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Gaussian Polynomials

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1 INTRODUCTION

Let A be a commutative ring, and let x be an indeterminate over A. For a polynomial $f \in A[x]$, denote by c(f) - the so called *content of f* - the ideal of A generated by the coefficients of f.

The content ideal of f satisfies certain multiplicative properties:

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i. $c(fg) \subset c(f)c(g)$ for any $g \in A[x]$.

ii. $c(f)^n c(fg) = c(f)^n c(f)c(g)$ for any $g \in A[x]$ with $\deg g = n$.

This equality is known as Dedekind - Mertens Theorem (Gilmer 1967).

Containment i. becomes equality in certain cases. A well known result (Gilmer 1967, 1972) states that $c(fg) = c(f)c(g)$ for all $g \in A[x]$ when $c(f)$ is an invertible ideal. More generally this equality holds when $c(f)$ is a locally principal ideal (Tsang 1965, Vasconcelos 1970, D.D. Anderson & Kang 1995).

We introduce the following definition:

Definition: A polynomial $f \in A[x]$ is called *a Gaussian polynomial* if $c(fg) = c(f)c(g)$ for any $g \in A[x]$.

Given the results mentioned above it is natural to ask the following questions:

Question 1: Let A be a commutative ring and let $f \in A[x]$ be a Gaussian polynomial. Is $c(f)$ an invertible ideal of A?

Question 2: Let A be a commutative ring and let $f \in A[x]$ be a Gaussian polynomial. Is $c(f)$ a locally principal ideal of A?

A version of Question 1 appears in Vasconcelos's unpublished notes (1970) where it is attributed to Kaplansky. Versions of these questions also appear in Tsang (1965), D.D. Anderson (1995), D.D. Anderson & Kang (1995), Glaz & Vasconcelos (1995), and Heinzer & Huneke (1995). Before embarking on an elaborate discussion on this topic we should mention that without any qualifications on the ring A or on the coefficients of the Gaussian polynomial f, even the weaker of these questions has a negative answer.

Example: Let (A, m) be a local Artinian ring with maximal ideal m satisfying $m^2 = 0$. Then $c(fg) = c(f)c(g)$ for any two polynomials f and g in A[x]. To construct a specific example where $c(f)$ is not a locally principal ideal, let k be a field, let t and u be indeterminates over k, and denote by T and U the images of t and u in $k[t,u]/(t,u)^2$. Let $A = k[T,U]_{(T,U)}$, then $f(x) = Tx^2 + Ux + T$ is a polynomial with content (T,U).

In order to avoid situations like this we restrict our outlook to domains A. In this case, for a finitely generated ideal of A, the property of being invertible and that of being locally principal coincide, and we phrase the merged questions in form of a conjecture:

Conjecture: Let A be a domain and let $f \in A[x]$ be a Gaussian polynomial. Then c(f) is an invertible ideal of A.

This article surveys the work done about and around this conjecture. Its center, Sections 3 and 4, is a reformulation, to the extent that is possible, of Glaz & Vasconcelos (1995) results, couched in the more "non Noetherian" language of prestable ideals rather then the "Noetherian" language of Hilbert functions of the original work. Section 2 presents a history of the results related to this conjecture, while Section 5 consists of a brief exposition of Heinzer & Huneke's (1995) work extending Glaz & Vasconcelos (1995) results.

2 BACKGROUND

Historically, there were two kinds of inquiries related to the conjecture posed in the introduction. The first kind of inquiry attempted to find rings (not necessarily domains) for which the conjecture is valid. In this respect only two results are available.

THEOREM 2.1 (Tsang 1965, D.D. Anderson & Kang 1995) Let A be a commutative ring, let $f = ax^m + bx^n$, with $m < n$, and a and b regular elements. Assume that $c(fg) = c(f)c(g)$ for all binomials $g \in A[x]$. Then c(f) is an invertible ideal of A.

THEOREM 2.2 (Vasconcelos 1970) Let A be a domain and let $f = ax + b$ be a Gaussian polynomial. Then c(f) is an invertible ideal of A.

The second kind of inquiry assumes the Gaussian and/or related properties on every pair of polynomials f and g in A[x], and asks what does this property imply about the ring itself.

