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Chapter 1

FINITE CONDUCTOR RINGS WITH ZERO-DIVISORS

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1. INTRODUCTION

The finite conductor property of a domain R, that is the finite generation of the conductor ideals (I:J) for principal ideals I and J of R, came into prominence with the publication of Mc Adam's work [36]. The definition of a finite conductor domain appears in an early unpublished version of Mc Adam's manuscript, but it appears in print for the first time in [11]. The notion embodies, in its various aspects, both factoriality properties and finiteness conditions. Indeed, the class of domains where (I:J) is itself principal is precisely that of Greatest Common Divisor (GCD) domains, while the requirement that (I:J) be finitely generated is a necessary condition for the coherence of a domain. For that reason the finite conductor property makes frequent, explicit or implicit, appearance in the literature in two kinds of, occasionally intermingling, investigations: those involving factoriality and those concerned with finiteness, coherent-like conditions, of domains. Regarding investigations involving factoriality : GCD domains were investigated in their own right for a variety of structural properties, as an aspect of properties of various ring constructions, and as a source of generalizations to other classes of rings. Articles [2, 3, 5, 6, 9, 10, 11, 12, 15, 18, 32, 33, 45, 51] provide just a partial list of references on the subject. Providing a list of references that will do justice to the research done in the subclass of the GCD domains which are UFDs is a formidable task (see [4] for an account and bibliography in this direction). We will just mention here the work of Gilmer, and Gilmer and Parker [17], [18], which has a direct bearing on the central theme of this article. Gilmer and Parker [18] determine conditions under which group rings are UFDs (and GCDs). Gilmer [17] uses this characterization to provide examples of rings which are non Noetherian UFDs, therefore separating the finite conductor condition from Noetherianess. The rings constructed in Gilmer's examples are all coherent. The second kind of investigation involves the interconnection between various finiteness, coherent-like, conditions and the finite conductor property. Here also the literature is too vast to cite in its entirety, for once all research done in coherent rings falls into this category (see [22] for an account on research in coherent rings till 1989). Other coherent-like conditions include Mori domains, PVMD domains, v-domains, v-coherent domains, DVF domains, quasi coherent domains, and several others. [15] defines all these notions and exhibits some of the relations between them. Some articles relating directly between one of the mentioned coherent-like conditions and finite conductor domains are [7, 12, 13, 15, 37, 40, 50]. Quasi coherence in domains, a property falling between finite conductor and coherence, which will play an important role in our investigation, was defined by Barrucci, D.F. Anderson and Dobbs in [7], as part of this kind of investigation.

Our interest in the relation between the finite conductor property and coherence was aroused when we encountered the following statement regarding a domain R, in Gabelli and Houston's paper [15]: "To our knowledge, there are no known examples which prove that these properties are distinct". Gabelli and Houston's paper investigate a number of coherent-like conditions in pullback rings. The nature of the conditions found in [15], necessary to insure that pullback rings are coherent, quasi coherent or finite conductor domains does not yield a ready example differentiating between coherence and the other two properties. The same phenomenon, for D+M constructions, occurs [12].

This survey article centers around the work done in Glaz [25] whose original purpose was to generate various examples of non coherent finite conductor domains. These examples are reproduced here in Sections 5 and 6. The investigation [25] branched out in a more general inquiry on the relation between the finite conductor property and coherence. We follow this inquiry through the extension of the notions of finite conductor and quasi-coherence to rings with zero-divisors. The introduction of zero divisors into play simplified the work of producing counterexamples (see the examples of Section 2), but also introduced a new complexity when attempting to generate positive results. The ring theoretic techniques which worked so well in case the ring is a domain did not suffice to tackle rings with zero divisors. Homological algebra methods, an interplay between finiteness and flatness, turned out to be more useful in this case. Sections 2 and 3 display several basic ascent and descent results under flat extensions, and clarify the relation between these properties for rings of small weak and global dimension. Section 4 highlights the work done in G-GCD rings. G-GCD rings, a class of finite conductor rings defined by Glaz [25], generalize GCD domains, G-GCD domains defined by the Andersons [2], and coherent regular rings. We explore the interplay between finiteness, flatness and projectivity of ideals in G-GCD rings; the nature of their localizations, minimal prime spectrum, total ring of quotients; and their behavior under flat extensions. We end the section with an example [25] of a total ring of quotients which is not a finite conductor ring. Section 5 explores the interplay between the (quasi) coherence of a ring R and finite conductor properties of the polynomial ring R[x]. In particular if R is an integrally closed coherent domain then R[x] is quasi coherent [25]; and if R is a coherent regular ring then R[x] is actually a G-GCD ring [25]. In view of the difficulties involved in ascending the coherence of a ring R to the polynomial ring R[x] (see [22] and [49] for accounts on stable coherence), it is interesting to note that for such a large class of coherent rings at least the quasi coherence (and G-GCD) property ascends to R[x]. We also exhibit an example of a local domain R of w.dim R=2 which is finite conductor (GCD) domain but not coherent [25]. In Section 6 we explore the conditions under which a fixed subring R^G preserves the quasi coherence and finite conductor properties of the ring R. We display an example of a

