Prüfer Rings

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Dedicated to Robert Gilmer

1 Introduction

In the introduction to his book, *Multiplicative Ideal Theory* [26], Robert Gilmer states: "It is possible to enumerate a few concepts which are central in our development of multiplicative ideal theory. Quotient rings and rings of quotients fall into this category, they are basic to all subsequent considerations; invertible ideals also constitute a basic tool in the presentation of the theory. *A third concept which plays a central role in the development of the classical ideal theory is that of a Prufer domain*".

Prufer domains were defined in 1932 by H. Prufer [56], as domains in which every finitely generated ideal is invertible. In 1936, Krull [49] named these rings in Prufer's honor and proved the first of the many equivalent conditions that make an integral domain Prufer (see Theorem 1.1). Although, the new concept started to slowly appear in the literature, it reached its central role which it enjoys today in the sixties and seventies, due, in no small part, to Robert Gilmer's publications and their impact on research in commutative algebra. On one hand, Prufer rings and related ring conditions feature in many of Robert's about 200 articles, investigating a large variety of ring properties, including the connection of the Prufer condition to other properties of interest, see for example [5, 6, 18-36]. Many commutative algebraists followed in Robert's footsteps, using some of his methods and examples, including his emphasis on the Prufer domain notion. On the other hand, in 1968, the first version of Robert's book: *Multiplicative Ideal Theory* [26], was published by Queen's University Press, collecting all results and references known to date on Prufer rings and emphasizing their central role in ring theory research. This was followed by the publication of revised versions of this book in 1972 [29], and in 1992 [36]. This book became, and continues to be, one of the most influential books for research in commutative algebra, and with it the centrality of the Prufer domain notion becomes consolidated.

The theorem below collects the equivalent definitions to the Prufer domain notion, that appear in *Multiplicative Ideal Theory* [26, 29, 36].

Theorem 1.1 *Let R be a domain. The following conditions are equivalent:*

1. *R is a Prufer domain.*
2. *Every two-generated ideal of R is invertible.*
3. *R_P is a valuation domain for every prime ideal P of R.*
4. *R_P is a valuation domain for every maximal ideal P of R.*
5. *Each finitely generated non-zero ideal I of R is a cancellation ideal, that is $IJ = IK$ for ideals J and K implies $J = K$.*
6. *If I, J and K are finitely generated ideals of R such that $IJ = IK$, and $I \neq 0$, then $J = K$.*
7. *R is integrally closed and there is an integer $n > 1$ such that for every two elements a, $b \in R$, $(a, b)^n = (a^n, b^n)$.*
8. *R is integrally closed and there exists an integer $n > 1$ such that for every two elements a, $b \epsilon R$, $a^{n-1}b \in (a^n, b^n)$.*
9. *Each ideal I of R is complete, that is $I = \cap IV_\alpha$ as V_α run over all the valuation overrings of R.*
10. *Each finitely generated ideal of R is an intersection of valuation ideals.*
11. *If I, J, and K are non-zero ideals of R, then $I \cap (J + K) = (I \cap J) + (I \cap K)$.*
12. *If I, J, and K are non-zero ideals of R, then $I(J \cap K) = IJ \cap IK$.*
13. *If I and J are non-zero ideals of R, then $(I + J)(I \cap J) = IJ$.*
14. *If I and J are non-zero ideals of R, and K is a finitely generated non-zero ideal of R, then $(I + J) : K = (I : K) + (J : K)$.*
15. *For any two elements a, $b \in R$, $(a : b) + (b : a) = R$.*
16. *If I and J are two finitely generated non-zero ideals of R, and K is a non-zero ideal of R, then $K : (I \cap J) = (K : I) + (K : J)$.*
17. *R is integrally closed and each overring of R is the intersection of localizations of R.*
18. *R is integrally closed and each overring of R is the intersection of quotient rings of R.*
19. *Each overring of R is integrally closed.*
20. *R is integrally closed and the prime ideals of any overring of R are extensions of the prime ideals of R.*
21. *R is integrally closed and for each prime ideal P of R, and each overring S of R, there is at most one prime ideal of S lying over P.*
22. *For any two polynomials f and $g \in R[x]$, $c(fg) = c(f)c(g)$; where for a polynomial $h \in R[x]$, $c(h)$ denotes the ideal of R generated by the coefficients of h.*

Recall that for a ring R, an *overring of* R is a subring of the total ring of quotients of R containing R. The equivalence of condition 1 and 2 is due to Prufer 1932 [56]; 3 and 4 are due to Krull 1936 [49]; 5, 6, 7 and 8 are due to Jensen 1963 [46], and Gilmer 1965 [19]; 9 and 10 are due to Gilmer and Ohm 1965 [18]; 11 - 16 are due to Jensen 1963 [46]; 17 - 21 are due to Butts and Phillips 1965 [10], and Gilmer 1966 [20, 21]; 22 is due to Tsang 1965 [58], and Arnold and Gilmer 1967 [5, 22].

