COMMUNICATIONS IN ALGEBRA, 16(9), 1791-1811 (1988)

FACTORIALITY AND FINITENESS PROPERTIES OF SUBALGEBRAS OVER WHICH $k[x_1, \ldots, x_n]$ IS FAITHFULLY FLAT

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Let $C = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k, and let $k \subset B \subset C$ be a k algebra over which C is flat or faithfully flat. We consider several factoriality and finiteness properties that B inherits from C under the faithfully flat condition.

The setting $k \subset B \subset C$, in the context of factoriality and finiteness properties, inherited by B, was considered by many authors under a variety of restrictions on the extension $B \subset C$, or on the nature of k and B.

The classical case goes back to Hilbert's 14th problem [20]. The task in this case was to determine whether $B = C \cap K$, where K is a field containing k, and contained in $L = k(x_1, \ldots, x_n)$, is

*The research on this paper was partially supported by the National Science Foundation Grant DMS 8507944.

a finitely generated k algebra. Zariski solved the question in the affirmative, provided tr $\deg_k B \leq 2$ [20].

P. M. Cohn [7] shows that if $k \subset B \subset k[x_1]$, and B is integrally closed, then $B = k[f]$, for some $f \in C$. More generally, for arbitrary n, and B a Dedekind domain (necessarily of tr $\deg_k B \leq 1$), Zaks [27] shows that $B = k[f]$, for some $f \in C$.

The case $n = 2$, or $n = 3$ but tr $\deg_k B \leq 2$, was considered under several restrictions. Miyanishi [17],[18] shows that if $n = 2$, k an algebraically closed field of characteristic 0, and C flat and finite over B, then B is a polynomial ring in two variables over k. Russell [24] derives the same conclusion for B, given that $n = 2$, k is perfect, B and $B \otimes \bar{k}$, where $\bar{k}$ denotes the algebraic closure of k, satisfy several finiteness and factoriality properties, and $k(x,y)$ is a separable extension of the field of quotients of B. In [18], Miyanishi also considers the case $n = 3$, and several heavy restrictions on k, B and the extension $B \subset C$. Miyanishi's and Russell's work described here is representative of a school of algebraic geometers exploring the factoriality properties of finitely generated subalgebras of polynomial rings in two variables over fields.

In [2], [3] and [4] Anderson considers the case $n = 2$ and B a finitely generated k algebra generated by monomials, over which C is integral. Those algebras B are characterized and their factoriality properties exhibited in the calculations of cl(B).

Except for Miyanishi's work cited above [18], there are only two cases known to us where tr $\deg_k^B > 2$ is considered, [8] and

[21]. In [8], Evyatar and Zaks consider the case where B is a so-called factorable subring of C containing k; that is, whenever an element of B factors in C, then all its factors lie already in B. Such rings are frequently called in the literature inert subrings or inert embeddings (see, for example, [1]). They prove that if $\text{tr deg}_k B = n$, then $B = C$, and provide an example that shows that if this is not the case, then B does not have to be a polynomial ring.

In [21], Nagata and Otsuka prove the finite generation of B, for k a universally catenary Nagata domain with certain analytical irreducibility properties, C a generalization of a polynomial ring, and B an intermediate algebra satisfying several properties. This result, which is useful to us in this paper, is discussed in more detail in Section 1.

The setting $k \subset B \subset C$, where C is flat or faithfully flat over B, appears in two famous problems, the Jacobian problem (see for example Wang [25] and Wright [26]), and the coefficient ring problem (see for example Abhyankar, Heinzer & Eakin [1] and Hochster [13]).

Our motivation lies in the investigation carried out by Glaz, Sally and Vasconcelos [11]. In [11], the setting is $A \subset B \subset A[x_1]$, for an arbitrary ring A, and $A[x_1]$ flat or faithfully flat over B. The case where A is a field, in particular P. M. Cohn's result, played an important role in the general investigation. Hence, we consider the setting $k \subset B \subset C = k[x_1, \ldots, x_n]$, $n > 1$ and C faithfully flat over B.

Throughout this paper the following notation will be fixed: k denotes a field, $k \subset B \subset C = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k. $L = k(x_1, \ldots, x_n)$, and K denotes the field of quotients of B. $\overline{B}$ denotes the integral closure of B in L (and, hence, in C).