non coherent UFD fixed ring [25], and utilizing a result and idea of Gilmer and Parker [17,18], we provide a sequence of non coherent UFD group rings with strictly increasing Krull dimensions.
The article is a blend of results appearing in the literature and a number of new examples and results that are nevertheless, in keeping with the survey article tradition, given here without proofs.

2. PRELIMINARIES

Let R be a commutative ring. For two ideals I and J of R, (I:J) denotes the conductor of J into I, that is $(I:J) = \{r\in R \ / \ rJ\subset I\}$. If I=aR and J=bR we write (a:b) for (I:J). $\mu(I)$ denotes the cardinality of a minimal set of generators of I.

Definition 1 R is called a *finite conductor ring* if $aR\cap bR$ and (0:c) are finitely generated ideals of R for all elements a, b and c of R. R is a *called quasi coherent ring* if $a_1R\cap...\cap a_nR$ and (0:c) are finitely generated ideals of R for any finite set of elements c and $a_1,...,a_n$ of R.

Proposition 1 [25] *Let R be a ring. The following conditions are equivalent:*
1. R is a finite conductor ring.
2. Any ideal I of R with $\mu(I) \leq 2$ is finitely presented.
3. (a:b) is a finitely generated ideal of R for all elements a and b of R.

Let Q(R) denote the total ring of quotients of R. Then each of the conditions of Proposition 1 is equivalent to its analogous statement where the elements of R are replaced by the elements of Q(R), ideals of R are replaced by fractionary ideals of R, but "colons" are still taken in R.

In case R is a domain we have an additional equivalent property, which extends to the quasi coherent case as well. For a fractionary ideal I of R denote by $I^{-1}= \{a\in Q(R) \ / \ aI\subset R\}$.

Proposition 2 [25] *Let R be a domain. Then R is a finite conductor ring (respectively a quasi coherent ring) if and only I^{-1} is finitely generated for any ideal* I of R with $\mu(I) \leq 2$ (respectively $\mu(I)< \infty$).

Just like in Proposition 1, Proposition 2 remains valid if the ideal I is replaced by a fractionary ideal I of R.

I^{-1} admits several generalizations to rings with zero divisors, the traditional ones being $(R:_{Q(R)}I)$ and $I^* = Hom_R(I,R)$. Both of these modules coincide if I contains a non zero divisor. Nevertheless, there is no non-domain equivalent of Proposition 2. What can be said is that if R is a finite conductor ring (respectively a quasi coherent ring), and I an ideal generated by non zero divisors with $\mu(I) \leq 2$ (respectively $\mu(I) <\infty$), then I^{-1} is finitely generated. Note also that if R is a quasi coherent domain and I is a *divisorial ideal of finite type*-- that is $I =(J^{-1})^{-1}$ for a finitely generated ideal J of R, then I is finitely generated.

Example 1 It is relatively easy for a ring or domain not to be finite conductor. Such examples abound in the literature. A systematic way one can obtain such examples is by considering a Krull domain R possessing a non finitely generated height one prime ideal (such an ideal is an intersection of two principal fractionary ideals). A simple example of this kind is $R=k[x_ix_j]$, where k is a field and $\{x_i\}$ countably many indeterminates over k. Let L be the quotient field of R. Then $R=k[x_i]\cap L$, and is therefore a Krull domain. The ideal P generated by all x_0x_i is a non finitely generated height one prime ideal of R. R has infinite Krull dimension. An example along the same lines of a Krull domain of Krull dimension three was constructed in [14].

