjecture posed in [5], as to necessary and sufficient conditions for a group ring of an abelian group to be a coherent regular ring.

1. Coherence

Let R be a ring and let M be a flat R module. We say that *rank* $M=1$ if $\Lambda^2 M=0$, and that *rank* $M=n$ if $\Lambda^n M$ has rank equal to 1; otherwise, we put rank $M=\infty$. If M is a finitely generated projective R module for which rank is defined in the classical sense, the numerical value of rank M and of the classical rank of M coincide. If R is a domain this definition of rank coincides with the usual definition of rank, that is, rank M is equal to the cardinality of a basis of the free module $M \otimes_R K$, where K is the field of quotients of R.

For a general ring denote by K the total ring of quotients of R. If R is a domain and M is a flat R module then rank $M=1$ if and only if $M \otimes_R K \simeq K$, if and only if M is isomorphic to an R submodule of K. In general any flat R submodule of K, M, has rank $M=1$ since the flatness of M over R guarantees that the injection $M \to K$ yields an injection $\Lambda^2 M \to \Lambda^2 K = 0$ [9]. The converse is not necessarily true, but it is true in certain important cases as we will see in this section.

Let M be an R module and denote by $S_R(M)$ the symmetric algebra of M over R.

LEMMA 1.1. *Let R be a Noetherian ring and let M be a flat R module. Let* $\mathrm{Min}(R)=\{P_1, ..., P_n\}$ *be the set of minimal prime ideals of R and $I=\bigcap_{i=1}^n P_i$; then $S_R(M)$ is a coherent ring if and only if $S_{R/I}(M/IM)$ is a coherent ring.*

Proof. $S_{R/I}(M/IM) \simeq S_R(M) \otimes_R R/I \simeq S_R(M)/IS_R(M)$. Let $\phi: S_R(M) \to S_R(M)/IS_R(M)$ be the canonical map. ϕ is surjective. ϕ makes $S_R(M)/IS_R(M)$ into a finitely presented $S_R(M)$ module. $\ker \phi = IS_R(M)$ is a nilpotent ideal and, since $S_R(M)$ is a flat R module [9], $\ker \phi$ is a finitely presented ideal as well. It follows from [7, 13] that $S_R(M)$ is a coherent ring if and only if $S_R(M)/IS_R(M)$ is a coherent ring.

THEOREM 1.2. *Let R be a Noetherian ring of Krull* $\dim R = n < \infty$ *and let M be a rank one flat R module, then $S_R(M)$ is a coherent ring.*

Proof. By Lemma 1.1 we may assume that R is a reduced ring. We prove our claim by induction on $n = \text{Krull dim } R$.

For $n=0$ we have that R is an Artinian reduced ring; therefore, $R = K_1 \times \cdots \times K_n$, where K_i are fields. Then $M_i = M \otimes_R K_i = 0$ or $M_i \simeq K_i$ and

$$S_{K_i}(M_i) = \begin{cases} K_i & \text{if } M_i = 0 \\ K_i[t_i] & \text{if } M_i \simeq K_i \end{cases},$$

where t_i is an indeterminate over K_i. It follows that $S_R(M) = S_R(M) \otimes_R R = S_{K_1}(M_1) \times \cdots \times S_{K_n}(M_n)$ is a Noetherian ring.

For $n \geqslant 1$, set $\text{Min}(R) = \{P_1, ..., P_n\}$ and $S = R - \bigcup_{i=1}^n P_i$ the set of nonzero divisors of R, then $K = R_S = R_{P_1} \times \cdots \times R_{P_n}$ is the total ring of quotients of R and R_{P_i} are fields for $1 \leqslant i \leqslant n$.

For each i either $M_{P_i} = 0$ or $M_{P_i} \simeq R_{P_i}$ and assume that $M_{P_i} \neq 0$ for $1 \leqslant i \leqslant s$ and $M_{P_i} = 0$ for $s < i \leqslant n$ and $1 \leqslant s \leqslant n$. Then $M_S = M \otimes_R K \simeq R_{P_1} \times \cdots \times R_{P_s}$ and $S_K(M \otimes_R K) = S_R(M) \otimes_R K = S_{R_{P_1}}(M_1) \times \cdots \times S_{R_{P_n}}(M_n) = R_{P_1}[t_1] \times \cdots \times R_{P_s}[t_s] \times R_{P_{s+1}} \times \cdots \times R_{P_n}$, where t_i are indeterminates over R_{P_i} for $1 \leqslant i \leqslant s$. Since the map $S_R(M) \to S_R(M) \otimes_R K$ is injective we conclude that $S_R(M)$ is a reduced ring and that $S_K(M \otimes_R K)$ is contained in the total ring of quotients of $S_R(M)$.