In Section 1, we show that if C is faithfully flat over B, then B is a Noetherian regular ring of $\text{tr deg}_k B = \dim B \leq n$, and $K \cap C = B$. This allows us to prove the main theorem of this section, namely that B is a finitely generated k algebra. We conclude the section with an example that shows that if C is merely flat over B, B does not have to be a finitely generated k algebra, even when their fields of quotients are equal.

In Section 2, we consider the prefactoriality of B, and the nature of $\overline{B}$. We prove that if L is a normal extension of K satisfying that $\xi(C) \subset C$ for every automorphism ξ of L over K, then $\overline{B} = C$ and B is prefactorial. In case the extension $K \subset L$ is normal the prefactoriality of $\overline{B}$ is equivalent to its equality to C. We conclude the section with a relation between the prefactoriality and integral closure in L of B, and D, where D is a finitely generated, integrally closed k algebra, satisfying $B = S(I,D)$ the ideal transform of an ideal I of D.

Section 3 considers the question whether B, with C faithfully flat over B, is a polynomial ring. We prove that if C is integral over B, the faithful flatness of C over B is equivalent to the regularity of B. This theorem provides examples of subalgebras B over which C is faithfully flat (and integral) but which are not polynomial rings. We then consider subalgebras B of $k[x_1,x_2] = C$, generated by monomials, over which C is integral. We show that, in this case, if C is faithfully flat over B, B has to be a polynomial ring. We conclude with examples of algebras B of this type over which C is not faithfully flat, but which are close to sharing many of the finiteness and factoriality properties enjoyed by subalgebras over which C is faithfully flat.

SECTION 1. Finiteness.

LEMMA 1. Let $k \subset B \subset C = k[x_1, \ldots, x_n]$ be a k algebra with $\text{tr deg}_B C = r \geqslant 0$. Then there are r of the variables, say $x_1, \ldots, x_r$, which are algebraically independent over B, and an element $b \in B[x_1, \ldots, x_r]$ such that $B[x_1, \ldots, x_r, 1/b]$ is a finitely generated k algebra with integral closure in L equal to $C[1/b]$. In particular, if B is a Noetherian ring, then $\dim B \geqslant \text{tr deg}_k B$.

Proof: We prove our claim by induction on $r = \text{tr deg}_k B$.

If $r = 0$, then L is algebraic over K; thus, there exists an element $b \in B$ such that $bx_i \in \overline{B}$ for $1 \leq i \leq n$. We conclude that $C[1/b]$ is integral over $B[1/b]$. It follows from [5, p. 81] that $B[1/b]$ is a finitely generated k algebra.

If $\text{tr deg}_B C = r \geq 1$, then for some i, x_i is transcendental over B. Say $i = 1$. We have $k \subset B[x_1] \subset C$ and $\text{tr deg}_{B[x_1]} C = r - 1$. The claim now follows using the induction hypothesis.

Assume that B is Noetherian, we then have:

$n = \dim C[1/b] = \dim B[x_1, \ldots, x_r, 1/b] =$

$\dim B[x_1, \ldots, x_r, T]/(bT-1) \leq \dim B + (r+1) - 1$. Thus,

$\dim B \geq n - r = \operatorname{tr} \deg_k B$.

PROPOSITION 2. <u>Let $k \subset B \subset C = k[x_1, \ldots, x_n]$ be such that C is faithfully flat over B. Then:</u>

<u>(1) B is a Noetherian regular ring of $\dim B = \operatorname{tr} \deg_k B \leq n$.</u>

<u>(2) $B = K \cap C$.</u>

<u>In particular, B is integrally closed in K.</u>

<u>Proof</u>:

(1) Let I be an ideal of B, and let $a_1, \ldots, a_r \in I$ such that $IC = (a_1, \ldots, a_r)C$. then $I \otimes_B C = IC = (a_1, \ldots, a_r)C = (a_1, \ldots, a_r) \otimes_B C$. By the faithful flatness of C over B, we have that $I = (a_1, \ldots, a_r)$. Thus, B is a Noetherian ring.

Let m be a maximal ideal of B, and let n be a maximal ideal of C lying over m. Since C_n is faithfully flat over B_m, it follows that B_m is a regular ring of $\dim B_m \leq n$ [16, pp. 79, 155]. Thus, B is a regular ring of $\dim B \leq n$.

