Zero-Dimensional Commutative Rings
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17 Homological Dimensions of Localizations of Polynomial Rings

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1 INTRODUCTION

Let R be a commutative ring, and let x be an indeterminate over R. For a polynomial $f \in R[x]$, denote by $c(f)$ - the so called content of f - the ideal of R generated by the coefficients of f. Let $U = \{f \in R[x],\ f \text{ is monic}\}$, and $V = \{f \in R[x],\ c(f) = R\} = R[x] - \bigcup\{mR[x],\ m \text{ maximal ideal of } R\}$. U and V are multiplicatively closed subsets of $R[x]$. Set $R\langle x\rangle = R[x]_U$ and $R(x) = R[x]_V$. Then $R[x] \subset R\langle x\rangle \subset R(x)$, $R(x)$ is a localization (rings of fractions) of $R\langle x\rangle$, and both $R\langle x\rangle$ and $R(x)$ are flat R-modules.

Ever since $R\langle x\rangle$ played a prominent role in Quillen's solution to Serre's conjecture (Quillen 1976), and its succeeding generalizations to non-Noetherian rings (Brewer & Costa 1978, Lequain & Simis 1980), there has been a considerable amount of interest in the properties of $R\langle x\rangle$. This interest expanded to include similarly constructed localizations of $R[x]$. Notable among these constructions is the ring $R(x)$, which, through a variety of useful properties, provides a tool for proving results on R via passage to R(x).

The interest in the properties of R<x> and R(x) branched in many directions (Akiba 1980, Anderson 1976, 1977, Anderson, Anderson & Markanda 1985, Anderson, Dobbs & Fontana 1989, Arnold 1969, Brewer & Costa 1978, Brewer & Heinzer 1980, Ferrand 1982, Gilmer 1968, Gilmer & Heitmann 1980, Glaz 1989, 1991, Hinkle & Huckaba 1977, Huckaba & Papick 1980, 1981, LeRiche 1980, McDonald & Waterhouse 1981, Ratliff 1979). Several of these directions consider homological properties of these two rings.

The present article centers around the author's work (Glaz 1989, 1991) exploring the behavior of the weak and global dimensions of R<x> and R(x). It is a combination of survey of literature, the author's published results, and some new results and open problems.

When investigating homological dimensions, in order to obtain results of some significance, one often needs to either impose or highlight the already existing, underlying finiteness conditions of the rings involved. In order to maintain a clear distinction between the weak and global dimensions, we impose a weaker condition than Noetherianess; we ask that most rings involved in our investigation be coherent. This condition arises naturally in this setting since most rings of small weak or global dimension are automatically coherent. This holds true for semisimple rings (that is rings of global dimension 0) - these rings are actually Noetherian; hereditary rings (that is rings of global dimension 1); and Von Neumann regular rings (that is rings of weak dimension 0). Although not all rings of weak dimension 1 are coherent, the ones of most interest are: semihereditary rings are precisely those rings of weak dimension 1 which are coherent (Glaz 1989a). Section 2 of this paper explores the conditions under which R<x> and R(x) are coherent rings.

Historically, the interest in the homological properties of R<x> and R(x) had a ring-theoretic quality concentrating initially in exploring the condition under which R<x> or R(x) are Prüfer domains (that is semihereditary domains) and Dedekind domains (that is hereditary domains). This approach lead to investigations exploring divisibility properties of R<x> and R(x). Section 3 surveys the literature in this direction.

Sections 4 and 5 describe the author's results regarding the behavior of the weak and global dimensions of R<x> and R(x). The motivation for this work was the same as the motivation for the investigations described in Section 3, namely a wish to understand the conditions that make R<x> or R(x) semihereditary or hereditary (not necessarily domains). The ring theoretic approach led to investigations of divisibility, while the homological algebra approach described in Section 4 and 5 led to investigating behavior of weak and global dimensions, coherent regularity and Cohen-Macaulayness. These two kinds of investigations complement each other in shedding light on the nature of the rings R<x> and R(x).

Section 6 presents the author's results regarding coherent regularity, a property that "contains" the homological properties of the previous sections; and also provides a discussion and some new results regarding the property of being Cohen Macaulay.

2 COHERENCE

As mentioned in the introduction, the underlying finiteness condition on rings that is present throughout the investigations described in this paper is that of coherence. This section determines necessary and sufficient conditions for R<x> and R(x) to be coherent rings (Glaz 1989).