Research in commutative algebra involving Prufer or related conditions branched in the many directions glimpsed from the above characterizations. The present article is concerned with one of these directions: the extensions of the Prufer domain notion to rings with zero divisors, and the interesting and surprising ways in which the above characterizations changed in response to the various possible extensions. The centrality of the Prufer domain notion resulted in an abundance of extensions of this notion to both domains and general rings. Each one of the twentytwo characterizations above yielded several such extensions, and a number of others arose from different sources. For that, and for reasons of mathematical preference, we restrict our article to surveying the type of Prufer domain notion extensions that have a relation to homological algebra. By this we mean that either the methods employed in obtaining the results come from homological algebra, or if ring theoretic methods were applied, the results involve homological algebra notions. In particular, we consider the following extensions of the Prufer domain notion:

1. R is a semihereditary ring.
2. $w.gl.dimR \leq 1$.
3. R is an arithmetical ring.
4. R is a Gaussian ring.
5. R is a Prufer ring.

In Section 2 we exhibit the early extensions of the Prufer domain notion to rings with zero divisors, involving the definitions of arithmetical and Prufer rings. This approach employed the definition of an extension of the notion of a valuation domain to rings with zero divisors, that is Manis valuations. It is the work of a number of commutative algebraists, and culminates in Griffin's 1970 article [43]. Section 3 describes the extensions of the Prufer domain notion that arise in homological algebra, namely semihereditary rings and rings of weak global dimension at most one. Classical characterizations of these rings allow for comparisons between them and the two extensions described in Section 2. Section 4 focuses on the extension of the Prufer domain notion to Gaussian rings. This section covers a number of classical results, as well as recent work by Glaz 2005 [42]. [42] involves homological algebra methods for relating the extensions of the notion of a Prufer domain to semihereditary rings, rings of $w.gl.\dim R \leqslant 1$, and Gaussian rings; as well as provides a homological algebra characterization of coherent Gaussian rings. In Section 5 all five extensions mentioned above appear, and implication relations between them are clarified. This section surveys briefly recent work by Glaz 2005 [41], and current work of the authors of the present article, Bazzoni and Glaz [7]. In [7] Bazzoni and Glaz introduce a new element into the investigation. This element entails of a closer scrutiny of the five Prufer conditions mentioned above when imposed on the total ring of quotient of a ring. It is through this devise that exact conditions for reversing the implications between the five extensions were found.

2 Multiplicative Ring Extensions: Arithmetical and Prufer Rings

The earliest extension we know of the notion of a Prufer domain to rings with zero divisor, is the notion of an arithmetical ring, defined by Fuchs in 1949 [16]. Specifically:

Definition 2.1 *A ring R is called arithmetical if the lattice formed by the ideals of R is distributive, that is, any three ideals of R, I, J and K satisfy $I \cap (J + K) = (I \cap J) + (I \cap K)$.*

These rings provide an extension to the Prufer domain notion via property 11 of Theorem 1.1. Early investigations into some of the properties of arithmetical rings brought out more similarities to Prufer domains. A wealth of results, some of which we will cite in this article, and further references can be found in Jensen [47, 48], Butts and Smith [11], and Griffin [43], Glaz [41]. In Butts and Smith's article [11] arithmetical rings are called Prufer rings, and what we call nowadays Prufer rings are called α-rings. Theorem 2.2 below is a combination of results from the articles mentioned in this paragraph.

Theorem 2.2 *Let R be a ring. The following conditions are equivalent.*

1. *R is an arithmetical ring.*
2. *The ideals of R_P are totally ordered by inclusion for every prime ideal P of R.*
3. *The ideals of R_P are totally ordered by inclusion for every maximal ideal P of R.*
4. *Every finitely generated ideal of R is locally principal.*
5. *If I and J are ideals of R, and K is a finitely generated ideal of R, then $(I + J) : K = (I : K) + (J : K)$.*
6. *If I and J are two finitely generated ideals of R, and K is an ideal of R, then $K : (I \cap J) = (K : I) + (K : J)$.*
7. *Let L denote the set of all prime ideals P of R satisfying that the ideals of R_P are totally ordered by inclusion. For each ideal I of R, $I = \cap\, (IR_P \cap R)$, as P runs over all prime ideals in L.*
8. *If I and J are two ideals of R with J finitely generated and $I \subset J$, then there exists an ideal K such that $I = JK$.*

The following well know result shows the connection between conditions 2 and 3 of Theorem 2.2 and Krull's characterization of a Prufer domain given in Theorem 1.1, 3 and 4.