By Lemma 1, we have that $\operatorname{tr} \deg_k B \leq \dim B \leq n$. For the reverse inequality let m be a maximal ideal of B of maximal height, and set $r = \dim B = \dim B_m$. Since B is a regular ring there exists a regular system of parameters for B_m, $y_1, \ldots, y_r$ such that $k[y_1, \ldots, y_r] \subset B_m$, and

$k[y_1, \ldots, y_r]$ is a polynomial ring in r variables over k [16, p. 150]. It is now clear that

$\operatorname{tr} \deg_k B = \operatorname{tr} \deg_k B_m \geq r = \dim B.$

(2) The faithful flatness of C over B implies that $bC \cap B = bB$ for every $b \in B$ [16, p. 28]; thus, $B = K \cap C$.

Let $k \subset B \subset C = k[x_1, \ldots, x_n]$ with C faithfully flat over B. Since $B = K \cap C$, there is an integrally closed k algebra D, and an ideal I of D such that $B = S(I,D) = \bigcup_{n \geq 0} I^{-n}$ -the so-called ideal transform of I [20, p. 45]. Note that D and B have the same field of quotients. Thus, either $B = D[I^{-n}]$ for some n, and therefore it is a finitely generated k algebra, or we can define a strictly increasing sequence of integrally closed k algebras B_i, with $B_0 = D$, and B_{i+1} equal to the integral closure of $B_i[(IB_i)^{-1}]$, such that $B = \bigcup_i B_i$ [20, p. 48]. Using this fact, Nagata and Otsuka [21, Theorem 1] proved the following theorem:

THEOREM 3 (Nagata, Otsuka). Let k be a universally catenary, Nagata domain, satisfying that any local, integrally closed domain which is a localization of a finitely generated k algebra, is analytically irreducible. Let $C_1, C_2, \ldots, C_n$ be integrally closed domains which are finitely generated k algebras, and set $C = C_1 \oplus \ldots \oplus C_n$. Let $k \subset B \subset C$ be a domain with field of quotients K, satisfying:

(1) K is contained in the total ring of quotients of C.

(2) $B = K \cap C$.

(3) The canonical map Spec C $\to$ Spec B is surjective.

(4) For any maximal ideal m of B, mB_m is finitely generated.

Then B is a finitely generated k algebra.

We note here that our formulation of the Nagata-Otsuka theorem uses the terminology developed in [16]. A discussion about the properties of k can be found in [16, pp. 86, 231, 237]. In particular, any field, or any Nagata, local, integrally closed domain which is a localization of a finitely generated algebra over a field, satisfies the properties of k required in this theorem [16, Chapter 12] and [22, pp. 139, 140]. We can therefore conclude:

THEOREM 4. Let $k \subset B \subset C = k[x_1, \ldots, x_n]$, with k a field and C faithfully flat over B. Then B is a finitely generated k algebra.

Proof: Set $C = C_1$ in Theorem 3. Proposition 2 and the faithful flatness of C over B guarantee the requirements (1)-(4).

It should be noted that in the particular case that dim B $\leq$ 2, we have tr $\deg_K L \leq 2$ and, therefore, Zariski's theorem [20, p. 52] yields the finite generation of B over k as well.

If C is merely flat over B, then B does not have to be a finitely generated k algebra, even if their field of quotients coincide. We present here an example of this kind. This example is based on a three-dimensional construction shown to us by J.

Sally in a different context, and the refinement to the two-dimensional case due to W. Heinzer.

Example:

Let k denote the complex field. Let $E = k[x, xy^2 + y] \subset C = k[x, y]$. C is faithfully flat over E [11]. C is quasifinite over E, that is, every prime ideal P of C is a maximal as well as a minimal prime over $P \cap E$. It follows by Zariski's Main Theorem [23, p. 41] that C is flat over any ring containing $\bar{E}$ and contained in C. E is isomorphic to C and, therefore, a U.F.D.

We claim that $\bar{E} = E[xy] = k[x, xy^2 + y, xy]$. To see this, first note that $(xy)^2 + xy - x(xy^2 + y) = 0$; thus, E[xy] is integral over E. Now write $E[xy] = k[x][u, v]/(u^2 + u - xv)$. For any maximal ideal m of k[x][u, v], one can check locally that $u^2 + u - xv \notin m^2$; therefore, E[xy] is a regular ring, and as such, integrally closed; thus; $\bar{E} = E[xy]$. Let $w = xy + 1$, and let $B = \bar{E}[y^2w, y^3w, \ldots]$. If $B \neq C$ we have that C is flat, but not faithfully flat, over B, and B is not a finitely generated k algebra.