THEOREM 2.1 (Glaz 1989) Let R be a ring. The following conditions are equivalent:
1. R[x] is a coherent ring.
2. R<x> is a coherent ring.
3. R(x) is a coherent ring.

Recall that a ring R is called a stably coherent ring if for every positive integer n, the polynomial ring in n variables over R is a coherent ring along with R. The class of stably coherent rings includes a wide variety of rings, to name a few: Noetherian rings, Von Neumann regular rings, semihereditary rings, hereditary rings, coherent rings of global dimension two and several others (Glaz 1989a,1992, Vasconcelos 1972, 1976). If R is a stably coherent ring then clearly so are R<x> and R(x). The surprising aspect of Theorem 2.1 is that if R<x> or R(x) is a coherent ring then so is R[x]. This does not guarantee that R is stably coherent. In fact, it is still an open question whether the coherence of R[x] implies that R is a stably coherent ring.

3 DIVISIBILITY

This section describes some of the results obtained in investigating the divisibility properties of R<x> and R(x) (Arnold 1969, Anderson 1977, Anderson, Anderson & Markanda 1985, Anderson, Dobbs & Fontana 1989, Brewer & Costa 1978, Brewer & Heinzer 1980, Hinkle & Huckaba 1977, Huckaba & Papick 1980, LeRiche 1980). The initial concern was with determining conditions under which R<x> or R(x) is a Prüfer or a Dedekind domain. The investigation then branched into related divisibility properties. Of the variety of divisibility properties that where considered, we chose to present those most directly related to the original concern.

We start the discussion on the Prüfer property of R<x> and R(x) with the basic result:

THEOREM 3.1 (LeRiche 1980, Arnold 1969) Let R be a domain. Then:
1. R<x> is a Prüfer domain if and only if R is a Prüfer domain.
2. R(x) is a Prüfer domain if and only if R is a Prüfer domain.

Prüfer domains, that is semihereditary domains, can be characterized in many ways, and therefore admit many generalizations. We list a few of the classical characterizations of a Prüfer domain R, within the class of domains:
1. Every finitely generated (nonzero) ideal of R is invertible.
2. Every finitely generated ideal of R is locally principal.
3. Every localization of R by a prime ideal is a valuation domain (and therefore its prime spectrum as a poset under inclusion is a tree).

We consider the following generalizations: A ring R is called an arithmetical ring if every finitely generated ideal of R is locally principal. A ring R is called a Prüfer ring if every finitely generated regular ideal is invertible. A domain R is called a treed domain if its prime spectrum as a poset under inclusion is a tree, that is no maximal ideal of R contains incomparable prime ideals. A ring R is called a strongly Prüfer ring if every finitely generated ideal of R with zero annihilator is locally principal. The next three theorems provide the sufficient and necessary conditions for R<x> or R(x) to be an arithmetical ring, a Prüfer ring or a treed domain.

THEOREM 3.2 (Anderson, Anderson & Markanda 1985) Let R be a ring. Then :

1. R<x> is arithmetical if and only if R is arithmetical, Krull dim R $\leqslant$ 1, and every localization of R by a non-maximal prime ideal is a field.
2. R(x) is arithmetical if and only if R is arithmetical.

Theorem 3.2(1) was also partially proved by LeRiche (1980), and Theorem 3.2(2) was also proved by Anderson (1977).

THEOREM 3.3 (Anderson, Anderson & Markanda 1985) Let R be a ring. Then:

1. R<x> is a Prüfer ring if and only if R is a strongly Prüfer ring, Krull dim R $\leqslant$ 1, and every localization of R by a non-maximal prime ideal is a field.
2. R(x) is a Prüfer ring if and only if R is a strongly Prüfer ring.

Particular cases of Theorem 3.3(1) were also proved by LeRiche (1980) and by Brewer & Costa (1978). Particular cases of Theorem 3.3(2) were also proved by Arnold (1969), Hinkle & Huckaba (1977), and Huckaba & Papick (1980).

THEOREM 3.4 (Anderson, Dobbs & Fontana 1989) Let R be a domain. Then:

1. R<x> is treed if and only if R', the integral closure of R, is a Prüfer domain and Krull dim R $\leqslant$ 1. (It follows that R is treed.)
2. R(x) is treed if and only if R is treed and R' is a Prüfer domain.

Next we consider the Dedekind domain property.

THEOREM 3.5 (LeRiche 1980, Arnold 1969) Let R be a domain. Then:

1. R<x> is a Dedekind domain if and only if R is a Dedekind domain.
2. R(x) is a Dedekind domain if and only if R is a Dedekind domain.