Example 2 The classical finite conductor domains are Unique Factorization Domains (UFDs), and Greatest Common Divisor domains (GCD domains). In both cases $aR\cap bR$ is a principal ideal for all a and b in R, therefore these domains are also quasi coherent. Somewhat less known are the so called Generalized Greatest Common Divisor domains (G-GCD domains) defined in [2]. A domain R is a *G-GCD domain* if the intersection of two invertible ideals of R is an invertible ideal of R. As this property extends to the intersection of finitely many invertible ideals such a domain is quasi coherent.

Example 3 [25] There are finite conductor (quasi coherent) domains R for which not all $aR\cap bR$ are invertible ideals of R. One such Noetherian domain appears in [30].
$R=k[x^2,x^3,y,xy]$, where k is a field and x and y are indeterminates over k. $m=(x^2,x^3,y,xy)$ is a non invertible-- in fact $mm^{-1}=m$, maximal ideal of R, and $x^2R\cap x^3R=x^3m$ is not invertible. The same phenomenon occurs in any local Noetherian domain of Krull dimension 1 which is not a DVR.

Example 4 There are finite conductor (quasi coherent) rings which are not coherent rings. Based on a construction of Quentel [43], we let $R_i=C[x,y,z]$ be countably many copies of the polynomial ring in three variables over the complex field C. Let $R=\prod R_i$. Let $a=(a_i)$ and $b=(b_i)$ be elements of R. Since C[x,y,z] is a UFD there are elements c_i in C[x,y,z] with $(a_i:b_i)=c_iR_i$, then $(a:b)=cR$ for $c=(c_i)$. It follows that R is quasi coherent. On the other hand as seen in [43] (and [49] Example 8.11), R is not a coherent ring.

Example 5 [25] Let $R=\prod R_i$ and let $I=\prod I_i$ and $J=\prod J_i$ be two ideals of R, then $I\cap J=\prod(I_i\cap J_i)$. Similarly, for an element $c=(c_i)$ in R $(0:c)=\prod(0:c_i)$. Thus $I\cap J$ is a finitely generated ideal of R if and only if $\sup\{\mu(I_i\cap J_i)\}<\infty$; and (0:c) is finitely generated if and only if $\sup\{\mu((0:c_i))<\infty$. In particular if each R_i is a finite conductor ring (respectively a quasi coherent ring), then $R=\prod^n_{i=1}R_i$ is a finite conductor ring (respectively a quasi coherent ring).

3. FLATNESS

Flat Limits

Let $\{R_i\}$ be a directed system of rings with directed index set, and let $R=\lim R_i$. Recall that R is called *a flat direct limit of R_i*, if for every $j \geq i$, R_j is flat over R_i. R is called *a faithfully flat direct limit of R_i*, if for every $j \geq i$, R_j is faithfully flat over R_i.

Proposition 3 *Let $\{R_i\}$ be a directed system of rings with directed index set, and let $R=\lim R_i$ be a flat direct limit of R_i. If for every i, R_i is a finite conductor ring (respectively a quasi coherent ring), then R is a finite conductor ring (respectively a quasi coherent ring).*

Flat Ring Extensions

Proposition 4 [25] *Let A → B be a ring extension. If B is faithfully flat over A and B is a finite conductor ring (respectively a quasi coherent ring), then A is a finite conductor ring (respectively a quasi coherent ring).*

Example 6 [25] If A⊂B and B is merely flat over A the extension does not need to descend the finite conductor property, even when A and B are both domains and B is a Noetherian UFD. To see this let $A=k[x,yx,yw,y^2w,y^3w,...]\subset B=k[x,y]$, where k is a field, x and y are indeterminates over k, and w=yx+1. It is shown in [21] that B is flat over A. A is not a finite conductor domain as $(yx:x)=(yx,yw,y^2w,y^3w,...)$ is not a finitely generated ideal of A.