The following result is due to Tsang and Gilmer:

THEOREM 2.3 (Tsang 1965, Gilmer 1972) Let A be a domain for which every polynomial is Gaussian. Then A is a Prüfer domain.

D.D. Anderson (1977) characterizes rings for which every polynomial is Gaussian in terms of cancellation ideals. (I is a cancellation ideal if $JI + L:I = J + L:I$ for all ideals L and J).

Querre (1980) and Mott, Nashier & Zafrullah (1990) characterize integrally closed domains with fields of quotients K as those satisfying $(c(fg)^{-1})^{-1} = ((c(f)c(g))^{-1})^{-1}$.

D.D. Anderson & Kang (1995) characterize completely integrally closed domains as those satisfying $(c(fg)^{-1})^{-1} = ((c(f)c(g))^{-1})^{-1}$ for any polynomial f and any power series g. They also investigate domains where the above equality holds for all power series f and g. A corollary of their characterization is that Krull domains satisfy this equality.

3 PRESTABLE IDEALS AND PRIME CHARACTERISTIC

Sections 3 and 4 describe the results obtained in Glaz & Vasconcelos (1995). In that paper the authors are concerned with establishing the conjecture posed in the introduction. We note that since the ring A is a domain we may assume that the ring is local with maximal ideal m. In this section we deal with the case where the residue field of A is of prime characteristic. The results obtained can, then, be entirely described through the language of prestable ideals.

For an ideal I of a ring A denote by $\mu(I)$ the minimal number of generators of I. We start by quoting a result of Eakin & Sathaye:

THEOREM 3.1 (Eakin & Sathaye 1976) Let (A,m) be a local ring with maximal ideal m and infinite residue field, and let I be an ideal of A. Let n and r be positive integers and suppose that $\mu(I^n) < \binom{n+r}{r}$. Then there are elements $y_1,\ldots,y_r \in I$ such that $I^n = (y_1,\ldots,y_r)I^{n-1}$.

In order to be able to use this theorem we need to make a change of rings ensuring that the ring we deal with has infinite residue field. The change of rings is:

$$A \to A(t) = A[t]_{mA[t]}, \text{ where t is a variable over A.}$$

This change of rings may not carry back and forth the Gaussian property of a polynomial, but does carry back and forth properties that have to do with the number of generators of ideals of A. Since we will utilize the Gaussian property of a polynomial f to yield results on the number of generators of c(f), it will be harmless to assume, with a little bit of care, that the residue field of A is infinite. In this case the Gaussian property of the polynomial f yields:

$$\mu(c(f)^n) = \mu(c(f^n)) \leq dn + 1, \text{ where } d = \deg f.$$

Thus for $n > 2d - 3$, $\mu(c(f)^n) < \binom{n+2}{2}$, and therefore by Theorem 3.1:

$$c(f)^n = (a,b)c(f)^{n-1} \text{ for some a and b in } c(f).$$

This useful equality yields, for example, the fact that the grade of c(f) is at most two. For definitions and basic results on non Noetherian grade see (Alfonsi 1977, 1981) and (Glaz 1989). Since the grade of an invertible ideal is one, this is a step in the right direction. To make more extensive use of this type of equality and the techniques developed when working with them we introduce several definitions and results.

Let I and J be ideals in a ring A. J is called *a reduction*, or reduction ideal, of I if $I^n = JI^{n-1}$ for some integer n. Northcott & Rees (1954, 1954) proved that if I is an ideal of a Noetherian local ring A, then I has a minimal reduction J, in the sense that no proper subideal of J is a reduction of I; and if, in addition, the residue field of A is infinite the minimal number of generators of a minimal reduction is an invariant of I, so called *the analytic spread* of I. This conclusion remains valid for non Noetherian rings provided that the pertinent ideals are finitely generated (Eakin & Sathaye 1976).

In the above discussion we had seen that the content of a Gaussian polynomial f has a reduction generated by two elements. In our aim to come as close as possible to invertibility, which in this case means to being principal, we single out the case where an ideal has a reduction generated by one element.