To see that $B \neq C$ we employ [10, Appendix 2].

Note that $k[x, w]_{(w)}$ is a discrete valuation ring of the form $k(x) + wk[x, w]_{(w)}$, and that $V = k[x]_{(x)} + wk[x, w]_{(w)}$ is a rank two valuation domain with x a generator of the maximal ideal of V, and w contained in the height one prime ideal of V. Hence, $w/x^n \in V$ for each positive integer n. Since $y = (w - 1)/x$ and

$w - 1$ is a unit of V, we have $y \notin V$, and therefore $C \not\subseteq V$. On the other hand, $y^n w = ((w - 1)^n/x^n) \cdot w \in V$ for each positive integer n. Therefore $B \subseteq V$, and $B \neq C$.

Along the same lines, the following three-dimensional example can be constructed: $E = k[x, y, xz^2 + z] \subset C = k[x, y, z]$ and $B = k[x, y, xz^2 + z, xz, yz, yz^2, \ldots]$. Then C is flat over B, and B is not a finitely generated k algebra.

SECTION 2. Prefactoriality and Integral Closure

Let $k \subset B \subset C = k[x_1, \ldots, x_n]$ with C faithfully flat over B. In Proposition 2, we proved that B is a regular ring. Under certain restrictions on the extension $K \subset L$ we can obtain that B is prefactorial with $\overline{B} = C$.

THEOREM 5. Let $k \subset B \subset C = k[x_1, \ldots, x_n]$ with C faithfully flat over B. Assume that L is a normal extension of K. Denote by $G = \mathrm{Aut}(L/K)$ the group of all automorphisms of L over K, and assume that $\xi(C) \subset C$ for every $\xi \in G$. Then $\overline{B} = C$ and B is prefactorial.

Proof: We will first show that under these assumptions $\overline{B} = C$. Since L is algebraic over K, we have $\dim B = \dim \overline{B} = n$, and L is the field of quotients of $\overline{B}$.

Since B and C are Noetherian rings, $\mathrm{tr\,deg}_B C = 0$, and C is a faithfully flat finitely generated B algebra, we have that for every prime ideal P of C, $\mathrm{ht}(P) = \mathrm{ht}(P \cap B)$ [16, pp. 79, 85].

Thus, C is quasifinite over B. It follows by Zariski's Main Theorem [23, p. 41] that $\overline{B}_P \cap \overline{B} = C_P \cap \overline{B}$ for every prime ideal P of C and, thus, C is a flat epimorphism of $\overline{B}$ [15, p. 112]. Since a faithfully flat epimorphism is an isomorphism, in order to show that $\overline{B} = C$, it suffices to show that no prime ideal of $\overline{B}$ blows up in C.

Let P be a prime ideal of $\overline{B}$, set $p = P \cap B$, and let Q be a prime ideal of C lying over p. Then $P' = Q \cap \overline{B}$ is a prime ideal of $\overline{B}$ lying over p, which does not blow up in C. Since any two prime ideals of $\overline{B}$ lying over the same prime ideal of B are conjugate to each other by some automorphism of L over K [16, p. 34], and $\xi(C) \subset C$ for any such automorphism ξ, we conclude that $PC \neq C$.

We will now show that B is prefactorial. Let K' be the fixed field of G, then either K = K' (if L is separable over K), or L is normal and separable over K', and K' is a purely inseparable extension of K. This last case may happen for some fields k, with $ch(k) = p > 0$.

Let p be a prime ideal of B of $ht(p) = 1$. Since B is a Krull domain, p is a divisorial ideal. Since B is a regular ring, p is an invertible and, therefore, projective ideal of B. It follows that $pC = p \otimes_B C$ is a projective ideal of C and, therefore, principal. Let $pC = fC$ for some $f \in pC$.

For $\xi \in G$ we have $pC = \xi(pC) = \xi(fC) = \xi(f)C$; therefore, f and $\xi(f)$ generate the same ideal of C and $\xi(f) = uf$ for some $u \in k$.