Localizations

Proposition 5 [25] *Let S be a multiplicatively closed subset of R. If R is a finite conductor ring (respectively a quasi coherent ring) then R_S is a finite conductor ring (respectively a quasi coherent ring).*

In particular every localization of a finite conductor ring (respectively a quasi coherent ring) at a maximal ideal is a finite conductor ring (respectively a quasi coherent ring). The converse holds for a ring R with finitely many maximal ideals m_i, since then the ring $T=\prod R_{m_i}$ is a finite conductor ring (respectively a quasi coherent ring) which is faithfully flat over R; but does not hold in general. Nagata-Harris example [39],[27] (see [22] pages 51- 54 for details) provides a non finite conductor ring (not a domain) whose localizations at every maximal ideal is a field or a DVR. An example of a non finite conductor domain whose every localization at a maximal ideal is a finite conductor ring is given [37].

Rings of Small Weak and Global Dimensions

We now turn our attention to rings of small weak and global dimensions. Rings R of w.dim R=0 are precisely the Von Neumann regular rings and as such coherent. The next result clarifies the situation for rings or weak dimension 1.

Proposition 6 [25] *Let R be a ring of w.dim R=1. The following conditions are equivalent:*
1. R is a semihereditary ring.
2. R is a coherent ring.
3. (0:c) is a finitely generated ideal of R for every element c of R.
In particular a domain R of w.dim R=1 is a coherent ring.

Thus for a ring of weak dimension 1 the finite conductor, the quasi coherent, and the coherent conditions coincide. If the weak dimension of R is two this is not necessarily true.

Theorem 7 [25] *Let R be a ring of w.dim R=2. If R is a finite conductor ring then R is a quasi coherent ring.*

We remark that if w.dim R=2, the finite conductor, and hence quasi coherence, property does not necessarily imply coherence, even in case R is a local domain. This is shown in Example xxx

As a consequence we can clarify the relation between the three properties for rings of small global dimension. 1 and 2 of Corollary 8 are well known.

Corollary 8 *Let R be a ring. Then:*
1. gl.dim R=0 if and only if R is semisimple. Hence in this case R is Noetherian.
2. gl.dim R=1 if and only if R is hereditary. Hence in this case R is coherent.
3. If gl.dim R=2, R is coherent if and only if (0:c) is finitely generated for every element c of R.

4. G-GCD RINGS

In this section we explore the properties of G-GCD rings defined in [25]. These rings generalize GCD domains, G-GCD domains defined and explored in [2], and coherent regular rings. The motivation behind our definition is the result of Theorem 7, and indeed a finite conductor ring R of w.dim R=2 will be an example of a G-GCD ring.

Let R be a ring, and denote by Q(R) the total ring of quotient of R. For a fractionary ideal of R, I, let I^{-1}={ a∈Q(R) / aI⊂R}. A fractionary ideal I of R is called **invertible** if II^{-1}=R. Write $a_1b_1+...+a_nb_n=1$, with a_i∈I and $b_i \in I^{-1}$, i≥1. Then it is clear that I is finitely generated-- by $a_1,...,a_n$, and that I contains a non zero divisor-- the multiplication of all the denominators of b_i. It is well known (see for example [47]) that a fractionary ideal I is invertible if and only if I is a projective R module containing a non zero divisor.

Definition 2 [25] A ring R is called *a G-GCD ring* if the following two conditions hold:
C1. Every principal ideal of R is projective.
C2. The intersection of any two finitely generated flat ideals of R is a finitely generated flat ideal of R.

A ring satisfying C1 is sometimes known in the literature as a pp ring. Note that C1is equivalent to: (0:c) is a finitely generated ideal and cR is a flat ideal for every element c of R. In the presence of C1, C2 becomes equivalent to: The intersection of any two principal (fractionary) ideals of R is a finitely generated flat (fractionary) ideal of R.

Also note that if R is a domain the above definition coincides with the definition of a G-GCD domain. It is clear that G-GCD rings are quasi coherent rings.

In the following propositions we collect some properties of a G-GCD ring.

Proposition 9 [25] *Let R be a G-GCD ring. Then the following hold:*
1. R is a reduced ring and R_P is a GCD domain for every prime ideal P of R.
2. R is integrally closed in its total ring of quotients.
3. Min R, the set of all minimal prime ideals of R, is compact in the induced Zariski topology.
4. Q(R), the total ring of quotients of R, is a Von Neumann regular ring.

G-GCD rings are well behaved with respect to faithfully flat extensions.