Proposition 2.3 *Let R be a domain. Then R is a valuation domain if and only if the ideals of R_P are totally ordered by inclusion for every prime (respectively maximal) ideal P of R.*

Definition 2.4 *Let R be a commutative ring, and denote by $Q(R)$, the total ring of quotients of R. A (fractionary) ideal I of R is invertible if $II^{-1} = R$, where $I^{-1} = \{r \in Q(R) : rI \subset R\}$.*

An invertible ideal is finitely generated and contains a regular element. For an ideal I of R there is a strong relation between invertibility, projectivity, and the property of being locally principal, namely:

Theorem 2.5 *Let R be a ring, and let I be an ideal of R. Then:*
1. *If I is invertible, then I is projective.*
2. *If I is projective, then I is locally principal.*
3. *If I is finitely generated and regular then:*
I is invertible if and only if I is projective if and only if I is locally principal. In particular, the three conditions are equivalent for a finitely generated ideal of a domain R.

The second early extension of the Prufer domain notion is the definition of a Prufer ring. We believe its first appearance in the literature was under the name α-ring in Butts and Smith's article [11]. α-rings were named Prufer rings by Griffin [43].

Definition 2.6 *A ring R is called a Prufer ring if every finitely generated regular ideal of R is invertible.*

Prufer rings appear in the literature in many sources, too numerous to cite fully. We will follow Griffin's article [43] for some of the early results, and mention other as needed in the exposition. Glaz [41], Butts and Smith [11], Huckaba [45], and Larsen and McCarthy [50] provide a more extensive treatment and references on the subject. Griffin [43] extends many of the characterizations of a Prufer domain stated in Theorem 1.1 to Prufer rings. He does this by using an extension of the valuation domain notion to rings with zero divisors, the notion of a Manis valuation [54], and by replacing localizations by prime ideals with other quotient rings associated with prime ideals. In order to exhibit Griffin's results on Prufer rings we digress briefly to outline the definition and few properties of a Manis valuation ring.

Definition 2.7 *A valuation is a map υ from a ring Q onto a totally ordered group G and a symbol ∞, such that for all x and y in Q:*
1. $\upsilon(xy) = \upsilon(x) + \upsilon(y)$.
2. $\upsilon(x+y) \geqslant min\{\upsilon(x), \upsilon(y)\}$.
3. $\upsilon(1) = 0$ *and* $\upsilon(0) = \infty$.

The ring $V = V_\upsilon = \{x \in Q : \upsilon(x) \geqslant 0\}$, together with the ideal $P = P_\upsilon = \{x \in Q : \upsilon(x) > 0\}$, denoted (V, P), is called a Manis valuation pair (of Q). V is called a Manis valuation ring (of Q). G is called the value group of V.

(V, P) is a Manis valuation pair of Q, where V is a subring of a ring Q and P is a prime ideal of V if and only if for every $x \epsilon Q - V$, there exists a $y \epsilon P$ such that $xy \epsilon V - P$. In spite of the similarity to the notion of a valuation domain, there are many important ways in which Manis valuation pairs fail to satisfy the basic analogous properties of valuation domains. For example, there exist Manis valuation rings that are not Prufer rings. And, if (V, P) is a Manis valuation pair, P may not be the unique regular maximal ideal of V, and it may happen that P is not even a maximal ideal of V. These anomalies seem to feed on each other, as for example, we see in the following result from Boisen and Larsen [8]:

Theorem 2.8 *Let R be a ring and let P be a prime ideal of R. The following conditions are equivalent:*

1. *(R, P) is a Manis valuation pair and R is a Prufer ring.*
2. *R is a Prufer ring and P is the unique regular maximal ideal of R.*
3. *R is a Manis valuation ring and P is the unique regular maximal ideal of R.*

In [30] Gilmer provides an interesting example of an integrally closed Manis valuation ring, which is not a Prufer ring.

Example 2.9 *An example of a ring R for which each ideal generated by a finite set of regular elements is invertible, R is a Manis valuation ring, but R is not a Prufer ring.*

Let $D = k[x, y]$ be a polynomial ring in two indeterminates over a field k, and let $\{m_\lambda\}$ be the set of maximal ideals of D not containing y. Let $N = \oplus(D/m_\lambda)$, and let $R = D + N$, the idealization of D by N. In [30] it is shown that each ideal of R generated by regular elements is principal, and hence invertible. R is not a Prufer ring since the regular ideal of R generated by x and y is not R-invertible. By the criteria described in the previous paragraph one can check that R is a Manis valuation ring.

