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Hereditary Localizations of Polynomial Rings

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1. INTRODUCTION

Let R be a commutative ring, and let x be an indeterminate over R. For a polynomial $f \in R[x]$, denote by $c(f)$—the so-called *content of* f—the ideal of R generated by the coefficients of f. Let $U = \{f \in R[x], f \text{ monic}\}$ and $V = \{f \in R[x],\ c(f) = R\} = R[x] - \bigcup \{mR[x],\ m \text{ maximal ideal of } R\}$. U and V are multiplicatively closed subsets of $R[x]$. Set $R\langle x\rangle = R[x]_U$ and $R(x) = R[x]_V$. Then $R[x] \subset R\langle x\rangle \subset R(x)$, $R(x)$ is a localization of $R\langle x\rangle$, and both $R\langle x\rangle$ and $R(x)$ are faithfully flat R modules.

Ever since $R\langle x\rangle$ played a prominent role in Quillen's solution to Serre's conjecture [25], and its succeeding generalizations to non-Noetherian rings [6, 20], there has been a considerable amount of interest in the properties of $R\langle x\rangle$. This interest expanded to include similarly constructed localizations of $R[x]$. Notable among these constructions is the ring $R(x)$, which, through a variety of useful properties, provides a tool for proving results on R via passage to $R(x)$.

The interest in the properties of $R\langle x\rangle$ and $R(x)$ branched in many directions [1–5, 7, 8, 12, 14–16, 21, 23, 26]. Several of these directions consider homological properties of these two rings. Ferrand [8] and McDonald and Waterhouse [23] investigate finitely generated projective modules over $R(x)$. The behavior of the weak dimensions of $R\langle x\rangle$ and $R(x)$ is investigated through the exploration of ascent and descent of Von Neumann regularity, semihereditarity and related conditions, and coherent regularity of the extensions $R \to R\langle x\rangle$ and $R \to R(x)$. In [21], Le Riche determines conditions for the semihereditarity of $R\langle x\rangle$. D. D. Anderson, D. F. Anderson, and Markanda [4], and Huckaba and Papick [15, 16] consider conditions related to semihereditarity such as being a Prüfer or Prüfer-like ring, for both ring constructions. In Glaz [12], we consider the

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exact relations between the weak dimensions of the three rings involved, with applications to ascent and descent of Von Neumann regularity, semi-hereditarity, and coherent regularity. So far, the behavior of the global dimensions of $R\langle x\rangle$ and $R(x)$ has been touched only incidentally, in the presence of Noetherianness which forces its equality to the weak dimension. Thus, [12] considers semisimplicity, that is, global dimension equal to zero, while [4] and [21] consider the property of being a Dedekind domain, that is, a domain of global dimension one. Both semisimple rings and Dedekind domains are Noetherian.

In this paper we consider some aspects of the behavior of the global dimensions of $R\langle x\rangle$ and $R(x)$, without any Noetherianness assumption on the ring R.

Let R and S be two rings of finite global dimension. We say that the extension $R \to S$ *ascends global dimension* if $\operatorname{gl.dim} R = n$ implies that $\operatorname{gl.dim} S \leqslant n$, the extension $R \to S$ *descends global dimension* if $\operatorname{gl.dim} S = n$ implies that $\operatorname{gl.dim} R \leqslant n$. If an extension $R \to S$ both ascends and descends global dimension, then $\operatorname{gl.dim} R = \operatorname{gl.dim} S$.

In Section 2 we prove that the extensions $R \to R\langle x\rangle$ and $R \to R(x)$ both ascend and descend hereditarity, that is, global dimension equal to one. In Section 3 we show that for $\aleph_0$-Noetherian rings these extensions descend any finite global dimension, and provide an example of a class of local, stably coherent, $\aleph_0$-Noetherian rings R of global dimension equal to two, for which the extension $R \to R\langle x\rangle$ does not ascend global dimension.

2. Hereditarity

The main result of this section, Theorem 5, proves that the extensions $R \to R\langle x\rangle$ and $R \to R(x)$ ascend and descend hereditarity. Before proving our main theorem, we require a short discussion and several preliminary results.

Throughout this paper we will maintain the following notation:

For a polynomial $f \in R[x]$, $L(f)$ will denote the leading coefficient of f, that is, if $f = a_n x^n + \cdots + a_0$, with $a_i \in R$ and $a_n \neq 0$ then $L(f) = a_n$. For an ideal I of $R[x]$, $L_n(I)$ will denote the ideal of R consisting of the leading coefficients of all the polynomials in I of degree less or equal to n, and $L(I)$ will denote the ideal of R consisting of the leading coefficients of all the polynomials in I. We have that $I \cap R = L_0(I) \subset L_1(I) \subset \ldots$, and $L(I) = \bigcup L_n(I)$.