Like a Prüfer domain, a Dedekind domain, that is a hereditary domain, has many equivalent characterizations. Rather then listing them we will mention some of the properties of a Dedekind domain. A Dedekind domain is a Noetherian Krull domain. If it has finitely many maximal ideals then it is a PID; if the number of maximal ideals is infinite it is a, so called, Hilbert ring, that is a ring for which every prime ideal is the intersection of maximal ideals. Considering a ring satisfying any of these properties to be a generalization of a Dedekind domain, we determine when R<x> and R(x) are Hilbert rings, PIDs, or Krull domains.

THEOREM 3.6 (Brewer & Heinzer 1980, Anderson, Anderson & Markanda 1985) Let R be a ring. Then:

1. If R is a Noetherian Hilbert ring then so is R<x>.
2. R(x) is a Hilbert ring if and only if R is a Hilbert ring and every prime ideal of R(x) is an extension of a prime ideal of R.
   If R is Noetherian then R(x) is a Hilbert ring if and only if R is a Hilbert ring and Krull dim R $\leqslant$ 1.

A particular case of Theorem 3.6(1) was independently proved by LeRiche (1980).

THEOREM 3.7 (LeRiche 1980, Anderson, Anderson & Markanda 1985) Let R be a domain. Then:

1. R<x> is a PID if and only if R is a PID.
2. R(x) is a PID if and only if R is a Dedekind domain.

THEOREM 3.8 (Anderson, Anderson & Markanda 1985) Let R be a domain. The following are equivalent:

1. R is a Krull domain.
2. R<x> is a Krull domain.
3. R(x) is a Krull domain.

We will conclude this section by considering several related classical divisibility properties, namely the properties of being a GCD, a UFD, or an Euclidean domain.

THEOREM 3.9 (Anderson, Anderson & Markanda 1985) Let R be a domain. Then:

1. R<x> is a GCD (respectively a UFD, respectively an Euclidean domain) if and only if R is a GCD (respectively a UFD, respectively an Euclidean domain).
2. R(x) is a GCD domain if and only if R is a G-GCD domain (that is the intersection of every two nonzero principal ideals of R is an invertible ideal of R).
   R(x) is a UFD if and only if R is a π-domain (that is every nonzero principal ideal of R is a product of prime ideals).
   R(x) is an Euclidean domain if and only if R is a Dedekind domain.

For a variety of other divisibility properties, the reader is advised to consider the papers mentioned in the introduction of this section.

4 WEAK DIMENSION

In this section we explore the relation between the weak dimension of R and that of R<x> or R(x) (Glaz 1989). Using the notion of non-Noetherian grade, we pinpoint exact relations between these weak dimensions, provided that R is a stably coherent ring of finite weak dimension. As corollaries, we determine necessary and sufficient conditions for R<x> and R(x) to be Von Neumann regular and semihereditary.

We discuss briefly non-Noetherian grade, as defined by Alfonsi (1977, 1981). Let R be a ring, let M be a finitely presented R module, and let N be any R module. Then $\mathrm{grade}_R(M,N) \geqslant n$ if there exists a faithfully flat R algebra S, which may be taken to be a polynomial extension of R, and elements $f_1,\dots,f_n \in (0:_S M\otimes_R S)$, the annihilator of $M\otimes_R S$ in S, which form an $N\otimes_R S$ regular sequence. The largest such integer n is $\mathrm{grade}_R(M,N)$. If no largest integer n exists, put $\mathrm{grade}_R(M,N) = \infty$. For a general R module M, $\mathrm{grade}_R(M,N) \geqslant n$ if for every $y \in M$, $(0:_R y)$ contains a finitely generated ideal I_y satisfying $\mathrm{grade}_R(R/I_y,N) \geqslant n$. Finally, let (R,m) be a local ring with maximal ideal m, and let M be any R module. The <u>depth of M</u> is defined as depth M = $\mathrm{depth}_R\, M = \mathrm{grade}_R(R/m,M)$.

In general, if R is a ring with R[x] coherent, the weak dimensions of R<x> and R(x) are "nicely" bounded.

THEOREM 4.1 (Glaz 1989) Let R be a ring for which $R[x]$ is coherent. Then:

1. $\text{w.dim } R \leqslant \text{w.dim } R\langle x\rangle \leqslant \text{w.dim } R + 1$.
2. $\text{w.dim } R \leqslant \text{w.dim } R(x) \leqslant \text{w.dim } R + 1$.