For other early examples of this phenomenon the reader is referred to an example by Griffin appearing in Huckaba's book [45, Chapter VI, Example 7], and an example by Boisen and Larsen [8].

We conclude our short digression into the Manis valuation notion by pointing out a case where the notion of a Manis valuation behaves in a desirable domain-like fashion.

Theorem 2.10 *Let V be a ring in which every regular ideal is generated by its set of regular elements (such a ring is called a Marot ring). The following conditions are equivalent:*

1. *V is a Manis valuation ring, and a Prufer ring.*
2. *V is a Manis valuation ring.*
3. *For each regular element x in Q, the total ring of quotients of V, either x or x^{-1} lie in V.*

It is worth noting that the notion of a Manis valuation is just one of the possible extensions of the notion of a valuation domain to rings with zero divisors. Among the several candidates that were considered in the early seventies, only one other is still occasionally called in the literature a valuation ring. This is the notion called in [45] a *chained ring*, namely a ring R whose set of ideals is totally ordered by inclusion. Note that a chained ring is a local arithmetical ring. A chained ring is a Marot ring, and therefore it is a Manis valuation ring and a

Prufer ring. The converse does not hold in general as the following example from [45] shows:

Example 2.11 *An example of a ring R which is not a chained ring, but it is a Manis valuation ring and a Prufer ring.*

Let R be a total ring of quotient that is not chained and let P be a prime ideal of R. Then R becomes a Manis valuation ring and a Prufer ring by defining a valuation map υ satisfying $\upsilon(x) = \infty$ if $x \epsilon P$ and $\upsilon(x) = 0$ if $x \epsilon R - P$.

Next we consider two quotient rings of a ring R associated to a prime ideal P of R.

Definition 2.12 *Let R be a ring and denote by S a multiplicatively closed subset of R.*

The regular quotient ring of R with respect to S, denoted by $R_{(S)}$, is the localization of R by the set of regular elements in S.

The large quotient ring of R with respect to S, denoted by $R_{[S]}$, is the set $R_{[S]} = \{z \; \epsilon \; Q(R) : zs \text{ is in } R \text{ for some } s \text{ in } S\}$

We have the following containments $R \subset R_{(S)} \subset R_{[S]} \subset Q(R)$.

If P is a prime ideal of R and $S = R - P$, we denote the above quotients by the usual $R_{(P)}$ and respectively $R_{[P]}$.

A ring R in which all the regular elements which are not units are contained in a prime ideal P, with $R_{[P]}$ a Manis valuation ring, is called a Manis prevaluation ring.

We are now ready to state Griffin's [43] fifteen conditions equivalent to the definition of a Prufer ring:

Theorem 2.13 *Let R be a ring. The following conditions are equivalent:*

1. *R is a Prufer ring.*
2. *Every two-generated regular ideal is invertible.*
3. *$R_{[P]}$ is a Manis valuation ring for every maximal ideal P of R.*
4. *$R_{(P)}$ is a Manis prevaluation ring for every maximal ideal P of R.*
5. *For every maximal ideal P of R, the regular ideals of R which are contained in P are totally ordered by inclusion.*
6. *If I is a finitely generated regular ideal of R, and J and K are ideals of R with $J = IK$, then $J = K$.*
7. *R is integrally closed and for any two elements a and b in R with a regular, there exists and integer $n > 1$ for which $(a, b)^n = (a^n, b^n)$.*
8. *If I is a finitely generated regular ideal of R, and J is an ideal of R contained in I, then there exists an ideal K such that $J = KI$.*
9. *If $J$and K are ideals of R, one of which is regular, and I is an ideal of R, then $I(J \cap K) = IJ \cap IK$.*
10. *If I and J are ideals of R, one of which is regular, then $(I + J)(I \cap J) = IJ$.*
11. *If I, J, and K are ideals of R with I regular and K finitely generated, then $(I + J) : K = (I : K) + (J : K)$*
12. *If I, J, and K are ideals of R with K regular and I and J finitely*

generated, then $I : (J \cap K) = (I : J) + (I : K)$.

13. *If* I, J, *and* K *are ideals of* R *one of which is regular, then* $I \cap (J + K) = (I \cap J) + (I \cap K)$.
14. *Every overring of* R *is flat.*
15. *Every overring of* R *is integrally closed.*

A comparison between Theorem 2.13 and Theorem 1.1 shows the changes which occurred in conditions 1 - 21 of Theorem 1.1 as a result of this extension of the Prufer domain notion. In spite of the similarities, Prufer rings are much less "tame" then Prufer domains, since the invertibility condition affects the regular ideals, while other ideals seem to have almost random behavior. We also note that Theorem 2.13 lacks a condition analogous to condition 22 of Theorem 1.1. This lack is the driving force behind the recent and current investigations described in Sections 4 and 5.