For an ideal I of $R[x]$, $c(I)$—*the content of I*— will denote the ideal of R generated by the coefficients of all the polynomials in I. We recall the content formula [9]; for $f, g \in R[x]$ with $\deg g = m$ we have $c(f)^{m+1} c(g) = c(f)^m c(fg)$. In particular, if $c(f) = R$ then $c(g) = c(fg)$, and if R is a domain and $c(f)$ is an invertible ideal of R, then $c(f)\,c(g)=c(fg)$.

Max $R=\{M\in \text{Spec } R, m \text{ is a maximal ideal of } R\}$.

Min $R=\{p\in \text{Spec } R, p \text{ is a minimal ideal of } R\}$ and MiniMax $R=$ Min $R\cap$ Max R.

Finally, K will stand for the total ring of quotients of R, and L for the total ring of quotients of $R\langle x\rangle$. Note that L is also the total ring of quotients of $R(x)$.

PROPOSITION 1. *Let R be a Von Neumann regular ring; then the following conditions are equivalent*:

1. *R is a hereditary ring.*
2. *$R\langle x\rangle$ is a hereditary ring.*
3. *$R(x)$ is a hereditary ring.*

Proof. $1\to 2$. Let I be an ideal of $R\langle x\rangle$. We aim to show that I is projective. Since R is a Von Neumann regular ring, so is $R\langle x\rangle$ [12, Corollary 2], and thus, $R\langle x\rangle/I$ is a flat R module. By [11, Theorem 1.2.15], I is a pure ideal of $R\langle x\rangle$; therefore, for every maximal ideal M of $R\langle x\rangle$ either $I_M=0$ or $I_M=R\langle x\rangle_M$. We conclude that $I=I^2$.

Write $I=JR\langle x\rangle$, where J is an ideal of $R[x]$ containing no monic polynomial. Since I is a proper ideal of $R\langle x\rangle$, $c(J)$, the content of J, is a proper ideal of R. To see this, note that since R is a Von Neumann regular ring, every prime ideal of R is maximal; hence, the prime ideals of $R\langle x\rangle$, necessarily maximal, are of the form $mR\langle x\rangle$ for $m\in \text{Spec } R$. As $I\subset mR\langle x\rangle$ for some prime ideal m of R, we have $J\subset mR[x]$ and $c(J)\subset m$.

Since R is a hereditary ring, $c(J)$ is a projective ideal; therefore, $c(J)$ is equal to a direct sum of principal ideals of R [18, Theorem 4]. As every principal ideal of R is generated by an idempotent, we have $c(J)=\bigoplus Re_i, e_i\in c(J), e_i^2=e_i$, and $e_ie_j=0$ for $i\neq j$. Now: $I=I^2\subset Ic(J)R\langle x\rangle= I(\bigoplus Re_i)\,R\langle x\rangle=(\bigoplus Re_i)\otimes_R IR\langle x\rangle=\bigoplus (Re_i\otimes_R IR\langle x\rangle)=\bigoplus Ie_i\subset I$; thus $I=\bigoplus Ie_i$, and to show that I is projective it suffices to show that Ie_i is projective for every i.

Consider the ideal $Ie_i\oplus R\langle x\rangle(1-e_i)$. We claim that $Ie_i\oplus R\langle x\rangle(1-e_i)$ is not contained in any maximal ideal of $R\langle x\rangle$. Assume that $Ie_i\oplus R\langle x\rangle(1-e_i)\subset mR\langle x\rangle$ for some prime ideal m of R; then $1-e_i\in mR\langle x\rangle$ and $Ie_i\subset mR\langle x\rangle$. As $1-e_i\in mR\langle x\rangle$, $e_i\notin mR\langle x\rangle$; therefore, $I\subset mR\langle x\rangle$. But then $e_i\in c(J)\subset m\subset mR\langle x\rangle$. This contradiction shows that $Ie_i\oplus R\langle x\rangle(1-e_i)=R\langle x\rangle$, and thus, Ie_i is projective.

$2\to 3$. Since $R(x)$ is a localization of $R\langle x\rangle$ we have gl.dim $R(x)\leqslant$ gl.dim $R\langle x\rangle\leqslant 1$, and $R(x)$ is a hereditary ring.