Since f is algebraic over K, the number of distinct conjugates of f is finite, say

$f = \xi_1(f)$, $u_2 f = \xi_2(f)$, . . ., $u_r f = \xi_r(f)$. Let

$g = (\prod_{i=1}^{r} \xi_i(f))^q = u\, f^{rq}$, where $u = (u_2 \ . \ . \ . \ u_r)^q \in k$ and $q = 1$ if $K' = K$, $q = p^\nu$ for large ν if $K' \neq K$ and $ch(k) = p > 0$. Then $g \in K \cap C = B$; therefore, there exists a positive integer s such that $f^s \in B$. But $f^s \in pC \cap B = p$. We claim that $p = \sqrt{f^s B}$. To see this, let $h \in p \subset pC$, then $h = fc$ for some $c \in C$, $h^s = f^s c^s$ and, therefore $c^s \in B$. Therefore, $p \subset \sqrt{f^s B}$, and we have equality.

REMARK. The referee pointed out that various parts of Theorem 5 can be proved by other methods as well. For example:

(1) To deduce that $C = \overline{B}$ one can proceed as follows: since $G = \mathrm{Aut}(L/K)$ is a finite group, $C^G = \{f \in C \ / \ \xi(f) = f$ for every $\xi \in G\} \subset C$ is an integral extension [5, p. 68]. Since L is a normal extension of K, a computation similar to the one carried out in the last paragraph of the proof of Theorem 5 shows that $B \subset C^G$ is a purely inseparable extension. Thus, $C = \overline{B}$.

(2) Once the equality $C = \overline{B}$ is established we can conclude the prefactoriality of B using [1, Corollary 2.14].

It is interesting to note that in the set up $k \subset B \subset C$ with C faithfully flat over B, and L, a normal extension of K, the prefactoriality of $\overline{B}$ is equivalent to its equality to C.

PROPOSITION 6. <u>Let $k \subset B \subset C = k[x_1, . . ., x_n]$ with C faithfully flat over B. Assume that L is a normal extension of K, and that $\overline{B}$ is prefactorial. Then $\overline{B} = C$.</u>

<u>Proof</u>: Let P be a prime ideal of $\overline{B}$ with ht P = 1 and set $P = \sqrt{g\overline{B}}$ for some $g \in \overline{B}$. Let Q be a prime ideal of C minimal over gC, then ht Q = 1. We claim that $Q \cap \overline{B} = P$. To see this let $Q' = Q \cap \overline{B} \supset g\overline{B}$; thus, $P \subset Q'$. Since ht $(Q \cap B) = 1$ and $Q' \cap B = Q \cap B$, we have ht $Q' = 1$ and $P = Q'$.

L is a finitely generated algebraic, and hence finite, extension of K; therefore, $\overline{B}$ is a Krull domain [<u>16</u>, p. 296]. Thus, $\overline{B}_P \subset C_Q$ are two discrete valuation rings with the same field of quotients L and, hence, $\overline{B}_P = C_Q$. It follows that

$$C = \bigcap_{\substack{Q \in \mathrm{Spec}(C) \\ ht(Q) = 1}} C_Q = \bigcap_{\substack{P \in \mathrm{Spec}(\overline{B}) \\ ht(P) = 1}} \overline{B}_P = \overline{B}.$$

Returning to the representation of B as an ideal transform S(I,D) for an ideal I of a finitely generated integrally closed k algebra D, we can relate the factoriality of D and that of B and, in certain cases, between $\overline{B}$ and $\overline{D}$. the integral closure of D in L. This is done in Propositions 7 and 8.

PROPOSITION 7. <u>Let $k \subset B \subset C = k[x_1, . . ., x_n]$ with C faithfully flat over B. If D is prefactorial (respectively, a U.F.D.), then so is B, and B = D.</u>

Proof: Assume that D is either prefactorial or a U.F.D. Let $d \in D$ with $I^s \subset dD$ for some $s \geq 1$. Then $(1/d)I^s \subset D$; therefore, $1/d \in B \subset C$. It follows that $d \in k$ and ht $I > 1$. Since D is Noetherian and integrally closed, this implies that $B = S(I, D) = D$ [20, p. 41].

PROPOSITION 8. Let $k \subset B \subset C = k[x_1, . . ., x_n]$ with C faithfully flat over B. If L is a finite separable extension of K, then $\bar{B} = S(I\bar{D}, \bar{D})$, where $\bar{D}$ denotes the integral closure of D in L.