Proposition 10
1. Let {R_i} be a directed system of rings with directed index set, and let R = lim R_i be a

faithfully flat direct limit of R_i. If each R_i is a G-GCD ring, then R is a G-GCD ring.
2. Let A → B be a ring extension with B faithfully flat over A. If B is a G-GCD ring then A is a G-GCD ring.

At first glance it seems that we could define two stronger generalizations of a G-GCD domain by requiring that principal ideals of R are projective and replacing the second condition by either of the following:
C2'. The intersection of two finitely generated projective ideals of R is a finitely generated projective ideal of R.
C2". The intersection of two invertible ideals of R is an invertible ideal of R.

The next result, somewhat surprisingly, shows that neither of these requirements generates a new class of rings.

Theorem 11 [25]
1. Let R be a ring whose principal ideals are projective and let I be a finitely generated flat ideal of R. Then I is a projective ideal. In particular, R is a G-GCD ring if and only if C1 and C2. are satisfied.
2. Let R be a ring whose principal ideals are projective. If aR∩bR is finitely generated projective for any two non zero divisors a and b of R, then aR∩bR is finitely generated projective for any two elements a and b of R. In particular, R is a G-GCD ring if and only if C1 and C2" are satisfied.

In [3] D.D. Anderson and Markanda defined a GCD ring as a ring R in which every two non zero divisors have a greatest common divisor. Like in the domain case this condition is equivalent to aR∩bR is principal for any two non zero divisors a and b in R. If principal ideals of such a ring R are projective then, by Theorem 11, aR∩bR is principal for any two elements a and b in R. We, therefore, include this condition in the definition of a GCD ring. A ring R is *a GCD ring* if principal ideals of R are projective and the intersection of any two principal ideals of R is a principal ideal of R. With this definition, GCD rings are G-GCD rings, and thus satisfy the properties of Proposition 4.1. It also holds that faithfully flat direct limits ascend the GCD property. But it might not be the case that a faithfully flat extension A → B descends the GCD property as IB being principal for an ideal I of A, does not guarantee that I is principal.

As befitting a generalization of GCD domains, not all coherent rings are G-GCD rings (Example 3), neither are all G-GCD rings coherent (Examples 4, xx, xxx, and xxxx), but there is an important class of coherent rings which are G-GCD rings. Recall that a ring R is called *regular* if every finitely generated ideal of R has finite projective dimension. This notion, which agrees with the classical definition of regularity in case the ring is Noetherian, has been extensively studied for coherent rings as well (see [22] for an extensive treatment and bibliography). Coherent rings of finite weak dimension are regular rings, though the converse does not necessarily hold.

Proposition 12 [25] *Let R be a coherent regular ring then R is a G-GCD ring.*

In view of the evidence accumulated so far we hazard the following conjecture:

Conjecture 1 *Let R be a finite conductor regular ring then R is a G-GCD ring*.

Let R be a ring and let Q(R) be its total ring of quotients. In some sense Q(R) is simpler then R itself: if R is a reduced ring then Q(R) retains only the minimal prime ideals of R; if R is a-- not necessarily coherent, G-GCD ring, then Q(R) is a Von Neumann regular ring, and thus coherent. Hence it seems interesting to note that a ring R which is a total ring of quotients-- that is every element of R is either a unit or a zero divisor, does not necessarily have to be a finite conductor ring. The construction of the following example originates in Quentel's paper [41]. Because of various errors in this, otherwise excellent, paper (some of which were corrected in [42]), we refer the reader to the fully corrected version in [22, Chapter 4, Section 2].

Example 7 [25] Let K be a countable, algebraically closed field, let I be an arbitrary infinite set, and let N be the natural numbers. For two sets A and B denote by A^B the set of all set maps from B to A. Let $S = W(R) \subset K^{IxN}$ be the algebra constructed on page 118 of [22]. It was shown in [22, 42] that S satisfies the following properties:
1. S is a reduced ring.
2. S = Q(S).
3. Min S is compact.
4. S is not Von Neumann regular.
It is shown in [25] that a ring S satisfying the above 4 properties contains an element c such that (0:c) is not a finitely generated ideal of S. Therefore S is not a finite conductor domain. (We remark in passing that since R is a reduced, but not a Von Neumann regular ring, Krull dim R>0.)