$3 \to 1$. Let I be an ideal of R. We aim to show that I is projective. Since $R(x)$ is a hereditary ring, $IR(x)$ is a projective ideal of $R(x)$. As $R(x)$ is a Von Neumann regular ring [12, Corollary 2], $IR(x) = \bigoplus R(x)e_i$ with $e_i \in IR(x)$, $e_i^2 = e_i$, and $e_i e_j = 0$ for $i \neq j$. By [4, Theorem 2.4], $e_i \in R$ and, therefore, $e_i \in IR(x) \cap R = I$. The ideal $J = \bigoplus Re_i \subset I$ satisfies $J \otimes_R R(x) = I \otimes_R R(x)$. Since $R(x)$ is a faithfully flat R module, $J = I$ and, thus, I is projective.

COROLLARY 2. *Let R be a commutative ring; then the following hold:*

1. *If R is a hereditary ring then so is L, the total ring of quotients of $R(x)$.*

2. *If $R(x)$ is a hereditary ring then so is K, the total ring of quotients of R.*

Proof. 1. Since R is a hereditary ring, it is in particular a coherent ring of finite weak dimension and, therefore, K is a Von Neumann regular ring [11, Theorem 4.2.18]. As gl.dim $K \leqslant$ gl.dim $R \leqslant 1$, K is a hereditary ring. By Proposition 1, $K(x)$ is a hereditary ring.

Let $A = \{r \in R,\ r$ is a nonzero divisor in $R\}$ and $B = \{f/g \in R(x),\ f/g$ is a nonzero divisor in $R(x)\}$; then $B = \{f/g,\ f, g \in R[x],\ c(g) = R$ and $(0:_R c(f)) = 0\} \supset A$. Thus, $L = R(x)_B = (R(x)_A)_B = K(x)_B$ and gl.dim $L \leqslant$ gl.dim $K(x) \leqslant 1$. Thus, L is a hereditary ring.

2. Assume that $R(x)$ is a hereditary ring; write $K(x) = R(x)_A$ to obtain that $K(x)$ is a hereditary ring. Since $R(x)$ is in particular semi-hereditary, it follows that so is R [12, Corollary 3] and, therefore, K is a Von Neumann regular ring. By Proposition 1 we conclude that K is a hereditary ring.

PROPOSITION 3. *Let R be a hereditary ring; then the following hold:*

1. Krull dim $R \leqslant 1$.

2. *Every ideal $m \in$* Max $R -$ MiniMax R *is finitely generated.*

3. *Every ideal $p \in$* Min R *is of the form $p = \bigoplus Re_i$ with $e_i \in p$, $e_i^2 = e_i$, and $e_i e_j = 0$ for $i \neq j$. In particular, any two distinct minimal prime ideals of R are comaximal.*

Proof. 1. Let $p \in$ Spec R; then w.dim $R_p \leqslant$ w.dim $R \leqslant$ gl.dim $R \leqslant 1$.

If w.dim $R_p = 0$ then depth $R_p = 0$ [12, Lemma 3]. Since R_p is a domain [11, Corollary 4.2.4], R_p is a field and, thus, p is a minimal prime ideal of R.

If w.dim $R_p = 1$, then R_p is a valuation domain [11, Corollary 4.2.6] and gl.dim $R_p =$ w.dim $R_p = 1$; thus, pR_p is finitely generated [29], and free. It follows that R_p is a discrete valuation ring and $\operatorname{ht} p \leqslant 1$. Thus, Krull dim $R \leqslant 1$.

2. Let $m \in \operatorname{Max} R - \operatorname{MiniMax} R$; then m is a projective ideal not contained in any minimal prime ideal of R. By [30, Proposition 1.6] m is finitely generated.

3. Let $p \in \operatorname{Min} R$. R is hereditary and, therefore, pK is a proper projective ideal of the Von Neumann regular ring K. Let $pK = \bigoplus Ke_i$ with $e_i \in pK$, $e_i^2 = e_i$, and $e_i e_j = 0$ for $i \neq j$. Since every localization of R is a domain $e_i \in R$; thus, $e_i \in pK \cap R = p$. It follows that $p = pK \cap R = \bigoplus Re_i$. Now if p and q are two distinct prime ideals of R, write $p = \bigoplus Re_i$ with $e_i \in p$, $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$. Since $p \neq q$, $e_i \notin q$ for some i; thus, $1 - e_i \in q$ and $1 = e_i + (1 - e_i) \in p + q$. We conclude that p and q are comaximal.