Let A be a local ring. An ideal I of A is called *stable* if there is an element $a \in I$ such that $I^2 = aI$. I is called *prestable* if some power of I is stable. For a general ring A, an ideal I is stable (respectively prestable) if it is locally stable (respectively prestable). Stable and prestable ideals were investigated by Lipman (1971), Sally &

Vasconcelos (1974), Eakin & Sathaye (1976), among others.

THEOREM 3.2 (Eakin & Sathaye 1976) Let (A,m) be a local ring with maximal ideal m and infinite residue field, and let I be a finitely generated ideal of A. Then the following conditions are equivalent:

i. There is an element $a \in I$ such that $I^n = aI^{n-1}$ for some n.
ii. I is a prestable ideal of A.
iii. There is an integer b(I) such that, for every n, $\mu(I^n) \leq b(I)$.

It follows that prestable ideals are precisely those possessing a reduction generated by one element. Moreover it is possible to determine this property through bounds on the number of generators of powers of the ideal.

How close is a prestable ideal to being invertible? For one, a prestable ideal has grade one, but more then that:

THEOREM 3.3 (Eakin & Sathaye 1976) Let A be an integrally closed domain and let I be a finitely generated ideal of A. Then I is prestable if and only if I is invertible.

A weaker version then the questions posed in the introduction will be:

Question 3: Let A be a domain and let $f \in A[x]$ be a Gaussian polynomial. Is c(f) a prestable ideal of A?

Contrary to the properties of invertibility or of being locally principal, prestability of the content ideal of a polynomial does not guarantee its Gaussianess.

Example: Let $A = Z[2i]$ and $f(x) = 2x + 2i \in A[x]$. Then $c(f) = (2,2i)$. Let $g(x) = 2ix + 2$. $c(f)^2 = c(f)c(g) = (4,4i)$ but $c(fg) = 4iA$, so that f is not a Gaussian polynomial. On the other hand $c(f)Z[i] = 2Z[i]$ is principal, and therefore prestable, in $Z[i]$. It follows that c(f) is prestable in $Z[2i]$.

The following is a case where Question 3 has an affirmative answer, leading, via Theorem 3.3 to a large class of rings where the conjecture holds:

THEOREM 3.4 (Glaz & Vasconcelos 1995) Let A be a local domain with residue field of characteristic $p > 0$, and let f be a Gaussian polynomial. Then c(f) is a prestable ideal of A. If, in addition, A is also integrally closed, then c(f) is a principal ideal of A.

Proof: The proof given by Glaz & Vasconcelos (1995, Theorem 3.1) couched in the language of Hilbert functions can be modified to prove this formulation of the result. Taking $n = p^m$ we note that the summands of the coefficients of f^n which are not divisible by p consist of the n-th powers of the coefficients of f. We conclude, by Nakayama Lemma, that the n-th powers of the coefficients of f generate $c(f^n)$. Since f is Gaussian, $c(f)^n$ is generated by at most $d+1$ elements, where d is the degree of f. Passing from A to A(t) and using Theorem 3.1 we conclude that c(f)A(t), and hence that c(f) is prestable.

Note that in particular finitely generated *Z* algebras satisfy locally the conditions of this theorem. Also note that the proof actually shows that if the maximal ideal of A contains any positive integer the conclusions of the theorem hold, so that in dealing with the local characteristic zero case we need only be concerned with domains containing the rational numbers *Q*.

4 HILBERT FUNCTIONS AND CHARACTERISTIC ZERO

In this section we present the results of Glaz & Vasconcelos (1995) pertaining to rings of characteristic zero. These cases require a closer analysis of the "shape" of the number of generators of the content of a Gaussian polynomial and its powers, therefore techniques involving bounds alone, as those employed when working with prestability, do not suffice. We need to introduce a class of Hilbert functions. Before discussing those Hilbert functions, their relation to prestability, and the conclusions we were able to obtain through their use, we present a general result whose proof required none of the techniques described in Section 3 and 4.

THEOREM 4.1 (Glaz & Vasconcelos 1995) Let A be a local domain, and let f be a Gaussian polynomial with c(f) generated by two elements. Then c(f) is an invertible ideal of A.