Let I be an ideal of $S_R(M)$ then:

$$\begin{aligned} I_S = I \otimes_R K &= I(R_{P_1}[t_1] \times \cdots \times R_{P_n}) \\ &= IR_{P_1}[t_1] \times \cdots \times IR_{P_n} = I_{P_1} \times \cdots \times I_{P_n}. \end{aligned}$$

These equalities follow from the flatness of $S_{R_{P_i}}(M_{P_i})$ and $S_K(M \otimes_R K)$ over $S_R(M)$.

Assume that I is a finitely generated ideal of $S_R(M)$. We aim to show that I is a finitely presented ideal and, therefore, $S_R(M)$ is a coherent ring.

Reduction to the case where $I_{P_i} \neq 0$ for all $1 \leqslant i \leqslant n$.

Assume that $I_{P_i} = 0$ for some $1 \leqslant i \leqslant n$. Let $P_1, ..., P_k \in \text{Min}(R)$ satisfy $I_{P_i} = 0$ for $1 \leqslant i \leqslant k$ and that $I_{P_i} \neq 0$ for $k < i \leqslant n$ and $1 \leqslant k \leqslant n$. Since $I_{P_i} = 0$ for $1 \leqslant i \leqslant k$ and I is a finitely generated ideal of $S_R(M)$ there exists an element $t \in R - \bigcup_{i=1}^n P_i$ such that $tI = 0$. Since $S_R(M)$ is a reduced ring we have that $I \cap tS_R(M) = 0$. Set $L = I \oplus tS_R(M)$. L is a finitely generated ideal of $S_R(M)$, $L_{P_i} \neq 0$ for all $1 \leqslant i \leqslant n$ and since $tS_R(M)$ is finitely presented, if L is finitely presented so is I.

Reduction to the case where $I \cap R = J$ contains a nonzero divisor.

Assume that $I_{P_i} \neq 0$ for all $1 \leqslant i \leqslant n$, then $I_{P_i} = R_{P_i}$ for $s < i \leqslant n$ and $I_{P_i} = f_i R_{P_i}[t_i] \neq 0$ for $1 \leqslant i \leqslant s$, where $f_i \in R_{P_i}[t_i]$. $I_S = I \otimes_R K = f_1 R_{P_1}[t_1] \times \cdots \times f_s R_{P_s}[t_s] \times R_{P_{s+1}} \times \cdots \times R_{P_n} = f(R_{P_1}[t_1] \times \cdots \times R_{P_n})$, where $f = (f_1, ..., f_s, 1, ..., 1)$ and f is a nonzero divisor in $R_{P_1}[t_1] \times \cdots \times R_{P_n}$. It follows that $fS_R(M) \simeq S_R(M)$ and $f^{-1}S_R(M)$ is an

$S_R(M)$ submodule of the ring of quotients of $S_R(M)$. We also have $I_S = (fS_R(M))_S$; therefore, there exist nonzero divisors a, $b \in R$ such that $bf \in I$ and $aI \subset fS_R(M)$. Set $L = af^{-1}I$, then L is a finitely generated ideal of $S_R(M)$, $ab \in L \cap R$ and $fL = aI$ implies that if L is finitely presented so is I.

Final case.

Set $J = I \cap R$. As J contains a nonzero divisor we have that Krull dim $R/J < n$. By the induction hypothesis $S_R(M)/JS_R(M) = S_{R/J}(M/JM)$ is a coherent ring and we have an exact sequence of $S_R(M)$ modules: $0 \to JS_R(M) \to I \to I/JS_R(M) \to 0$. Since R is a Noetherian ring we have that $JS_R(M)$ is a finitely presented ideal of $S_R(M)$. $I/JS_R(M)$ is a finitely presented $S_R(M)/JS_R(M)$ module and, therefore, it is finitely presented $S_R(M)$ module. It follows that I is a finitely presented ideal.