We now cite a result of Marot [22] which we will use in the proof of our main theorem. The reader can find proofs of various versions of this result in [11, Corollary 4.2.20; 30, Theorem 1.8; 22, Proposition 3.4].

PROPOSITION 4. *Let R be a commutative ring with total ring of quotients K; then the following conditions are equivalent:*

1. *R is a hereditary ring.*

2. *K is a hereditary ring and every ideal of R not contained in any minimal prime ideal of R is projective.*

We are now ready for the main result of this section.

THEOREM 5. *Let R be a commutative ring; then the following conditions are equivalent.*

1. *R is a hereditary ring.*
2. *$R\langle x \rangle$ is a hereditary ring.*
3. *$R(x)$ is a hereditary ring.*

Proof. $1 \to 2$. Assume that R is a hereditary ring. By Corollary 2 we have that L is a hereditary ring. Hence, by Proposition 4, we have to show that if an ideal I is not contained in any minimal prime ideal of $R\langle x \rangle$, then I is projective. Since Krull dim $R \leqslant 1$, $R\langle x \rangle$ is a semihereditary ring [12, Corollary 4]; therefore, to achieve our goal it suffices to show that I is finitely generated. We will first consider the cases where I is a prime ideal of $R\langle x \rangle$ and then proceed to the general case.

By [21, Theorem 3.7] Krull dim $R\langle x \rangle \leqslant 1$. The minimal prime ideals of $R\langle x \rangle$ are of the form $qR\langle x \rangle$, where $q \in \operatorname{Min} R$; therefore, a prime ideal

I which is not minimal is either equal to $mR\langle x\rangle$ for $m\in \operatorname{Max} R - \operatorname{MiniMax} R$ or it is equal to $PR\langle x\rangle$, where P is a prime ideal of $R[x]$ containing no monic polynomial, $P\cap R = p$, $p\in \operatorname{Min} R - \operatorname{MiniMax} R$, and $pR[x]\subsetneqq P$.

Case 1. $I = mR\langle x\rangle$, $m\in \operatorname{Max} R - \operatorname{MiniMax} R$.

In this case I is finitely generated by Proposition 3.

Case 2. $I = PR\langle x\rangle$, P prime ideal of $R[x]$ containing no monic polynomial, $P\cap R = p$, $p\in \operatorname{Min} R - \operatorname{MiniMax} R$, and $pR[x]\subsetneqq P$.

We first remark that such a prime ideal P contains a polynomial f, with $c(f) = R$. To see this we will follow an argument of Ferrand [8]. Since $PR\langle x\rangle$ is a maximal ideal of $R\langle x\rangle$, $P\not\subset mR[x]$ for every $m\in \operatorname{Max} R$. For every $m\in \operatorname{Max} R$ choose $f_m\in P$ with $c(f_m)\not\subset m$; then the sets $D(c(f_m)) = \{p\in \operatorname{Spec} R,\ c(f_m)\not\subset p\}$ are open in $\operatorname{Spec} R$ and cover $\operatorname{Spec} R$. Since $\operatorname{Spec} R$ is quasi-compact, there are $f_1, ..., f_s\in \{f_m\}_{m\in \operatorname{Max} R}$ such that $\operatorname{Spec} R = \bigcup_{i=1}^{s} D(c(f_i))$. Let $d > \deg f_i$, then $f = \sum x^{d^i} f_i\in P$ and $c(f) = R$.

We now proceed with the proof of Case 2.

Let $p = L_0(p)\subset L_1(p)\subset ...$, and $L(P) = \bigcup L_i(P)$ be the ideals of leading coefficients of polynomials in I corresponding to the notation preceding Proposition 1. We claim that $L(P)\not\subset q$ for any $q\in \operatorname{Min} R$. Assume the contrary; let $L(P)\subset q$ for some $q\in \operatorname{Min} R$, then $p = L_0(P)\subset L(P)\subset q$ and $p = q$. Let $g\in P$ and write $g = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_i\in R$, $0\leqslant i\leqslant n$, then $L(g) = a_n\in L_n(P)\subset p$. Since p is a minimal prime ideal of R, there exists a $b\in P - p$ such that $ba_n = 0$ [11, Lemma 3.3.4, Corollary 4.2.4, Theorem 4.2.3(4)]. Then $bg\in P$ and either $ba_{n-1} = 0$ or $ba_{n-1} = L(bg)\in L_{n-1}(P)\subset p$. In either case, $a_{n-1}\in p$. Continue to obtain $g\in pR[x]$ and $P = pR[x]$. Thus, $L(P)\not\subset q$ for any $q\in \operatorname{Min} R$.