5. POLYNOMIAL RINGS

In this section we explore the relation between the (quasi) coherence of a ring R and the finite conductor properties of the polynomial ring R[x].

It is well known that if R is a UFD (respectively a GCD domain), then R[x] is a UFD (respectively a GCD domain) (see for example [16]). In [2], the Andersons proved that if R is a G-GCD domain then so is R[x]. Their proof made use of a result of Querre [44], which we cite below:

Theorem 13 [44] *Let R be an integrally closed domain, and let I be a divisorial ideal of R[x]. Then:*
1. $I \cap R = J \neq 0$ implies that I=JR[x].
2. $I \cap R = 0$ implies that there is a polynomial f in R[x] and a divisorial ideal J of R such that I=fJR[x].

Along the same lines we can prove:

Proposition 14 *Let R be an integrally close quasi coherent domain, and let I be a divisorial ideal of finite type of R[x], then I is a finitely generated ideal of R[x].*

Recall that a coherent ring R is called *stably coherent* if the polynomial rings $R[x_1,...,x_n]$ are coherent for every n. It seems that the crucial step in establishing that a ring is stably coherent is the ability to show that R[x] is coherent (although the known proofs do not proceed by induction on the number of variables like in the Noetherian case). It is known that Von Neumann regular rings, semihereditary rings, hereditary rings, and coherent rings of global dimension two are stably coherent rings. (see [22, Chapter 7] for an extensive treatment and bibliography on this topic). Thus coherent rings R of w.dim R≤1, and coherent rings R of gl.dim R≤2, are stably coherent. Soublin [46] provided an example of a coherent ring R (not a domain) of w.dim R=2, for which R[x] is not coherent. Alfonsi [1] refined Soublin's example to the case R is a domain. All the rings mentioned are coherent regular rings. It follows that it is not easy for a coherent ring R to ascend coherence to R[x]. It is therefore interesting to note that large classes of coherent rings R asccend some coherent like properties to R[x].

Theorem 15 [25] *Let R be an integrally closed coherent domain. Then R[x] is a quasi coherent domain.*

It is not clear to what extent the result of Theorem 15 can be extended to rings with zero divisors, R, even in case the zero divisors are controlled (i.e. when principal ideals of R are projective). As Querre's Theorem 13 seems to play a major role in the proof of Theorem 15, a step toward answering this question-- and a question of interest in its own right, will be to obtain an extension of Querre's Theorem to rings with zero divisors. Since the circulation of [25] in preprint form I have been shown other proofs of Theorem 15, which do not involve Querre's Theorem. These proofs depended heavily on techniques and definitions that were considered so far only for domains. At the moment both Querre's Theorem as well as other domain oriented means of proving Theorem 15 do not seem to have non domain extensions, nevertheless in case R is a coherent regular ring a way was found to bypass the difficulty, and ascend the G-GCD property to R[x].

We cite a Lemma that is of interest in its own:

Lemma 16 [25] *Let R be a ring whose principal ideals are projective, then R[x] satisfies the same property. In addition, if f∈R[x] then (0:f)=eR[x] for an idempotent e in R.*

Using these Lemma and the conditions found in [1] for acyclicity of complexes we have:

Theorem 17 [25] *Let R be a coherent regular ring, then R[x] is a G-GCD ring.*

In view of the evidence accumulated in this section we will hazard a conjecture:

Conjecture 2 *Let R be a G-GCD ring, then R[x] is a G-GCD ring.*

We now exhibit an example of a local domain R of w.dim R=2 which is a non coherent GCD domain.

Example 8 [25] We first consider Soublin's example [46]. Let S_i=Q[[t,u]] be countably many copies of the power series ring in two variables t and u over the rational numbers Q, and let S=$\prod S_i$. It is shown in [46] that S is a coherent ring of w.dim S=2 and that the polynomial ring S[x] is not a coherent ring. Since S is a coherent regular ring, S[x] is a G-GCD ring, but S[x] is neither a domain, nor a local ring and w.dim R[x]=3. According to [1], there is a localization of S, S_P, such that S_P[x] is not a coherent ring. As a localization of a regular coherent ring, S_P is a domain. Since $S_P[x]$ is not coherent w.dim S_P=2. Thus w.dim $S_P[x]$=3. To knock down the weak dimension by one and obtain a local ring we consider the ring $R=S_P(x)=S_P[x]_{PS[x]}$. Since $S_P[x]$ is not a coherent ring, R is not a coherent ring [23], w.dim R=w.dim S_P=2 [23]. Clearly R is a GCD domain.