COROLLARY 1.3. *Let R be a Noetherian ring of finite Krull dimension and let M be a rank one flat R module, then $S_R(M)$ is a stably coherent ring.*

Proof. Let $t_1, ..., t_n$ be indeterminates over $S_R(M)$, then:

$$\begin{aligned} S_R(M)[t_1, ..., t_n] &= S_R(M) \otimes_R R[t_1, ..., t_n] \\ &= S_{R[t_1, ..., t_n]}(M \otimes_R R[t_1, ..., t_n]). \end{aligned}$$

We suspect the finite dimensionality condition in Theorem 1.2 to be superfluous. Here is an instance in which this condition may be dropped:

THEOREM 1.4. *Let R be a Noetherian ring, let M be a ring and let ϕ: $R \to M$ be an injective, unital, ring homomorphism making M a rank one flat R module; then $S_R(M)$ is a coherent ring.*

Proof. Let $\mathrm{Min}(R) = \{P_1, ..., P_n\}$ and set $I = \bigcap_{i=1}^n P_i$. Note that the ring homomorphism $\bar{\phi} : R/I \to M/IM$ defined by $\bar{\phi}(r) = \phi(r) + IM$, $r \in R$, is an injective unital ring homomorphism. By Lemma 1.1 we may therefore assume that R is a reduced ring and that ϕ is the inclusion map.

Let $S = R - \bigcup_{i=1}^n P_i$; then the total ring of quotients of R, $K = R_{P_1} \times \cdots \times R_{P_n}$ and R_{P_i} are fields. Since $1 \in M$, $M_{P_i} \neq 0$ for $1 \leqslant i \leqslant n$ and $M_{P_i} = R_{P_i}$; therefore, $R \subset M \subset K$ as rings sharing the same identity. It follows from [9] that the inclusion map $R \to M$ is a flat epimorphism. We, therefore, have $S_R(M) = R + tM[t]$, where t is an indeterminate, and we can set the following cartesian square:

$$\begin{array}{ccc} S_R(M) = R + tM[t] & \longrightarrow & M[t] \\ \downarrow & & \downarrow \\ R & \longrightarrow & M \end{array}$$

Since R is a Noetherian ring and M is a flat epimorphism of R, M and, hence, $M[t]$, are Noetherian rings. It follows from [6] that $S_R(M)$ is a coherent ring.

Carrig [3] provided the following example: Let R be a non-Noetherian valuation domain with non-finitely generated maximal ideal m; then $S_R(m)$ is not a coherent ring. This example shows that we are not likely to find any significant cases in which the Noetherian hypothesis in Theorem 1.2 and Theorem 1.4 can be relaxed. Nevertheless, usually properties related to coherence held by Noetherian rings are held by flat direct limits of Noetherian rings. This way we can construct an example of an infinite dimensional, non-Noetherian, coherent ring R and a rank one flat R module M such that $S_R(M)$ is a coherent ring: Let A be a Noetherian ring of finite Krull dimension, let N be a rank one flat A module and let $t_1, t_2, \ldots$ be indeterminates. Let $R = \varinjlim A[t_1, \ldots, t_n]$. R is a non-Noetherian infinite dimensional coherent ring, $M = N \otimes_A R$ is a rank one flat R module and $S_R(M) = \varinjlim S_R(N)[t_1, \ldots, t_n]$ is a coherent ring.

2. Regularity

In this section we consider the homological property of the regularity condition. Let R be a ring; we say that an R module M admits an *infinite finite presentation* if there exists an exact sequence $\cdots \to F_1 \to F_0 \to M \to 0$ with F_i finitely generated and free R modules. If R is a coherent ring any finitely presented R module M admits an infinite finite presentation. In order to be able to separate, in a sense, the coherence property and the homological property of the regularity condition we prove the following lemma, which is known in case R as a coherent ring.

Lemma 2.1. *Let R be a ring and let M be an R module admitting an infinite finite presentation, then*:

(1) $\text{w.dim}_R M = \text{proj.dim}_R M$.

(2) *If R is a local ring with maximal ideal m, then* $\text{proj.dim}_R M \leqslant n$ *if and only if* $\text{Tor}_R^{n+1}(M, R/M) = 0$.

Proof. (1) We need only to show that if $\text{w.dim}_R M = n < \infty$, then $\text{proj.dim}_R M \leqslant n$. Let $\cdots \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \to M \to 0$ be an infinite finite presentation of M. Then $K_{n-1} = \ker d_{n-1}$ is a flat and finitely presented R module and, therefore, a projective R module and $\text{proj.dim}_R M \leqslant n$.