R is a hereditary ring; therefore $L(P)$ is a projective ideal not contained in any minimal prime ideal of R, and, thus, $L(P)$ is finitely generated. It follows that the chain $L_0(P)\subset L_1(P)\subset \cdots$ stops. Let u be the first non-negative integer such that $L_u(P)\subset q$ for some $q\in \operatorname{Min} R$; then $p = L_0(P)\subset L_u(P)\subset q$ and $p = q$. For $i > u$ we have that $L_i(P)\not\subset q$ for any $q\in \operatorname{Min} R$ and, therefore, $L_i(P)$ is finitely generated. We have thus far obtained the following situation. $p = L_0(P) = \cdots = L_u(P)\subsetneqq L_{u+1}(P)\subseteq \cdots \subseteq L_n(P) = L_{n+1}(P) = \cdots = L(P)$ and $L_{u+j}(P)$ are finitely generated.

Let $f_1, ..., f_s$ be polynomials in P whose leading coefficients generate $L_{u+1}(P), ..., L_n(P)$, and such that if $L(f_i)$ is a generator of $L_j(P)$ then $\deg f_i\leqslant j$.

Let $g\in P$; by an argument similar to the proof of Hilbert basis theorem [19, Theorem 69], we obtain that $g = \sum_{j=1}^{s} r_j f_j + g'$, with $r_j\in R[x]$, $\deg g'\leqslant u$, then $g'\in P$ and $L(g')\in L_u(P) = p$. Repeating the argument of the previous paragraph we obtain $g'\in pR[x]$; thus:

$P \subset (f_1, ..., f_s) R[x] + pR[x] \subset (f_1, ..., f_s, f) R[x] + pR[x] \subset P$ and $P = (f_1, ..., f_s, f) R[x] + pR[x]$.

Finally, we claim that $I = PR\langle x \rangle = (f_1, ..., f_s, f) R\langle x \rangle$. For that, it suffices to show that for every maximal ideal M of $R\langle x \rangle$ we have $PR\langle x \rangle_M = (f_1, ..., f_s, f) R\langle x \rangle_M$.

$PR\langle x \rangle$ is a maximal ideal of $R\langle x \rangle$; we have, therefore, to prove the following:

(i) For $M = PR\langle x \rangle$, $PR\langle x \rangle_M = (f_1, ..., f_s, f) R\langle x \rangle_M$.

(ii) For $M \neq PR\langle x \rangle$, $(f_1, ..., f_s, f) R\langle x \rangle \not\subset M$.

To prove (i) we write $R\langle x \rangle_{PR\langle x \rangle} = R[x]_P = R_p[x]_{PR_p[x]}$. Since p is a minimal prime ideal of R, R_p is a field; thus, $pR_p = 0$, so $pR\langle x \rangle_{PR\langle x \rangle} = 0$. The claim is now clear.

To prove (ii) let $M = QR\langle x \rangle$ with Q a prime ideal of $R[x]$ containing no monic polynomial. Let $M \cap R = Q \cap R = q$; then either $q \in \operatorname{Max} R$, in which case $Q = qR[x]$ and $M = qR\langle x \rangle$, or $q \in \operatorname{Min} R - \operatorname{MiniMax} R$, in which case $qR[x] \subsetneq Q$ and $p \neq q$. In the first case, if $(f_1, ..., f_s, f) R\langle x \rangle \subset qR\langle x \rangle$, we in particular have that $R = c(f) \subset q$, which is not possible. In the second case, if $(f_1, ..., f_s, f) R\langle x \rangle \subset QR\langle x \rangle$ we have $L(f_i) \in L(Q)$ for $1 \leqslant i \leqslant s$; thus, $L_i(P) \subset L(Q)$ for all i, in particular, $p = L_0(P) \subset L(Q)$. We also have $q = Q \cap R = L_0(Q) \subset L(Q)$. Since p and q are comaximal we obtain $L(Q) = R$. Let $1 = \sum r_i u_i$ with $r_i \in R$ and $u_i \in L(Q)$ and let $g_i \in Q$ with $L(g_i) = u_i$; then for $n > \deg g_i = n_i$, $g = \sum r_i x^{n-n_i} g_i \in Q$ and is monic. This contradiction concludes Case 2.

Case 3. I is a general ideal of $R\langle x \rangle$ which is not contained in any minimal prime ideal of $R\langle x \rangle$.