6. FIXED RINGS

In this section we explore the relation between the finite conductor and quasi coherence properties of a ring R and that of the fixed ring R^G, for a group of automorphisms G of R. Let R be a ring, let G be a group of automorphisms of R, and denote by R^G the fixed ring of R. $R^G=\{a\in R \ / \ g(a)=a \text{ for all } g\in G\}$. The conditions under which a coherent ring R descends coherence to R^G were explored in [24]. A crucial restriction involves the existence of a module retraction map $\alpha:R \rightarrow R^G$, that is α is an R^G module homomorphism satisfying $\alpha(a)=a$ for all a in R^G. If a module retraction map from R to R^G exists we say that R^G is *a module retract* of R. Note that the existence of the module retraction map α implies that R contains R^G as an R^G module direct summand. It follows that R^G is a pure R^G submodule of R, and that no proper ideal of R^G blows up in R. Bergman [8] pointed out the existence of such a map in two cases:

1. G is a finite group and o(G), the order of G, is a unit in R.
2. G is *a locally finite group*--that is for every $a\in R$ the orbit of a, Ga, has finite cardinality n(a); and n(a) is a unit in R for every $a\in R$.

Proposition 18 [25] *Let R be a finite conductor ring (respectively a quasi coherent ring). Then R^G is a finite conductor ring (respectively a quasi coherent ring) in the following cases:*
1. G is a locally finite group and R is a flat R^G module.
2. R^G is a module retract of R and R is a flat R^G module.
3. R^G is a module retract of R and R is a finitely generated R^G module.

Proof. If G is a locally finite group then R is an integral extension of R^G, thus in both case 1 and case 2, R is a faithfully flat R^G module, and we can use Proposition 3.2.
To show 3, let $\alpha:R \rightarrow R^G$ be a module retraction map and let $r_1,...,r_n$ be a set of generators of R as an R^G module. Let $a\in R^G$ and let $(0:_R a)$ be generated as an ideal of R by $a_1,...,a_m$. We claim that $\{\alpha(a_ir_j)\ i=1,...,m\ j=1,...,n\}$ generate $(0:_{R^G} a)$ as an ideal of R^G. $a\alpha(a_ir_j)=\alpha(aa_ir_j)=0$, thus $\alpha(a_ir_j)\in(0:_{R^G} a)$. If $b\in(0:_{R^G} a)$, we write $b=\sum_i b_ia_i$ with b_i in R, and $b_i=\sum_j c_{ij}r_j$ with c_{ij} in R^G. Then $b=\alpha(b)=\sum_i\sum_j\alpha(a_ir_j)c_{ij}$. A similar argument shows that if $a_1,...,a_n$ are elements of R^G and $a_1R\cap...\cap a_nR$ is a finitely generated ideal of R, then $a_1R^G\cap...\cap a_nR^G$ is a finitely generated ideal of R^G.

The conditions exhibited in Proposition 18 under which the finite conductor , quasi coherence and coherence properties descent from R to R^G, although not shown to be optimal, are shown to be a pretty tight fit by the multitude of examples provided in [24] of coherent rings R which do not descend coherence to R^G because R^G is not a finite conductor ring. The most convincing, and the simplest example is: $R=k[x,y,\{z_i\}]$, where k is a field with ch $k\neq 2$ and x,y and $\{z_i\}$ for infinitely many i, are indeterminates over k. G=<g> with g the automorphism of R which leaves k fixed and $g(x)=-x$, $g(y)=-y$, $g(z_i)=-z_i$, for all i.

$o(G)=2$ is a unit in R, and so R^G is a module retract of R. R is a coherent domain, but $R^G=k[x^2,y^2,xy,\{z^2_i\},\{xz_i\},\{yz_i\}]$ is not a finite conductor domain as the ideal $(xy:x^2)=(y^2,xy,\{yz_i\})$ is not finitely generated in R^G. Nevertheless there is an example were neither of the conditions of Proposition 18 hold but R descends UFDness to R^G, without descending coherence. This is a variation on the famous Nagarajan's example [38].