(2) We need only to show that if $\text{Tor}_R^{n+1}(M, R/m) = 0$ then $\text{proj.dim}_R M \leqslant n$. The claim is proved by induction on n. For $n = 0$ the result follows from [2]. For $n \geqslant 1$, let $\cdots \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \to M \to 0$ be an

infinite finite presentation of M and let $K_{n-1} = \ker d_{n-1}$; then $0 = \mathrm{Tor}_R^{n+1}(M, R/m) = \mathrm{Tor}_R^1(K_{n-1}, R/m)$. By the induction hypothesis K_{n-1} is projective and proj. $\dim_R M \leqslant n$.

We first consider the regularity of symmetric algebras.

PROPOSITION 2.2. *Let R be a Noetherian regular local ring with maximal ideal m and* Krull dim $R = n$. *Let M be a rank one flat R module. If M is finitely generated, then* w. dim $S_R(M) = n + 1$. *If M is not finitely generated then* w. dim $S_R(M) = n$.

Proof. We have two cases, either $M \neq mM$ or $M = mM$.

If $M \neq mM$ then M is a cyclic R module. To see this let K be the field of quotients of R, let $x = a/b \in M - mM$ and $y = c/d \in M$ with $a, b, c, d \in R$ and $b \neq 0$, $d \neq 0$. Then $(cb)x - (ad)y = 0$. Since M is a flat R module there exist elements $b_{ij}, b_{2j} \in R$ and $y_j \in M$, $1 \leqslant j \leqslant s$, such that $(cb)b_{1j} - (ad)b_{2j} = 0$ for all j, and $x = \sum b_{1j} y_j$, $y = \sum b_{2j} y_j$. Since $x \notin mM$ there exist $1 \leqslant j_0 \leqslant s$ such that b_{1j_0} is a unit in R, then $y = (b_{2j_0}/b_{1j_0})x$ and $M = Rx$. It follows that $S_R(M) = R[t]$ and by [8] w. $\dim_R S_R(M) = n + 1$.

If $M = mM$ Nakayama's Lemma guarantees that M is not finitely generated. We identify $S_R(M)$ with the subring of $K[t]$, $S_R(M) = R + Mt + M^2t^2 + \cdots$. Let P be a maximal ideal of $S_R(M)$ and set $p = P \cap R$. We distinguish between two cases $p = m$ and $p \neq m$.

If $p = m$ then $mS_R(M) = m + Mt + M^2t^2 + \cdots \subset P$ and $P = mS_R(M)$. Pick $x_1, ..., x_n$ to be a regular system of parameters for m, then the Koszul complexes $K.(x_1, ..., x_n, R)$ provide a finite free resolution of length n for R/m, which is minimal [10]. Tensoring this resolution over R with $S_R(M)$ and then over $S_R(M)$ with $S_R(M)_P$ we obtain a minimal finite free resolution of the $S_R(M)_P$ module $S_R(M)_P/PS_R(M)_P$; thus, proj. $\dim_{S_R(M)_P} S_R(M)_P/PS_R(M)_P = n$ and for every finitely generated ideal I of $S_R(M)_P$ we have that $\mathrm{Tor}^{n+1}_{S_R(M)_P}(S_R(M)_P/I, S_R(M)_P/PS_R(M)_P) = 0$. As $S_R(M)_P$ is coherent we conclude by Lemma 2.1 that proj. $\dim_{S_R(M)_P} S_R(M)_P/I \leqslant n$ and w. dim $S_R(M)_P = n$.

If $P \cap R = p \neq m$ then Krull dim $R_p < n$. If M_p is a finitely generated R_p module then w. $\dim_{R_p} M_p =$ w. dim $R_p + 1 \leqslant n$. If M_p is not a finitely generated R_p module then, by the above argument, w. $\dim(S_R(M))_P =$ w. $\dim(S_{R_p}(M_p))_P \leqslant n$. In either case, w. $\dim(S_R(M))_P \leqslant n$.

We conclude that w. dim $S_R(M) = n$.

THEOREM 2.3. *Let R be a Noetherian ring of finite Krull dimension and let M be a rank one flat R module, then*:

(1) *If R is a regular ring then $S_R(M)$ is a coherent regular ring, and* w. dim $R \leqslant$ w. dim $S_R(M) \leqslant$ w. dim $R + 1$.