Let $I = JR\langle x \rangle$ with J an ideal of $R[x]$ containing no monic polynomial; then $J \not\subset qR[x]$ for any $q \in \operatorname{Min} R$, and, thus, $c(J) \not\subset q$ for any $q \in \operatorname{Min} R$.

As in Case 2, consider the chain $L_0(J) \subset L_1(J) \subset \cdots$, and $L(J) = \bigcup L_i(J)$. We claim that $L(J) \not\subset q$ for any $q \in \operatorname{Min} R$. Too see this, assume that $L(J) \subset q$ for some $q \in \operatorname{Min} R$, and let $g \in J$; write $g = a_n x^n + \cdots + a_0$ with $a_i \in R$. Repeat the argument given in the similar situation of Case 2 to obtain that $a_i \in q$ for $0 \leqslant i \leqslant n$ and, thus, $c(J) \subset q$, which is a contradiction. As in Case 2 we can now conclude that $L(J)$ is a finitely generated ideal of R, and let $f_1, ..., f_s \in J$, whose leading coefficients generate $L(J)$. Then $(f_1, ..., f_s) R[x] \not\subset qR[x]$ for any $q \in \operatorname{Min} R$ and, thus, $(f_1, ..., f_s) R\langle x \rangle \not\subset qR\langle x \rangle$ for any $q \in \operatorname{Min} R$.

Consider the ring $R\langle x \rangle/(f_1, ..., f_s) R\langle x \rangle$. The prime ideals of this ring correspond to prime ideals of $R\langle x \rangle$ containing $(f_1, ..., f_s) R\langle x \rangle$. Since none of those are minimal they are precisely the kind of prime ideals of $R\langle x \rangle$ described in Cases 1 and 2 and are, therefore, finitely generated. It follows that $R\langle x\rangle/(f_1, ..., f_s)\,R\langle x\rangle$ is a Noetherian ring and, thus, $I/(f_1, ..., f_s)\,R\langle x\rangle$ is finitely generated. We conclude that I itself is finitely generated.

$2 \to 3$. Clear.

$3 \to 1$. Assume that $R(x)$ is a hereditary ring. By Corollary 2, K is a hereditary ring. By Proposition 4, we have only to show that if I is an ideal of R not contained in any minimal prime ideal of R, then I is projective. The ideal $IR(x) \not\subset qR(x)$ for any $q \in \operatorname{Min} R$. Since $R(x)$ is hereditary $IR(x)$ is a projective ideal of $R(x)$ which is not contained in any minimal prime ideal of $R(x)$, and, thus, $IR(x)$ is finitely generated. Since $R(x)$ is a faithfully flat R module, I is finitely generated. Since R is a semihereditary ring [12, Corollary 3], I is projective.

COROLLARY 6 [4, 21]. *Let R be a commutative ring; then the following conditions are equivalent.*

1. *R is a Dedekind domain.*
2. *$R\langle x\rangle$ is a Dedekind domain.*
3. *$R(x)$ is a Dedekind domain.*

3. EXAMPLES

When considering global dimensions higher than one, we cannot obtain the ascend and descend results of Theorem 5. According to the Hilbert–Syzygies theorem [30, Theorem 0.14] we always have $\operatorname{gl.dim} R\langle x\rangle \leqslant \operatorname{gl.dim} R+1$ and $\operatorname{gl.dim} R(x) \leqslant \operatorname{gl.dim} R+1$. If the ring R is $\aleph_0$-Noetherian, we can do better than that.

Recall that a ring R is called $\aleph_0$-*Noetherian* if every ideal of R is countably generated.

PROPOSITION 7. *Let R be an $\aleph_0$-Noetherian ring; then the extensions $R \to R\langle x\rangle$ and $R \to R(x)$ descend global dimension.*

Proof. Assume that $\operatorname{gl.dim} R\langle x\rangle = n < \infty$ and let I be an ideal of R. Since R is $\aleph_0$-Noetherian, every submodule of a countably generated R module is countably generated [17, Lemma 1]; we, therefore, have an exact sequence, $\cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} I \longrightarrow 0$, with F_i countably generated and free and $K_i = \ker d_i$ countably presented. Tensor this sequence with $-\otimes_R R\langle x\rangle$. As $\operatorname{gl.dim} R\langle x\rangle = n$ we have that $K_i \otimes_R R\langle x\rangle$ is a projective $R\langle x\rangle$ module for some $i \leqslant n-2$. It follows from [27, Part I, 3.1.4] that K_i is a projective R module, and, thus, $\operatorname{proj.dim}_R I \leqslant n-1$. We conclude that $\operatorname{gl.dim} R \leqslant \operatorname{gl.dim} R\langle x\rangle$.

