PRÜFER CONDITIONS IN RINGS WITH ZERO-DIVISORS

SARAH GLAZ

Department of Mathematics
University of Connecticut
Storrs, CT 06269
glaz@uconnvm.uconn.edu

1. INTRODUCTION

In his article: "Untersuchungen über die Teilbarkeitseigenschaften in Körpern" J. Reine Angew. Math. 168, 1 - 36, 1932 [21], Heinz Prüfer introduced a new class of integral domains, namely those domains R in which all finitely generated ideals are invertible. He also proved that to verify this condition, it suffices to check that it holds for all two-generated ideals of R. This was the modest beginning of the notion of a Prüfer domain, a notion which made, and continues to make, a significant impact on research in non-Noetherian commutative ring theory. Heinz Prüfer (1896 - 1934) in his short life, had no opportunity to see the rings named in his honor by Krull ([17], 1936). It is not an exaggeration to say that today there is no conference on a non Noetherian ring theory topic where the notion of a Prüfer domain does not make an appearance.

Prüfer domains acquired, through the years, a great many equivalent characterizations, each of which can, and was, extended to rings with zero divisors in a number of ways. The purpose of this article is to explore the relations between several extensions of the notions of a Prüfer domain to rings with zero divisors. These generalizations fall into the category which may be called homological characterizations, though work on all the rings mentioned in this article entails a blend of ring theoretic and homological techniques. We consider the following extensions of the Prüfer domain:

1. R is a semihereditary ring.
2. w.dim R ≤ 1
3. R is an arithmetical ring
4. R is a Gaussian ring.
5. R is a Prüfer ring.

In Section 2, we provide the necessary background material, including definitions, a short historical survey, and the motivation for this investigation. Section 3 is devoted to implications and counterexamples. We use a blend of known and new results to show the implications: $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$. We provide examples that show that none of the implications is, in general, reversible. We also explore necessary and sufficient conditions under which some of the implications become reversible.

2. BACKGROUND

A domain D is called a *Prüfer domain* if every finitely generated ideal of D is invertible. This is the original definition given by Prüfer in 1932 [21].

In 1936 Krull [17] turned his attention to Prüfer domains, he named them, and proved the first equivalent definition of such a domain, namely:

THEOREM 2.1 (Krull [17]) *D is a Prüfer domain if and only if every localization of D by a prime (respectively maximal) ideal of D is a valuation domain.*

For the purpose of later generalizations to rings with zero divisors we note that for a domain D to be a valuation domain it is necessary and sufficient that the set of all the ideals of D be totally ordered by inclusion.

Let R be a ring and let f be a polynomial in R[x]. c(f) – *the content of f* , is the ideal of R generated by the coefficients of f. For any two polynomials f and g in R[x], we have $c(fg) \subseteq c(f)c(g)$. f is called a *Gaussian polynomial* if $c(fg) = c(f)c(g)$ for every $g \in R[x]$. A ring R is called a *Gaussian ring* if every polynomial with coefficients in R is a Gaussian polynomial. Both definitions are due to Tsang [23].

In 1965, Tsang [23], and independently in 1967, Gilmer [9] provided the following elegant characterization of a Prüfer domain:

THEOREM 2.2 (Tsang [23], Gilmer [9]) *D is a Prüfer domain if and only if D is a Gaussian domain*.

Let R be a commutative ring, and denote by Q(R), the total ring of quotients of R. A (fractionary) ideal I of R, is *invertible* if $I I^{-1} = R$, where $I^{-1} = \{r \in Q(R) / rI \subset R\}$. An invertible ideal is finitely generated and contains a regular element.

For an ideal I of R there is a strong relation between invertibility, projectivity, and the property of being locally principal, namely:

THEOREM 2.3 *Let R be a ring, and let I be an ideal of R. Then:*

1. If I is invertible, then I is projective.

2. If I is projective, then I is locally principal.

3. If I is finitely generated and regular then:

I is invertible if and only if I is projective if and only if I is locally principal.

In particular, the three conditions are equivalent for a finitely generated ideal of a domain R.