For stably coherent rings of finite weak dimension we have a more accurate description of the relations between these weak dimensions.

THEOREM 4.2 (Glaz 1989) Let R be a stably coherent ring of $\text{w.dim } R = n < \infty$. Then:

1. $\text{w.dim } R(x) = \text{w.dim } R$.
2. If for every non-maximal prime ideal P of R, we have depth $R_P < n$, then $\text{w.dim } R\langle x\rangle = \text{w.dim } R$.
   Otherwise $\text{w.dim } R\langle x\rangle = \text{w.dim } R + 1$.

This theorem has a number of interesting corollaries.

Since depth R_P = ht P for any prime ideal P of a Noetherian regular ring, we obtain:

COROLLARY 4.3 (Glaz 1989) Let R be a Noetherian regular ring. Then $\text{w.dim } R\langle x\rangle = \text{w.dim } R$.

As Von Neumann regular rings are rings of weak dimension zero, we obtain:

COROLLARY 4.4 (Glaz 1989) Let R be a ring. The following are equivalent:

1. R is a Von Neumann regular ring.
2. $R\langle x\rangle$ is a Von Neumann regular ring.
3. $R(x)$ is a Von Neumann regular ring.

As semihereditary rings are (stably)coherent rings of weak dimension one, we obtain:

COROLLARY 4.5 (Glaz 1989) Let R be a ring. Then:
1. R<x> is a semihereditary ring if and only if R is a semihereditary ring of Krull dim R $\leqslant$ 1.
2. R(x) is a semihereditary ring if and only if R is a semihereditary ring.

Corollary 4.5(1) was also proved by LeRiche (1980). Theorem 3.1, the case of a Prüfer domain, can be immediately deduced from Corollary 4.5.

5. GLOBAL DIMENSION

This section explores the relation between the global dimension of R and that of R<x> or R(x) (Glaz 1991). Questions related to global dimension are, in general, more difficult to answer than those related to the weak dimension. For that reason our answers in this direction are not as complete as the ones given in the previous section. We solve the cases of dimensions zero and one and provide examples that show that rings of global dimension greater then one behave differently then rings with lower global dimension.

The case when gl.dim R = 0, that is when R is a semisimple ring, can be solved using Theorem 4.1.

COROLLARY 5.1 (Glaz 1989) Let R be a ring. The following are equivalent:
1. R is a semisimple ring.
2. R<x> is a semisimple ring.
3. R(x) is a semisimple ring.

The case where gl.dim R = 1, that is R is a hereditary ring, was solved through detailed observation into the nature of hereditary rings and the exact relation between prime ideals in R<x> and prime ideals in R . The final result is as follows:

THEOREM 5.2 (Glaz 1991) Let R be a ring. The following are equivalent:

1. R is a hereditary ring.
2. R<x> is a hereditary ring.
3. R(x) is a hereditary ring.

Theorem 3.5, the case of a Dedekind domain, can be immediately deduced from Theorem 5.2.

When considering higher global dimensions we cannot obtain the exact analog of Theorem 5.2. In general we can say that the global dimensions of both R<x> and R(x) are bounded above by gl.dim R + 1.

A ring is <u>$\aleph_0$-Noetherian</u> if every ideal of R is at most $\aleph_0$ generated. If a ring R is $\aleph_0$ -Noetherian, a submodule of an $\aleph_0$ generated module is $\aleph_0$ generated (Osofsky 1968). Consequently, the $\aleph_0$-Noetherian property of R is inherited by both R(x) and R<x>. If a ring is $\aleph_0$-Noetherian the global dimensions of R(x) and R<x> bound the global dimension of R.

THEOREM 5.3 (Glaz 1991) Let R be an $\aleph_0$-Noetherian ring. Then:

1. gl.dim R $\leqslant$ gl.dim R<x> $\leqslant$ gl.dim R + 1.
2. gl.dim R $\leqslant$ gl.dim R(x) $\leqslant$ gl.dim R + 1.

But gl.dim R<x> does not have to be equal to gl.dim R even in the presence of $\aleph_0$-Noetherianess. If a valuation domain R is $\aleph_n$-Noetherian, but not $\aleph_{n-1}$-Noetherian, then gl.dim R = n + 2 (Osofsky 1967); therefore one can construct valuation domains of global dimension 2 and any Krull dimension (Glaz 1991)). Using these constructions one exhibits a family of local, stably coherent, $\aleph_0$-Noetherian domains V of global dimension 2 with gl.dim V<x> = 3.