A similar argument yields gl.dim $R \leqslant$ gl.dim $R(x)$.

The extension $R \to R\langle x\rangle$ does not have to ascend global dimension even in the presence of $\aleph_0$-Noetherianess. Here we present an example of a class of local, stably coherent $\aleph_0$-Noetherian rings R of gl.dim $R=2$ but gl.dim $R\langle x\rangle = 3$.

EXAMPLE. *Let V be a valuation domain of* gl.dim $V = 2$; *then the following hold*:

1. gl.dim $V(x) = 2$.
2. *If* Krull dim $V \geqslant 3$, *then* gl.dim $V\langle x\rangle = 3$.
3. *If* Krull dim $V = 2$ *and for the prime ideal of height one of V, p, V_p is not a discrete valuation domain, then* gl.dim $V\langle x\rangle = 3$. *If V_p is a discrete valuation domain, then* gl.dim $V\langle x\rangle_M \leqslant 2$ *for every prime ideal M of $V\langle x\rangle$.*
4. *If* Krull dim $V \leqslant 1$, *then* gl.dim $V\langle X\rangle = 2$.

According to [24], Theorem A] (see also [30, Theorem 2.1]), for a valuation domain V to have global dimension two it is necessary and sufficient that V be $\aleph_0$-Noetherian—but not Noetherian unless gl.dim $V = 1$. Therefore, there exist valuation domains of global dimension two and any given, finite, Krull dimension [13, Sect. 7]. Valuation domains V are stably coherent rings; that is, the polynomial rings in finitely many variables over V are coherent rings [11, Corollary 7.3.4]; in particular, both $V\langle x\rangle$ and $V(x)$ are coherent rings. Note also that for every prime ideal p of a valuation domain V and any $b \in V - p$ we have $bp = p$. Therefore, by Nakayama's lemma, no prime ideal of V is finitely generated unless it is 0 or maximal.

By Proposition 7, we have $2 \leqslant$ gl.dim $V\langle x\rangle \leqslant 3$ and $2 \leqslant$ gl.dim $V(x) \leqslant 3$.

1. In this case $V(x)$ itself is a valuation domain [12, Corollary 3] and clearly $\aleph_0$-Noetherian. Thus, gl.dim $V(x) = 2$.

2. We first reduce to the case where Krull dim $V = 3$. Assume that Krull dim $V > 3$ and let p be a prime ideal of V of ht $p = 3$. Then V_p is a valuation domain of gl.dim $V_p = 2$ and Krull dim $V_p = 3$. By [4, Lemma 2.5], $V_p\langle x\rangle$ is a localization of $V\langle x\rangle$; thus, gl.dim $V_p\langle x\rangle \leqslant$ gl.dim $V\langle x\rangle$, and the reduction is complete.

Assume now that Krull dim $V = 3$ and let Spec $V = \{0 \subset p \subset q \subset m\}$. Krull dim $V[x] = 4$ [19, Theorems 39, 68] and, therefore, Krull dim $V\langle x\rangle = 3$ [21, Theorem 2.1]. Let Q be a prime ideal of $V[x]$ containing no monic polynomial, $Q \cap V = q$, and $qV[x] \subsetneqq Q$. ht $Q = 3$: thus $QV\langle x\rangle$ is a maximal ideal of $V\langle x\rangle$, and so Q contains a polynomial f with $c(f) = V$. It is clear that for every $a \in q$, the sequence $\{a, f\}$ is a regular sequence in

$V[x]$ and, therefore, it is a regular sequence in $V\langle x\rangle_{QV\langle x\rangle}$. In particular, depth $V\langle x\rangle_{QV\langle x\rangle}\geqslant 2$, and as w.dim $V\langle x\rangle_{QV\langle x\rangle}\leqslant 2$ [12, Lemma 4], we have that w.dim $V\langle x\rangle_{QV\langle x\rangle}=2$ [12, Lemma 3].

Assume that gl.dim $V\langle x\rangle=2$; then gl.dim $V\langle x\rangle_{QV\langle x\rangle}=2$.