A ring R is called a *semihereditary ring* if every finitely generated ideal of R is projective. The origin of this notion is obscure. The earliest source I encountered, where semihereditary rings are mentioned is [5]. In view of the above theorem, we conclude:

COROLLARY 2.4 *D is a Prüfer domain if and only if D is a semihereditary domain.*

It follows that both Gaussian rings and semihereditary rings extend the notion of a Prüfer domain to rings with zero divisors.

Prüfer's original definition was generalized almost verbatim by Butts and Smith [4]. R is a *Prüfer ring* if every finitely generated regular ideal of R is invertible. In 1970 ,Griffin [14], studies these rings and a number of related generalized notions. Among many other results his article includes the proof of fourteen equivalent definitions for a Prüfer ring, reminiscent of similar results known to hold for Prüfer domains (see [9], and [7]).

It remains to consider generalizations of Krull's equivalent definition of a Prüfer domain. There are two equally natural candidates for this purpose. The first is an arithmetical ring. A ring R is called an *arithmetical ring* if the set of ideals of every localization of R by a prime (respectively maximal) ideal of R are totally ordered by inclusion. This notion is due to Fuchs [8]. The second generalization is to a ring R with weak global dimension of R less or equal than one, $w.dim\ R \leq 1$. Those are the rings over which every module has weak dimension at most one. Equivalently, w.dim R ≤ 1 means that every ideal of R is flat. Another characterization of rings R with w.dim R ≤ 1 is that they are precisely the rings whose localizations at the prime ideals are all valuation domains (see [10]). This makes them good candidates for extensions of Prüfer domains as characterized by Krull. Hence both arithmetical rings, and rings of w.dim R ≤ 1 generalize Krull's characterization of a Prüfer domain. The difference between the two notions is that the localizations by prime ideals of an arithmetical ring need not be domains.

We therefore arrived at the full list of five generalizations mentioned in the introduction.

The motivation for our interest in these particular five generalizations was our interest in the notion of a Gaussian ring. This interest first took form of a joint work with W. Vasconcelos regarding a conjecture of Kaplansky: The content ideal of a Gaussian polynomial is an invertible (or locally principal) ideal. The reason behind the conjecture is that the converse holds [23]. Vasconcelos & myself answered the question affirmatively in a large number of cases [11, 12]. The affirmative answer was later extended by Heinzer & Huneke [15] to include all Noetherian domains. Recently the question was answered affirmatively for all domains by Loper & Roitman [18], and finally to non-domains provided the content ideal has zero annihilator by Lucas [19]. A flurry of related research ensued, particularly investigations involving Dedekind-Mertens Lemma and various extensions of the Gaussian property. [2] and [6] provide a survey of results obtained up to year 2000 and an extensive bibliography.

A related, but different, question is : How Prüfer-like is a Gaussian ring? Various aspects of the nature of Gaussian rings were investigated in Tsang's thesis [23], by D.D. Anderson [1], D.D. Anderson & Camillo [3], and Glaz [13]. While all of those works touch indirectly on the mentioned question, it is Glaz [13] that asks and provides some direct answers. We will cite relevant results later on in this article.

3. IMPLICATIONS AND COUNTEREXAMPLES

3.1 The relation between semihereditary rings and rings R of w.dim R $\leq$1.

THEOREM 3.1.1. *Semihereditary rings have w.dim R $\leq$1.*

Proof. This implication is well known. We sketch one possible proof here: Since every principal ideal of a semihereditary ring R is projective, we obtain that for every prime ideal P of R, R_P is a domain. Moreover, finitely generated ideals of R_P are projective and therefore free. Thus R_P is a Bezout domain, and hence a valuation domain. This implies that w.dim R = sup{w.dim R_P | P runs over all prime ideals of R}$\leq$ 1.

EXAMPLE 3.1.2. *A non semihereditary ring R with w.dim R $\leq$ 1.*

Let Q be the rational numbers, and let Q[x] be the ring of polynomials in

one variable over Q. Let R be the subring of ΠQ[x], the infinite product of Q[x], consisting of the sequence (x, 0, x^2, 0, x^3, 0, ...), and all sequences that eventually consist of constants.