We conclude this section by exhibiting the relation found in Griffin's article [43] between arithmetical, and Prufer rings.

Definition 2.14 *Let* R *be a ring. A maximal ideal of zero is an ideal (necessarily prime) maximal with respect to not containing regular elements.*

A ring is said to have arithmetical zero divisors if for every maximal ideal of zero P, *the ideals of* R_P *are totally ordered by inclusion.*

Theorem 2.15 *Let* R *be a ring with total ring of quotients* $Q(R)$. R *is arithmetical if and only if* R *is a Prufer ring and* $Q(R)$ *has arithmetical zero divisors.*

3 Homological Algebra Extensions: Rings With Weak Global Dimension Less Or Equal To One

Another early generalization of the Prufer domain notion is that of a semihereditary ring. This notion comes from homological algebra, and the earliest we saw it appear in the literature is in Cartan Eilenberg's book 1956 [12].

Definition 3.1 *A ring* R *is called semihereditary if all finitely generated ideals of* R *are projective.*

Various properties of semihereditary rings were and are considered by many authors. We will restrict ourselves to a number of properties that will place this type of rings in the family of Prufer domain extensions to rings with zero divisors. For the results in this section and additional results and references on semihereditary rings one may consult, for example, Endo [14], Marot [55], Griffin [43], and Glaz [37].

From the homological algebra point of view semihereditary rings belong to the class of rings of finite weak global dimension.

Rings of weak global dimension zero are the so called Von Neumann regular rings. These rings will play an important role in our article and we pause here to record some of their very interesting characterizations.

Theorem 3.2 *Let* R *be a ring. The following conditions are equivalent:*

1. *R is Von Neumann regular.*
2. *Every R-module is flat.*
3. *For every element x in R, there exists an element y in R, such that $x^2y = x$.*
4. *Every element of R can be expressed as a product of a unit and an idempotent.*
5. *Every finitely generated ideal of R is principal generated by an idempotent.*
6. *R_P is a field for every maximal ideal P of R.*
7. *R is a reduced self injective ring.*

We note that Von Neumann regular rings are coherent rings.

Semihereditary rings have weak global dimension less or equal to 1. They are precisely those rings of weak global dimension less or equal to one that are coherent (see Theorem 3.3). This means that Von Neumann regular rings are semihereditary rings. From Theorem 2.5 we deduce that the notion of a semihereditary ring is an extension of the Prufer domain notion. Theorem 3.3 below collects several characterizations of semihereditary rings. Note that like in Theorem 3.2, and contrary to the characterizations of Prufer and arithmetical rings, the emphasis in Theorem 3.3 is on homological properties rather then on multiplicative ring properties.

Theorem 3.3 *Let R be a ring. The following conditions are equivalent:*

1. *R is a semihereditary ring.*
2. *$w.gl.dimR \leqslant 1$ and R is a coherent ring.*
3. *$Q(R)$, the total ring of quotients of R, is Von Neumann regular, and R_P is a valuation domain for every maximal ideal P of R.*
4. *$w.gl.dimR \leqslant 1$ and $w.gl.dimQ(R) = 0$.*
5. *Every finitely generated ideal of R is a summand of an invertible ideal of R.*
6. *Every finitely generated submodule of a projective R-module is projective.*
7. *Every torsion-free R-module is flat.*

In general there are non-coherent rings of weak global dimension one (see for example, Glaz [41]). Nevertheless, a general ring of weak global dimension one can also be considered an extension of the Prufer domain notion to rings with zero divisors. The easiest way to see this is to consider the following characterizations of rings with $w.gl.\dim R \leqslant 1$ [37]:

Theorem 3.4 *Let R be a ring. The following conditions are equivalent:*

1. *$w.gl.dimR \leqslant 1$.*
2. *Every ideal of R is flat.*
3. *R_P is a valuation domain for all prime ideals P of R.*

We conclude this section by pointing out the implications between the four extensions discussed in sections 2 and 3, and some instances where these implications may be reversed. These, and additional results in the same direction, may be found in Glaz [41]. We obtain the following inclusions:

Semihereditary rings $\subset$ Rings with $w.gl.dimR \leqslant 1 \subset$ Arithmetical rings $\subset$ Prufer rings