EXAMPLE 5.4 (Glaz 1991) Let V be a valuation domain of gl.dim V = 2. Then the following hold:

1. gl.dim V(x) = 2.
2. If Krull dim V $\geqslant$ 3, then gl.dim V<x> = 3.
3. If Krull dim V = 2 and V_P is not a discrete valuation domain for the prime ideal P of height one of V, then gl.dim V<x> = 3. If V_P is a discrete valuation domain, then gl.dim V<x>$_M$ $\leqslant$ 2 for every prime ideal M of V<x>.
4. If Krull dim V $\leqslant$ 1, then gl.dim V<x> = 2.

This example raises a number of questions. To voice a few: Do we always have (at least in the presence of $\aleph_0$-Noetherianess) gl.dim R(x) = gl.dim R? Under what exact conditions will we have gl.dim R<x> = gl.dim R? Does the conclusion of Theorem 5.3 hold without the assumption that R is $\aleph_0$-Noetherian?

6. REGULARITY AND COHEN MACAULAYNESS

The notion that in some way encompasses and reflects all the homological properties described in the previous sections is that of non-Noetherian regularity. A ring R is called a regular ring if every finitely generated ideal of R has finite projective dimension. If R is a Noetherian ring this is just one formulation of the classical definition of regularity. The notion was first introduced in a non-Noetherian setting (particularly for local coherent rings) by Bertin (1971). Since then it was extended to general rings (Vasconcelos 1974, Glaz 1987), although it has not been much explored outside of the coherence setting. Every coherent ring of finite weak dimension is a regular ring (the converse does not necessarily hold even for local coherent rings). Therefore the class of coherent regular rings includes all classical non Noetherian rings, in particular Von Neumann regular, semisimple, semihereditary and hereditary rings. Regularity is well behaved under passage to R<x> or R(x).

THEOREM 6.1 (Glaz 1989) Let R be a ring for which R[x] is a coherent ring. The following are equivalent:
1. R is a regular ring.
2. R<x> is a regular ring.
3. R(x) is a regular ring.

The next step in the investigation is to trace the Cohen Macaulay behavior of R<x> and R(x). A ring R is a Cohen Macaulay ring if depth R_P = Krull dim R_P for every prime ideal P of R. It is well known that a Noetherian regular ring is a Cohen Macaulay ring. The first difficulty that we encounter with the non-Noetherian definition is that even "the best" coherent regular rings may not be Cohen Macaulay.
To construct examples we need this preliminary theorem.

THEOREM 6.2 (Glaz 1989) Let (R,m) be a local coherent regular ring with maximal ideal m. Then depth R = w.dim R.

Note that this theorem is the coherent equivalent of Serre's Theorem, which states that for a Noetherian regular ring R with maximal ideal m, gl.dim R = Krull dim R. To see this note that since R is a Noetherian regular ring, w.dim R = gl.dim R, and depth R is equal to the cardinality of a system of parameters for m, which is equal to ht m, and therefore to the Krull dim R.

EXAMPLE 6.3 A class of coherent regular rings which are not Cohen Macaulay.

Let n be any positive integer, k a finite field, and $x_1, \ldots, x_n$ indeterminates over k. Let V be any valuation overring of $k[x_1, \ldots, x_n]$. Then Krull dim V = n, but by the previous theorem, depth V = 1.

This example raises a question that is more ideological then mathematical. Can one find a definition of non-Noetherian Cohen Macaulayness for a ring R which satisfies:

1. If R is a Noetherian ring this definition coincides with the usual Noetherian Cohen Macaulay definition.
2. If R is a regular ring (coherent?) then R is Cohen Macaulay.

We can determine when R<x> and R(x) are Cohen Macaulay. Let R be a ring, and let p be a prime ideal of R with $pR[x] = P \subsetneq Q \subset R[x]$ and $Q \cap R = p$. R is called a <u>strong-S-ring</u> if ht P = ht p for all prime ideals p of R. A strong-S-ring satisfies that Krull dim R[x] = Krull dim R + 1 (Kaplansky 1974). Since grade does not change when passing to faithfully flat extensions, we obtain:

THEOREM 6.4 Let R be a ring. The following are equivalent:

1. R is a Cohen Macaulay strong-S-ring.
2. R<x> is a Cohen Macaulay ring.
3. R(x) is a Cohen Macaulay ring.

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