The following is a compilation of the known conditions [10, 20] under which a ring of w.dim R $\leq$ 1 is semihereditary.

THEOREM 3.1.3.(Glaz [10], Marot [20]) *Let R be a ring. The following conditions are equivalent:*

1. *R is a semihereditary ring.*
2. *R is a coherent ring of w.dim R$\leq$ 1.*
3. *w.dim R $\leq$1 and the total ring of quotients of R, Q(R), is Von Neumann regular.*

Proof. The implication 1 $\leftrightarrow$ 2 is well known and due, independently, to Marot [20], and Glaz [10]. The implication 1$\leftrightarrow$ 3 follows from a result of Marot [20], which states that a ring R is semihereditary if and only if Q(R) is Von Neumann regular, and R_P is a valuation domain for every prime ideal P of R.

3.2. The relation between rings R of w dim R $\leq$ 1 and arithmetical rings.

THEOREM 3.2.1 (Jensen [16]) *A ring with w.dim R $\leq$ 1 is an arithmetical ring.*

Proof. The proof is due to Jensen [16]. The remarks above regarding the nature of the localizations of both type of rings makes the proof of this implication clear.

We now cite a very useful characterization of arithmetical rings due to Jensen [16], which helped us find a counterexample to the converse of Theorem 3.2.1.

THEOREM 3.2.2 (Jensen [16]) *R is an arithmetical ring if and only if every finitely generated ideal of R is locally principal.*

EXAMPLE 3.2.3. *An arithmetical ring R with w.dim R > 1.*

R = Z_4, the ring of integers modulo 4. Since its only ideal is $2Z_4$, Z_4 is arithmetical. Note that Z_4 is not a reduced ring since $2^2 = 0$. By Theorem 3.2.4. below, we conclude that w.dim Z_4 > 1.

THEOREM 3.2.4. (Jensen [16]) *Let R be a ring. The following conditions are equivalent:*

1. w.dim R ≤ 1.

2. R is an arithmetical reduced ring.

3.3. The relation between arithmetical rings and Gaussian rings.

THEOREM 3.3.1. *An arithmetical ring is a Gaussian ring*.

Proof. Let f be a polynomial with coefficients in an arithmetical ring R. Then by Theorem 3.2.2. c(f) is a locally principal ideal. It is proved in [23] that such polynomials are Gaussian, therefore R is a Gaussian ring.

EXAMPLE 3.3.2. *A non-arithmetical Gaussian ring.*

Let k be a field, and let t and u be indeterminates over k. Let T and U be the images of t and u in $k[t,u]/(t, u)^2$, and let $R = k[T,U]_{(T, U)}$. R is a local ring with maximal ideal m = (T, U). Because $m^2 = 0$, one can easily check that R is Gaussian. But m is not principal so R is not arithmetical.

Of recent vintage, is the investigation carried out in Glaz [13], into the nature of the weak global dimension of a Gaussian ring. Among other results we found necessary and sufficient conditions for a Gaussian ring R to be of w.dim R $\leq$ 1, or semihereditary. After a brief preliminary discussion we cite the results obtained in this paper in Theorems 3.3.4. and 3.3.5. below.

Recall two conditions one may impose on the set of principal ideals of a ring R to gain some control over the behavior of its zero divisors. R is called a *PF ring* if the principal ideals of R are flat. This condition is equivalent to R being locally a domain [10]. A ring R is called a *PP ring*, or *weak Baer ring*, if the principal ideals of R are projective. The PP condition is stronger then the PF condition. The exact relation between the two conditions is given in Theorem 3.3.3., below. Min R denotes the set of all minimal prime ideals of R in the induced Zariski topology from Spec R.

THEOREM 3.3.3. (Glaz [10]). *Let R be a ring. The following conditions are equivalent:*

1. R is a PF ring and Min R is compact.
2. R is a PP ring.
3. R is a PF ring and Q(R), the total ring of quotients of R, is a Von Neumann regular ring.