The second implication follows from Theorem 3.4 (3) and Theorem 2.2(2). Glaz [41] provides a number of examples that show that these implications cannot be in general reversed. The discussion carried out in sections 2 and 3 provides instances in which we can reverse these implications. For example, we know that if a ring is coherent the first implication can be reversed; while if the total ring of quotients has arithmetical zero divisors, the last implication can be reversed. An instance where the middle implication may be reversed was provided by Jensen [47]:

Theorem 3.5 *Let R be a ring. The following conditions are equivalent:*
1. $w.gl.dimR \leqslant 1$.
2. *R is an arithmetical reduced ring.*

4 Recent Focus On An Old Extension: Gaussian Rings

Tsang [58], in her 1965 Ph.D. thesis, defined another extension of the Prufer domain notion to rings with zero divisors. This extension, which she called Gaussian rings, became a focus of intensive recent investigation, primarily because of its connection with the 40 years-old content conjecture of Kaplansky.

Definition 4.1 *Let R be a ring, and let x be a variable over R. Let f be a polynomial in $R[x]$.*

$c(f)$, the content of f, is the ideal of R generated by the coefficients of f.

In general $c(fg) \subset c(f)c(g)$ for every polynomial g in $R[x]$.

A polynomial f satisfying $c(fg) = c(f)c(g)$ for every polynomial g in $R[x]$ is called a Gaussian polynomial.

A ring R is called Gaussian if every polynomial with coefficients in R is a Gaussian polynomial.

Tsang [58] proceeded to prove many important and interesting results on Gaussian polynomials and Gaussian rings, some of which we will reproduce in this section. In particular she showed that a polynomial whose content ideal is invertible, or more generally, locally principal, is a Gaussian polynomial. Kaplansky's conjecture, referred to in the previous paragraph, states that the converse also holds; that is, the content ideal of a Gaussian polynomial must be an invertible, or locally principal, ideal. A number of authors contributed towards a solution of this conjecture: D.D. Anderson and Kang in 1995 [4], Glaz and Vasconcelos in 1997 - 98 [38, 39], Heinzer and Huneke in 1998 [44]. Loper and Roitman in 2005 [51] solved the question affirmatively for all domains. T.G. Lucas in 2005 [52] extended this solution to a partial positive answer for non-domains. One notes, that it is known that in general the answer is No (see Glaz and Vasconcelos [38, 39]). What is still unclear is to what extent is this conjecture and variations on this conjecture true. The following theorem summarizes the results known to date:

Theorem 4.2 *Let R be a ring, and let f be a Gaussian polynomial with coefficients in R, then:*
1. *If $(0 : c(f)) = 0$, then $c(f)$ is locally principal.*
2. *If $c(f)$ is a regular ideal, then $c(f)$ is invertible.*
3. *In particular if R is a domain, then $c(f)$ is invertible.*

A corollary of this theorem is that the notion of a Gaussian ring may be seen as an extension of the notion of a Prufer domain. This was indeed already proved independently by both Tsang [58], and Gilmer [22], and appears as condition 22 in Theorem 1.1. Moreover, the theorem above makes some questions, but not all, on the nature of Gaussian rings easier to answer. The following result summarizes some of the characterizations of Gaussian rings found in Tsang [58] , D.D Anderson and Camillo [3], Glaz [42], and Lucas [53]. D.D Anderson articles [1,2] deal with other aspects of the Gaussian ring property.

Theorem 4.3 *Let (R, m) be a local ring with maximal ideal m. The following conditions are equivalent:*
1. *R is a Gaussian ring.*
2. *For any finitely generated ideal I of R, $I/I \cap (0 : I)$ is a cyclic R-module.*
3. *For any two-generated ideal I of R, $I/I \cap (0 : I)$ is a cyclic R-module.*
4. *For any two elements a and b in R, the following two properties hold:*
   i. *$(a, b)^2 = (a^2)$ or (b^2)*
   ii. *If $(a, b)^2 = (a^2)$ and $ab = 0$, then $b^2 = 0$.*
5. *Every homomorphic image of R is an Armendariz ring, where a ring A is Armendariz if for any two polynomials with coefficients in A, f and g, $c(fg) = 0$ implies that $c(f)c(g) = 0$.*

It is interesting to note that the Gaussian property is a local property, and therefore shedding light on the local case provides much of the information needed to clarify the global case. For example, Tsang [58] showed that the prime ideals of a local Gaussian ring (R, m), are totally ordered by inclusion, therefore if (R, m) is not a domain, its nilradical is its unique minimal prime ideal. It follows that a local Gaussian ring modulo its nilradical is a valuation domain. In particular a reduced local Gaussian ring is a valuation domain. We conclude that a semi-local reduced Gaussian ring is a finite direct sum of Prufer domains.