Example 9 [25] Example of a local non coherent UFD of Krull dimension 2.
Let F be the field $F=Z_2(\{a_i\},\{b_i\})$, where Z_2 is the prime field of characteristic 2, and $\{a_i\}$ and $\{b_i\}$ are infinitely many variables over Z_2. Let $S=F[x,y]_{(x,y)}$, where x and y are indeterminates over F. Set $p_i=a_ix+b_iy$, and define an automorphism g of R by $g(x)=x$, $g(y)=y$, $g(a_i)=a_i+yp_{i+1}$, $g(b_i)=b_i+xp_{i+1}$, for all i. Let $G=<g>$, then $o(G)=2$, but 2, of course, is not a unit in S. Let $R_0=S^G$. (The original example of Nagarajan asks S to be $F[[x,y]]$). R_0 is a local Krull domain of Krull dimension 2. R_0 is not a coherent ring [25], but R_0 is a UFD [28]. (We remark that R_0 of the power series case satisfies the ascending chain condition for principal ideals [29], therefore if R_0 is a GCD domain, then it is a UFD. It is not known if this ring is even a finite conductor domain. Is seems to be very difficult to determine if even its height one prime ideals-- all of which are intersections of pairs of principal fractionary ideals, are finitely generated.)

If Krull dim R=n, and x is an indeterminate over R then
$n+1\leq$Krull dim $R[x]\leq 2n+1$ [16], thus $R_0[x_1]\subset R_0[x_1,x_2]\subset\ldots$, for R_0 the ring in Example 9 and $\{x_i\}$ infinitely many indeterminates over R_0, provides a chain of non coherent UFDs of strictly increasing Krull dimensions. A more interesting example of the same phenomenon can be constructed by using group rings.

Let G be an abelian group. G is said to be *cyclically Noetherian* if G satisfies the ascending chain condition for cyclic subgroups.

Let R be a ring, let G be an abelian group and denote by RG the group ring of G over R. Gilmer and Parker [18] provided the following characterization of UFD group rings.

Theorem 19 [18] *Let R be an integral domain and let G be a torsion free abelian group. Then RG is a UFD if and only if R is a UFD and G is cyclically Noetherian.*

In [17] Gilmer used this characterization to construct non Noetherian UFDs of arbitrary Krull dimensions. These rings are all of the form KG, where K is a field and therefore coherent rings by [20]. In Example 10 [25], we utilized the idea behind Gilmer's construction to exhibit a collection of non coherent UFDs group rings with strictly increasing Krull dimensions.

Example 10 [25] Let p be a fixed rational prime, and let $Q^{(p)}$ be the additive group of rationals whose denominators are non negative powers of p. Let σ be a p-adic integer which is not rational, and let σ_n be a sequence of rational integers with $\sigma_n\equiv\sigma(\text{mod } p^n)$ for all n. Choose independent elements a, b in $Q^{(p)}\oplus Q^{(p)}$ and put $a_n=p^{-n}(a+\sigma_n b)$ for all n. Let H be the group generated by b and the sequence $\{a_n\}$. For every integer $m\geq 2$, let $H_m=H$ if $m=2$ and $H_m=H\times F_{m-2}$ if $m>2$, where F_{m-2} is a free group of rank m-2. It is shown in [17, 18, see also 16 and 33] that H_m are torsion free cyclically Noetherian groups with rank H=2 and, thus, rank $H_m=m$ for $m>2$. Let R_0 be the fixed ring of Example 9. By Theorem 19 the rings $R_m=R_0H_m$ are UFDs. It is shown in [17] that for a group G with rank $G=t>0$,
Krull dim $RG\geq$Krull dim R+1. Thus Krull dim $R_2\geq 3$, For $m>2$,

$R_m=R_2F_{m-2}=R_2[x_1,x_1^{-1},...,x_{m-2},x_{m-2}^{-1}]$ is integral over the polynomial ring in m-2 variables $A_{m-2}=R_2[x_1+x_1^{-1},...,x_{m-2}+x_{m-2}^{-1}]$ [20]. Therefore
Krull dim R_2+m-2 ≤Krull dim A_{m-2}=Krull dim R_m, and so the Krull dimensions of R_m are strictly increasing. R_m are not coherent rings [25].

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