Given Theorem 4.1 and the fact that the content of a Gaussian polynomial has a reduction generated by two elements (see Section 3) we would like to conclude that the content ideal itself is generated by two elements and therefore invertible. In order to be able to do just that, in certain cases, we need not only introduce Hilbert functions but also make use of the powerful techniques developed by Noetherian ring theory relating Hilbert functions to depth, dimension and rank of the rings and modules involved. In other words it will be possible to describe the Hilbert functions involved and display their relation to prestability, but it will not be possible to modify the proofs using Hilbert functions to involve prestability alone.

Let (A,m) be a local ring with maximal ideal m and denote by $I = c(f)$ the content of a Gaussian polynomial f. The *Rees Algebra of I* is the algebra

$$A[IT] = A + IT + I^2T^2 + \ldots \subset A[T].$$

The Hilbert function we are interested in is the Hilbert function of the special fiber F(I) of the ring A[IT], that is of the graded ring

$$F(I) = A[IT] \otimes A/m = \oplus\, I^n/mI^n,\ n \geq 0.$$

This function is:

$$n \rightarrow \dim F(I)_n = \mu(I^n) = \mu(c(f^n)).$$

Because this function is a polynomial in n bounded by $dn + 1$, where d is the degree of f, we have that it is itself a polynomial of degree at most one, $e_0n + e_1$. By abuse of terminology we say that $e_0n + e_1$ is the *Hilbert polynomial of f*.

For basic properties of Hilbert functions the reader is referred to Matsumura (1986), Nagata (1962), and Bruns & Herzog (1993).

The relation between the "Hilbert polynomial--Rees Algebra" approach and the "reduction ideals--prestability" approach can de described as follows:

We first note that in the passage from A to A(t) described in Section 3 the Hilbert polynomial does not change, hence we may, for our purposes, assume the residue field of A to be infinite.

If one starts with the fact that I has a reduction generated by two elements then a minimal reduction of I will have one or two generators (in the first case I is prestable). The special fiber of the Rees Algebra of the minimal reduction of I will be a polynomial ring over the residue field of A, in one or two variables corresponding to, respectively one or two generators of the reduction, and the special fiber of the Rees Algebra of I will be a finite module over it. Since prestability essentially means that the number of generators of powers of I stabilizes, this case corresponds to $e_0 = 0$, while the case of a minimal reduction generated by two elements corresponds to e_0 being non zero.

If the starting point is the construction of the Rees Algebra of I and the Hilbert polynomial, then F(I) is finite over a polynomial ring in at most two variables over the residue field of A (its Noether normalization). The number of variables corresponds to the Krull dimension of F(I). The variables or variable can be lifted to a reduction of I with number of generators corresponding to the number of variables. The Krull dimension of F(I) is reflected in the Hilbert polynomial by the condition e_0 being zero, corresponding to Krull dimension one; or e_0 being non zero, corresponding to Krull dimension two. The special fiber of the lifted reduction of I is then precisely the corresponding polynomial ring.

Thus we conclude that prestability for our case is equivalent to $e_0 = 0$, and proofs involving the non zero or zero property of e_0 can be "translated" from the language of Hilbert functions to that of prestability and vice versa, but results that involve more subtle properties of e_0, like for example determining the numerical value of a non zero e_0, do not allow for such a translation.

Concentrating on the case where the rational numbers Q are contained in A we proved:

THEOREM 4.2 (Glaz & Vasconcelos 1995) Let A be a local ring containing Q and let f be a Gaussian polynomial with Hilbert polynomial $e_0 n + e_1$. Then $e_0 \leq 1$.

Theorem 4.2 essentially says that F(I) is a rank one module over the special fiber of a minimal reduction of I. This allows us to establish the conjecture for local Noetherian integrally closed domains containing the rational numbers and, utilizing Theorem 3.4, to extend the answer to include all Noetherian integrally closed domains.

THEOREM 4.3 (Glaz & Vasconcelos 1995) Let A be a local Noetherian integrally closed domain containing the rational numbers and let f be a Gaussian polynomial. Then c(f) is a principal ideal of A.