Since w.dim $V\langle x\rangle_{QV\langle x\rangle}=2$, $V\langle x\rangle_{QV\langle x\rangle}$ is not a valuation domain. As ht $QV\langle x\rangle=3$, Krull dim $V\langle x\rangle_{QV\langle x\rangle}=3$; therefore, $V\langle x\rangle_{QV\langle x\rangle}$ is not a Noetherian regular ring. According to [28; 30, Theorem 2.2], $V\langle x\rangle_{QV\langle x\rangle}$ has to be an umbrella ring. In particular, $V\langle x\rangle_{QV\langle x\rangle}$ has to contain a nonzero prime ideal P satisfying:

(i) $P=P(V\langle x\rangle_{QV\langle x\rangle})_P$, that is, for any $g\in V\langle x\rangle_{QV\langle x\rangle}-P$, $gP=P$.

(ii) $V\langle x\rangle/QV\langle x\rangle$ is a Noetherian regular local ring of Krull dimension equal to two.

Note that no such prime ideal P can be finitely generated unless it is maximal.

Property (ii) implies that such an ideal P will necessarily satisfy $\mathrm{ht}(P)=1$. Thus, either $P=pV\langle x\rangle_{QV\langle x\rangle}$ or $P=NV\langle x\rangle_{QV\langle x\rangle}$, where N is a nonzero prime ideal of $R[x]$ containing no monic polynomial and satisfying $N\cap V=0$. But $f(pV\langle x\rangle_{QV\langle x\rangle})\neq pV\langle x\rangle_{QV\langle x\rangle}$, and N is a finitely generated ideal of $V[x]$ [10, Corollary 4.12]. We conclude that gl.dim $V\langle x\rangle_{QV\langle x\rangle}\neq 2$; therefore, gl.dim $V\langle x\rangle=3$.

3. Let Spec $V=\{0\subset p\subset m\}$. Let P be a prime ideal of $V[x]$ containing no monic polynomial and satisfying $P\cap V=p$ and $pV[x]\subsetneq P$. Then P contains a polynomial f with $c(f)=V$ and $PV\langle x\rangle$ is a maximal ideal of $V\langle x\rangle$. As in the previous case we obtain that w.dim $V\langle x\rangle_{PV\langle x\rangle}=2$. Assuming that gl.dim $V\langle x\rangle=2$, we obtain that gl.dim $V\langle x\rangle_{PV\langle x\rangle}=2$. $V\langle x\rangle_{PV\langle x\rangle}$ is not a valuation domain, and since Krull dim $V\langle x\rangle_{PV\langle x\rangle}=2$, it is not an umbrella ring. Thus, $V\langle x\rangle_{PV\langle x\rangle}$ is a Noetherian regular ring. In particular, $pV\langle x\rangle_{PV\langle x\rangle}$ is finitely generated, and choosing a set of generators contained in p, we see that $pV\langle x\rangle_{pV\langle x\rangle}$ is finitely generated. Now $V\langle x\rangle_{pV\langle x\rangle}=V[x]_{pV[x]}=V_p[x]_{pV_p[x]}=V_p(x)$; thus, pV_p is finitely generated. It follows that V_p is a discrete valuation domain, and the conclusion follows.

On the other hand, if V_p is a discrete valuation domain, a case by case analysis of the localizations of $V\langle x\rangle$ by prime ideals shows that $V\langle x\rangle$ is locally of global dimension two. The prime ideals of height one of $V\langle x\rangle$ are $pV\langle x\rangle$ and $NV\langle x\rangle$, where N is a nonzero prime ideal of $V[x]$ containing no monic polynomial and satisfying $N\cap V=0$. In the first case, $V\langle x\rangle_{pV\langle x\rangle}=V_p(x)$. In the second case, $V\langle x\rangle_{NV\langle x\rangle}=V[x]_{NV[x]}$ is a Noetherian, local regular ring of Krull dimension one. Therefore, both localizations have global dimension one. The prime ideals of height two of $V\langle x\rangle$ are $mV\langle x\rangle$ and $PV\langle x\rangle$, where P is a prime ideal of $V[x]$ containing no monic polynomial, $P\cap V=p$ and $pV[x]\subsetneq P$. In the first case, $V\langle x\rangle_{mV\langle x\rangle}=V(x)$. In the second case, $V\langle x\rangle_{PV\langle x\rangle}=V_p[x]_{PV_p[x]}$ a localization of the Noetherian regular ring of Krull dimension two $V_p[x]$ by a height two prime ideal. In both cases the global dimension of the localizations is two.

4. In this case, $V\langle x\rangle$ is a semihereditary ring [12, Corollary 4], and $\aleph_0$-Noetherian; therefore, gl.dim $V\langle x\rangle\leqslant$ w.dim $V\langle x\rangle+1=2$ [17, Theorem 1], and, thus, gl.dim $V\langle x\rangle=2$.

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