We now note the conditions from [13], that allow reversal of implications involving the Gaussian condition.

THEOREM 3.3.4. (Glaz [13]) *Let R be a ring. The following conditions are equivalent:*

1. w dim R $\leq$ 1.
2. R is a Gaussian PF ring.
3. R is a Gaussian reduced ring.

THEOREM 3.3.5 (Glaz [13]) *Let R be a ring. The following conditions are equivalent:*

1. R is a semihereditary ring.
2. R is a Gaussian PP ring.
3. R is a Gaussian ring and Q(R) is a Von Neumann regular ring.

The two results cited above yield an unexpected bonus, the equivalence, over Gaussian rings, of conditions which are not generally equivalent:

The PF condition implies that a ring is reduced, but the converse is not generally true, as any reduced local ring which is not a domain shows. Theorem 3.3.4. implies that for Gaussian rings the two conditions coincide.

A similar relation holds between the two conditions "R is a PP ring" and "Q(R) is Von Neumann regular". The PP condition implies the Von Neumann regularity of the total ring of quotients. On the other hand the condition "Q(R) is Von Neumann regular" does not , in general, need to imply that principal ideals are projective. To see this let R be a Noetherian, local, reduced ring which is not a domain. Such a ring is, necessarily, not a PF ring. Because the ring is Noetherian it has finitely many minimal prime ideals. The total ring of quotients of R is the product of all the localizations of R by the minimal prime ideals of R. As each such localization is a field, Q(R) is a Von Neumann regular ring. Theorem 3.3.5. implies that the two conditions are equivalent when the ring is Gaussian.

By putting together some of the results and remarks regarding Gaussian and arithmetic rings, we obtain that if we impose on a Gaussian ring any of the zero divisor controlling conditions discussed in this section, be it : PF, PP, "Q(R) is Von Neumann regular" , or "R is reduced" we obtain that R is arithmetical. We do not, at the moment, know the precise conditions needed as an addition to the Gaussian property to make it equivalent to the arithmetic condition. But given the evidence accumulated so far, we dare make the following conjecture:

CONJECTURE 3.3.6. *A ring R is arithmetical if and only if R is a Gaussian ring and Q(R), the total ring of quotients of R, is an arithmetical ring.*

3.4. The relation between Gaussian rings and Prüfer rings.

THEOREM 3.4.1. *Let R be a Gaussian ring, then R is a Prüfer ring.*

Proof. There are several ways one can prove this fact, but it will be nice to deduce it easily from the latest article on Kaplansky's Conjecture, Lucas [19]. Lucas [19] proved a slightly stronger version of the following: Let R be a ring, and let f be a Gaussian polynomial whose content ideal c(f) contains a regular element, then c(f) is an invertible ideal. We conclude that all finitely generated regular ideals of a Gaussian ring are invertible.

EXAMPLE 3.4.2. *A non-Gaussian Prüfer ring*.

Let k be a countable, algebraically closed field, let J be an infinite set, and denote by k^J the set of all maps from J to k. Let N denote the set of natural numbers. Let $L = J \times N^N$ .Quentel([22], and see [10, page 120] for an error-free version) constructed an algebra $R \subseteq k^L$, which satisfies the following four properties:

1. R is a reduced ring.

2. R = Q(R).
3. Min R, the set of all minimal prime ideals of R in the induced Zariski topology, is compact.
4. R is not a Von Neumann regular ring.

Since R = Q(R), every element of R is either a unit or a zero divisor, and therefore R has no regular ideals. Thus R is a Prüfer ring. It is proved in [13] that R is not a Gaussian ring.

We are not sure which additional property will provide necessary and sufficient conditions for the reversal of the implication of Theorem 3.4.1. We just remark that Prüfer rings are very close to being Gaussian rings, namely:

THEOREM 3.4.3. *Let R be a Prüfer ring, and let f and g be two polynomials in R[x]. If c(f) is a regular ideal, then c(fg) = c(f)c(g).*

Proof. It is straightforward from Tsang's [23] result which states that polynomials with invertible content ideal are Gaussian.

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