In 2005 Glaz [42] investigated to what extent the semihereditary condition and the property $w.gl.dimR \leqslant 1$ are close to the Gaussian property. By Theorem 3.4 (3) a ring of $w.gl.dimR \leqslant 1$ is locally and therefore globally Gaussian.

Before we provide the answer given in Glaz [42], we recall two zero divisor controlling conditions on a ring.

Definition 4.4 *A ring R is called a PF ring if principal ideals of R are flat.*

A ring R is called a PP ring (or a weak Bear ring) if principal ideal of R are projective.

The following two results clarify some of the consequences of the PF and PP conditions for a ring (see Glaz [37]).

Theorem 4.5 *Let R be a ring. The following conditions are equivalent:*
1. *R is a PF ring.*
2. *R_P is a domain for every prime ideal P of R.*
3. *R_P is a domain for every maximal ideal P of R.*
4. *R is reduced and every prime ideal of R contains a unique minimal prime ideal.*
5. *R is reduced and every maximal ideal P of R contains a unique minimal prime ideal p. In this case*
$p = \{r \in R :$ there is a $u \epsilon\ R - p$ such that $ur = 0\}$
and $R_p = Q(R_P)$, the quotient field of R_P.

Theorem 4.6. *Let R be a ring. The following conditions are equivalent:*
1. *R is a PP ring.*
2. *For every element a in R, the ideal $(0 : a)$ is generated by an idempotent.*
3. *Every element of R can be expressed as a product of a non zero divisor and an idempotent.*

The PP condition on a ring is related to the PF condition in the following theorem from Glaz [37].

Theorem 4.7 *Let R be a ring. The following conditions are equivalent:*
1. *R is a PP ring.*
2. *R is a PF ring and $Min\ R$, the set of minimal prime ideals of R with the induced Zarisky topology, is compact.*
3. *R is a PF ring and $Q(R)$, the total ring of quotients of R, is Von Neumann regular.*

The reader is referred to [37] for a more extensive exposition on the relations between the conditions mentioned in Theorem 4.7, including a number of examples that show that the two conditions of Theorem 4.7 (2), as well as the two conditions of Theorem 4.7 (3), are independent of each other. We are now ready to bring the results found in Glaz [42] on the relation between the Gaussian property of a ring R and the property $w.gl.dimR \leqslant 1$.

Theorem 4.8 *Let R be a ring. The following conditions are equivalent:*
1. *$w.gl.dimR \leqslant 1$.*
2. *R is a Gaussian PF ring.*
3. *R is a Gaussian reduced ring.*

Theorem 4.9 *Let R be a ring. The following conditions are equivalent*:
1. *R is a semihereditary ring.*
2. *R is a Gaussian PP ring.*
3. *R is a Gaussian ring and $Q(R)$, the total ring of quotients of R, is a Von Neumann regular ring.*

The results of Theorems 4.8 and 4.9 were further used by Glaz [42] to find a classification for coherent Gaussian rings. Recall that a *regular ring* is a ring whose finitely generated ideals have finite projective dimensions.

Theorem 4.10 *Let R be a coherent Gaussian ring. Then either $w.gl.dimR \leqslant 1$, or $w.gl.dimR = \infty$. In particular, if R is a regular ring, then R is a semi-hereditary ring.*

Note that Theorem 4.10 states that a coherent regular Gaussian ring R has $w.gl.dimR \leqslant 1$, a statement that includes the possibility that such a ring is Von Neumann regular. For additional homological properties of coherent Gaussian rings see Glaz [42].

It is worth noting that if R is a Gaussian ring then, by Theorem 4.2 (2), every regular ideal of R is invertible. In addition, by Theorem 2.2 (4) and by Tsang's [58] result, any polynomial over an arithmetical ring is a Gaussian polynomial. We conclude that:

$$\text{Arithmetical rings} \subset \text{Gaussian rings} \subset \text{Prufer rings}$$

Glaz [41] provides examples that show that the above implications are, in general, not reversible.

5 Current Investigation: Prufer Conditions On Total Rings of Quotients

In this section we will describe briefly the current work of the authors Bazzoni and Glaz [7]. This work investigates the relations between all five Prufer conditions mentioned in the introduction. Recall that we have the following implications:

$$\text{Semihereditary rings} \subset \text{Rings of } w.gl.dimR \leqslant 1 \subset \text{Arithmetical rings} \subset \\ \text{Gaussian rings} \subset \text{Prufer rings}$$

In [7], we shifted emphasis from considering the consequences of imposing any of these conditions on the ring itself, to considering the consequences of imposing these conditions on the total ring of quotients of the ring. In other words, we are concerned with the question: What role does any of the five Prufer conditions on R have on a Prufer condition on $Q(R)$, and vice-versa?