Proof: We sketch the main steps of the proof given in Glaz & Vasconcelos (1995). Let J = (a,b) be a reduction of I = c(f). We aim to show that I = J and use Theorem 4.1 to conclude that I is an invertible ideal. For this purpose we may assume that dim R is larger than 1, and I/J is a nonzero module of finite length. We consider the injective map from A[JT] to A[IT] and denote its cokernel by C. It suffices to show that C = 0. Assuming that C is not zero we argue that dim C = 2 and mA[JT], where m is the maximal ideal of A, is the unique maximal ideal associated with C. This in turn yields that the dimension of C/m^rC remains 2 for all r. On the other hand, we have that, under the conditions of the theorem F(I) is a rank one module over F(J)-- a polynomial ring in two variables over the residue field of A. Tensoring the injection from A[JT] to A[IT] by A/m we conclude that the dimension of C/mC is less then two. This contradiction concludes the proof.

COROLLARY 4.4 (Glaz & Vasconcelos 1995) Let A be a Noetherian integrally closed domain and let f be a Gaussian polynomial. Then c(f) is an invertible ideal of A.

Proof: Follows from Theorem 4.3 and the remark below Theorem 3.4.

If the local domain A is Noetherian but not necessarily integrally closed we can still say some things about the content ideal of a Gaussian polynomial f, and in special cases conclude its prestability. Let A' be the integral closure of A in its quotient field. If A' is a Noetherian ring, like in the case where the Krull dimension of A is less or equal to two, we can conclude by passing to A' that c(f)A' is invertible and therefore that c(f) is a prestable ideal of A. In case A' is not Noetherian it is still a Krull domain with Noetherian prime spectrum (Heinzer 1973). This may allow for a modification of the proof of Theorem 4.3 to keep its conclusion, but we were unable to find the required modification. What we can say in this case is that c(f) is prestable at every localization by a prime ideal minimal over c(f); in particular, all prime ideals minimal over c(f) have height one.

5 THE RESULTS OF HEINZER & HUNEKE

In this section we describe briefly the results proved by Heinzer & Huneke (1995) which extend the class of rings for which Questions 1 and 2 have an affirmative answer.

A local Noetherian ring (A,m) with maximal ideal m is called *approximately Gorenstein* if for every integer $n > 0$ there is an ideal $I \subseteq m^n$ such that A/I is Gorenstein. For properties of approximately Gorenstein rings the reader is referred to Hochster (1977) and Heinzer & Huneke (1995). Heinzer & Huneke (1995) prove that the class of approximately Gorenstein rings includes the following rings:

1. Local Noetherian rings with depth greater or equal to two.
2. Noetherian integrally closed rings.
3. Locally analytically unramified Noetherian rings, that is every localization at a maximal ideal (A,m) satisfies that its m-adic completion is a reduced ring.
4. Locally excellent reduced Noetherian rings.
5. Polynomial or power series extensions in one variable over Noetherian reduced rings.
6. Polynomial or power series extensions in two variables over Noetherian rings.
7. Zero dimensional local Gorenstein rings.

Their answer to Questions 1 and 2 is as follows:

THEOREM 5.1 (Heinzer & Huneke 1995) Let A be a locally Noetherian approximately Gorenstein ring, and let f be a Gaussian polynomial. Then c(f) is a locally principal ideal of A. In particular, if c(f) contains a regular element then it is an invertible ideal of A.

When encountering polynomials that are not Gaussian, like for example the one in the example of Section 3, one can actually deduce their non Gaussianess by testing the multiplicative property of the content for small degree polynomials, usually up to the degree of the original polynomial. A question of interest is to determine a bound on the degree of polynomials which need to be tested in order to conclude that a polynomial is or is not Gaussian. Heinzer & Huneke provide an answer to this question for locally approximately Gorenstein rings:

THEOREM 5.2 (Heinzer & Huneke 1995) Let A be a locally approximately Gorenstein ring. Then a polynomial f of degree n is Gaussian if and only if
c(fg) = c(f)c(g) for all polynomials g of degree at most n.

In spite of the recent progress done toward validating the conjecture posed in the introduction, neither the Hilbert functions approach (Glaz & Vasconcelos 1995) nor the approximately Gorenstein rings approach (Heinzer & Huneke 1995) succeeded to solve the case where the domain A is local, Noetherian and one dimensional. Note that this is the case where all ideals of A are prestable. Moreover, the state of the conjecture is not known for non Noetherian rings beyond the results presented in Theorems 3.4 and 4.1.

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