(2) *If $S_R(M)$ is a coherent regular ring so is R.*

Proof. (1) $S_R(M)$ is a flat R module and contains R as a direct summand; therefore, $\text{w.dim}\, R \leqslant \text{w.dim}\, S_R(M)$. Now use Proposition 2.2 and Theorem 1.2.

(2) Note that $S_R(M)$ is a faithfully flat R module. Now use [5].

We note that if R is not a local ring we may actually have $\text{w.dim}\, S_R(M) = \text{w.dim}\, R + 1$ even in case M is not finitely generated. To see this let $R = Z$ and $M = \{x/a \mid x, a \in Z, 0 \neq a \text{ is square free}\} \subset Q$, the rationals. Then for any prime $p \in Z$ we have $M_p = Z_p(1/p)$ and $S_{Z_p}(M_p) = Z_p[t_p]$, where t_p is an indeterminate. Thus, $\text{w.dim}\, S_Z(M) = 2$.

We next turn our attention to the regularity of polynomial rings over coherent rings.

LEMMA 2.4. *Let R be a ring, let M be a finitely presented R module and let r be the smallest non-negative integer for which the r-Fitting ideal of M, $F_r(M) \neq 0$. If $F_r(M) = R$ then M is a projective R module. If* Spec(R) *is connected the converse is true as well.*

Proof. Assume that $F_r(M) = R$. Since $F_0(M) \subset F_1(M) \subset \cdots$ it follows that at every localization by a prime ideal P of R $F_i(M)_P = 0$ or R_P, for all i. By [14] this implies that M is projective.

Conversely, assume that M is projective and that Spec(R) is connected. Since M is finitely presented $F_i(M)$ are finitely generated ideals of R. Since M is locally free $F_r(M)_P = 0$ or R_P for every prime ideal P of R. It follows that $F_r(M) = F_r(M)^2$ and $F_r(M)$ is generated by an idempotent e. Since Spec(R) is connected $e = 0$ or 1 and $F_r(M) = R$.

PROPOSITION 2.5. *Let R be a coherent regular ring and let I be an ideal of the polynomial ring $R[t]$ admitting an infinite finite presentation; then* $\text{proj.dim}_{R[t]}\, I < \infty$.

Proof. We will first prove that for every maximal ideal P of $R[t]$, $\text{proj.dim}_{R[t]_P}\, I_P < \infty$. Let $P \cap R = p$. Since $(L_p)_P = L_P$ for every $R[t]$ module L we may assume that R is local with maximal ideal p. In this case P contains a monic polynomial $f \in R[t]$. I is a finitely presented $R[t]$ module; therefore, $I \otimes_R R[t]/fR[t] = I/fI$ is a finitely presented $R[t]/fR[t]$ and, hence, R module. Since R is a coherent regular ring $\text{proj.dim}_R\, I/fI = n - 2 < \infty$ and by [8] $\text{proj.dim}_{R[t]} I/fI \leqslant n - 1$. We have an exact sequence:

$$0 \longrightarrow I_P \xrightarrow{f} I_P \longrightarrow I_P/fI_P \longrightarrow 0,$$

which yields the long exact sequence:

$$\begin{aligned}\cdots &\longrightarrow \operatorname{Tor}^{n+1}_{R[t]_P}(I_P/fI_P, R[t]_P/PR[t]_P)\\ &\longrightarrow \operatorname{Tor}^{n+1}_{T[t]_P}(I_P, R[t]_P/PR[t]_P)\\ &\xrightarrow{\cdot f} \operatorname{Tor}^{n+1}_{R[t]_P}(I_P, R[t]_P/PR[t]_P)\\ &\longrightarrow \operatorname{Tor}^{n}_{R[t]_P}(I_P/fI_P, R[t]_P/RP[t]_P) \longrightarrow \cdots.\end{aligned}$$

Since proj. $\dim_{R[t]_P} I_P/fI_P \leqslant n-1$ multiplication by f in this exact sequence is an isomorphism. On the other hand, $f \in P$, thus multiplication by f, is the zero map. It follows that $\operatorname{Tor}^{n+1}_{R[t]_P}(I_P, R[t]_P/PR[t]_P)=0$ and by Lemma 2.1 proj. $\dim_{R[t]_P} I_P \leqslant n < \infty$.