We first note [7] that it is easy to answer one direction of this question:

Theorem 5.1 *If R is a ring satisfying any of the five Prufer conditions mentioned above, then $Q(R)$, the total ring of quotients of R, satisfies the same Prufer condition.*

Considering the reverse question, we first note that since a total ring of quotients has no proper regular ideals, any ring which is a total ring of quotients is a Prufer ring. Since there are rings which are not Prufer rings, we conclude that being a Prufer ring is not a property that descends from the total ring of quotients to the ring itself.

Although there are total rings of quotients that are not Gaussian rings (see Example 5.3), the Gaussian property still does not descend from the total ring of quotients to the ring (see Example 5.2).

Example 5.2 *A ring R with $Q(R)$ Gaussian, but R not Gaussian.*

Let R be a local, Noetherian, reduced ring which is not a domain. Such a ring cannot be Gaussian. Since R is Noetherian, $Min\ R$ is a finite set. Let $Min\ R = \{P_1, ..., P_n\}$. Then $Q(R) = R_{P_1} \times ... \times R_{P_n}$, and each R_{P_i} is a field. Thus $Q(R)$, as a direct product of fields, is Von Neumann regular and therefore Gaussian.

Example 5.3 *A non-Gaussian total ring of quotients.*

Let k be a countable, algebraically closed field. Let I be an infinite set and denote by K^I the set of all set maps from I to k. Let N denote the set of natural numbers. Quentel [57] (see Glaz [37], or Huckaba [45] for a corrected version of this ring construction), constructed an algebra $R \subset K^{I\times N^N}$, which is a reduced total ring of quotients that is not Von Neumann regular, but its $Min\ R$ is compact in the induced Zarisky topology. This ring can be shown to be non-Gaussian.

Example 5.4 *A Gaussian ring R, such that its total ring of quotients is not Von Neumann regular.*

Let R be the subring of the countable direct product $\prod Q[x]$, where $Q[x]$ is the polynomial ring in one variable x over the rational numbers Q, consisting of $(x, 0, x^2, 0, x^3, 0, ...)$, and all sequences that eventually consist of constants. It can be calculated that $w.gl.dimR \leqslant 1$, therefore R and its total ring of quotient $Q(R)$ are Gaussian. But $Q(R)$ is not Von Neumann regular by Theorem 3.3.

Additional examples, highlighting the Gaussian behavior of total rings of quotients appear in Bazzoni and Glaz [7].

Before answering the question posed in the first paragraph of this section we proved two results, Theorems 5.5 and 5.6, which are of interest in their own right. The proofs of the following results can be found in Bazzoni and Glaz [7].

Theorem 5.5 *Let R be a ring. Then R is an arithmetical ring if and only if R is a Gaussian ring and $Q(R)$ is an arithmetical ring.*

Theorem 5.6 *Let R be a ring. Then R is a Gaussian ring if and only if R is a Prufer ring and $Q(R)$ is a Gaussian ring.*

Theorems 5.5 and 5.6 provide very powerful tools for the understanding of the exact relations between a ring and its total ring of quotients under any of the five Prufer conditions considered in this article. We summarize these relations in Theorem 5.7. For a proof of Theorem 5.7 and other results on this topic see Bazzoni and Glaz [7]:

Theorem 5.7 *Let R be a ring, and let $Q(R)$ be its total ring of quotients. The following conditions hold:*

1. *R is a Gaussian ring if and only if R is a Prufer ring and $Q(R)$ is a Gaussian ring.*
2. *R is an arithmetical ring if and only if R is a Prufer ring and $Q(R)$ is an arithmetical ring.*
3. *R has $w.gl.dimR \leqslant 1$ if and only if R is a Prufer ring and $w.gl.dimQ(R) \leqslant 1$.*
4. *R is a semihereditary ring if and only if R is a Prufer ring and $Q(R)$ is a semihereditary ring.*
5. *In the implications between Prufer conditions displayed at the beginning of this section, we enumerate the Prufer conditions from left to right, starting with 1 for semihereditary rings, and ending with 5 for Prufer rings. Then, R has Prufer condition n if and only if R has Prufer condition $n+1$ and $Q(R)$ has Prufer condition n, for all $1 \leqslant n \leqslant 4$.*
6. *If $Q(R)$ is Von Neumann regular then all five Prufer conditions on R are equivalent.*

We note that Theorem 5.7 (2) can also be deduced by showing that if a total ring of quotient is arithmetical then it must have arithmetical zero divisors, and then employing Theorem 2.15. We also note that Theorem 5.7 (6) can easily be deduced from a result of Griffins' [43].

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