Next we prove that if for every maximal ideal P of $R[t]$ we have that proj. $\dim_{R[T]_P} I_P < \infty$, then proj. $\dim_{R[t]} I < \infty$. First note that the hypothesis implies that proj. $\dim_{R[t]_P} I_P < \infty$ for every prime ideal P of R. Let $\cdots \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \to I \to 0$ be an infinite finite presentation of I. Let P be a prime ideal of $R[t]$ satisfying proj. $\dim_{R[T]_P} I_P \leqslant n$, and let $K_{n-1} = \ker d_{n-1}$. We have that $(K_{n-1})_P$ is $R[t]_P$ projective. As $\operatorname{Spec}(R[t]_P)$ is connected there exists an integer r such that $F_i(K_{n-1})_P = 0$ for $0 \leqslant i < r$ and $F_r(K_{n-1})_P = R[t]_P$. Since $F_i(K_{n-1})$ are finitely generated ideals of $R[t]$ there exist a $g \in R[t]-P$ satisfying $F_i(K_{n-1})_g = 0$ for $0 \leqslant i < r$ and $F_r(K_{n-1})_g = R[t]_g$, where L_g denotes the localization of an $R[t]$ module L by the set consisting of all powers of g. It follows that for every prime ideal $Q \subset R[t]$ such that $g \notin Q$ we have $F_i(K_{n-1})_Q = 0$ for $0 \leqslant i < r$ and $F_r(K_{n-1})_Q = R[t]_Q$. Therefore, $(K_{n-1})_Q$ is a projective $R[t]_Q$ module and proj. $\dim_{R[t]_Q} I_Q \leqslant n$. Let $O_n = \{P \in \operatorname{Spec} R[t],$ proj. $\dim_{R[t]_P} I_P \leqslant n\}$. By the argument above the sets O_n are open, $O_0 \subset O_1 \subset \cdots$ and $\bigcup_{i=0}^{\infty} O_i = \operatorname{Spec}(R[t])$. Since $\operatorname{Spec}(R[t])$ is quasi-compact there is an integer n such that $\operatorname{Spec}(R[t]) = O_n$, and proj. $\dim_{R[t]_P} I_P \leqslant n$ for every prime ideal P of $R[t]$. Let N be an $R[t]$ module. Since I admits an infinite finite presentation we have by [4] that $(\operatorname{Ext}^{n+1}_{R[t]}(I, N))_P = \operatorname{Ext}^{n+1}_{R[t]_P}(I_P, N_P) = 0$ for all prime ideals P of $R[t]$; therefore, $\operatorname{Ext}^{n+1}_{R[t]}(I, N) = 0$ and proj. $\dim_{R[t]} I < \infty$.

THEOREM 2.6. *Let R be a stably coherent regular ring; then the polynomial ring in any number of variables over R is a coherent regular ring.*

Proof. According to [11] we may assume that the number of variables is finite. The theorem now follows from Proposition 2.5 by induction on the number of variables.

As a corollary of Theorem 2.6 we obtain:

THEOREM 2.7. *Let R be a ring and let G be an abelian group satisfying* *that the group ring RG is coherent, then RG is a regular ring if and only if R is a coherent regular ring which is uniquely divisible by the order of every element of G.*

Proof. RG is a free R module; thus, if RG is a coherent ring so is R. In [5] it was proved that if RG is a coherent regular ring then R is a coherent regular ring which is uniquely divisible by the order of every element of G.

Conversely, write $G = \varinjlim G_\alpha$, where $\{G_\alpha\}$ is the set of all finitely generated subgroups of G, then $RG = \varinjlim RG_\alpha$ and if $H \subset L$ are two subgroups of G then RL is free over RH. By [11] we may, therefore, assume that G is finitely generated. We now prove our claim by induction on $n = \text{rank } G$. If $n = 0$ G is a finite group. This case was solved in [5]. For $n \geqslant 1$ write $G = G' \times H$, where H is infinite cyclic. As RG' is coherent by the induction hypothesis RG' is a regular ring. Now $RG = (RG')H = RG'[t, t^{-1}]$, where t is an indeterminate. According to [5], $RG'[t]$ is a coherent ring and by Theorem 2.6 it is, therefore, regular. But $RG'[t, t^{-1}] = RG'[t]_U$, where U is the set of all powers of t; thus, RG is regular.

In particular, if R is a stably coherent ring then RG is a coherent ring for every abelian group G, [5], and the conclusion of Theorem 2.7 holds.

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