Proof: Since L is a finite separable extension of K, $\bar{B}$ is a finite B module [5, p. 64], and, hence, a Noetherian Krull domain. Let $x \in S(I\bar{D}, \bar{D})$, then $x(I\bar{B})^n \subset \bar{B}$ for some positive integer n. If $B \neq D$, then ht $IB > 1$ [20, p. 50]; therefore, $(I\bar{B}^n)^{-1} = \bar{B}$. It follows that $x \in \bar{B}$ [11, p. 5]. On the other hand, if $x \in \bar{B}$, then $x^s + b_{s-1}x^{s-1} + . . . + b_0 = 0$ for some $b_0, . . ., b_{s-1} \in B$. Pick $z \in I^r$ such that $b_jI^r \subset D$ for $0 \leq j \leq s-1$. Then $(xz)^s + (b_{s-1}z)(xz)^{s-1} + . . . + (b_0z)z^{s-1} = 0$ and $xz \in \bar{D}$. Thus, $xI^r \subset \bar{D}$ and $x \in S(I\bar{D}, \bar{D})$.

SECTION 3. Polynomial Rings.

Let $k \subset B \subset C = k[x_1, . . ., x_n]$, with C faithfully flat over B. In [18], Miyanishi proves that for $n = 2$, C integral over B, and k algebraically closed of ch(k) = 0, B is a polynomial ring in two variables over k. This is not true in general, even if C is integral over B. The following theorem will generate the required counterexample.

THEOREM 9. Let $k \subset B \subset C = k[x_1, \ldots, x_n]$ with C integral over B. Then C is faithfully flat over B if and only if B is a Noetherian regular finitely generated k algebra.

Proof: Note first that since C is integral over B, C is faithfully flat over B if and only if C is flat over B. By Proposition 2, and Theorem 4, if C is faithfully flat over B then B is a Noetherian regular finitely generated k algebra. The integrality of C over B is not necessary for this implication.

For the converse, let m be a maximal ideal of B, then B_m is a regular ring. To see that C_m is flat over B_m we either employ [16, p. 140], or prove directly that for a regular system of parameters

$f_1, \ldots, f_n$ of B_m, $\mathrm{Tor}^1_{B_m}(B_m/(f_1, \ldots, f_n) B_m, C_m) =$
$\mathrm{Tor}^1_{B_m}(B_m/mB_m, C_m) = 0$. It follows that C_n is a flat $B_{n \cap B}$ module for every maximal ideal n of C, and, therefore, C is a faithfully flat B module.

Example. Let k be an algebraically closed field of $ch(k) = p > 0$. Let $B = k[x^p, y^p, (x^p y^p + 1)x + y^{p+1}] \subset C = k[x,y]$.

Then $B \simeq k[T_1, T_2, T_3]/(T_3^p - (T_1^p T_2^p + 1)T_1 - T_2^{p+1})$. One can check locally that $f = T_3^p - (T_1^p T_2^p + 1)T_1 - T_2^{p+1} \notin m^2$ for any maximal ideal m of $k[T_1, T_2, T_3]$ such that $f \in m$ and, thus, B is a regular ring.

Since C is integral over B, we have that C is faithfully flat over B. That B is not isomorphic to a polynomial ring was proved by Miyanishi and Russell in [19, p. 286]. In fact, Miyanishi and Russell [19], proved by a different method that B is regular, and that B is a U.F.D. The case where p = 2, was proved, yet by a different method by Lang [14].

We now turn our attention to a special type of subalgebras B of C = k[x,y], where faithful flatness implies that B is a polynomial ring, regardless of the field k.

Let k be a field, and let $k \subset B \subset C = k[x,y]$, be a subalgebra of C generated by monomials, with C integral over B. We have that B is a finitely generated k algebra. B is a graded ring with the natural grading; in fact, B is bihomogeneous; that is, if $\Sigma\, a_{ij}x^iy^j \in B$, then each $a_{ij}x^iy^j \in B$. These algebras were studied by Anderson in [2], [3] and [4].

We first determine which of those algebras satisfying C is faithfully flat over B.

PROPOSITION 10. _Let $k \subset B \subset C = k[x,y]$, with C integral over B and B generated by monomials, then C is faithfully flat over B if and only if B is isomorphic to a polynomial ring in two variables over k._

Proof: If B is isomorphic to a polynomial ring, then by Theorem 9, C is faithfully flat over B.

If C is faithfully flat over B, then B is integrally closed, it follows from [2, p. 217], that via a change of variables which does not change the origin, B is either isomorphic to C or $B = k[x^n, xy^j, x^2y^{\overline{2j}}, \ldots, x^{n-1}y^{\overline{(n-1)j}}, y^n]$ where $0 < j < n$, g.c.d $(j, n) = 1$, and overscoring denotes mod n. We will show that if B is not isomorphic to C, then for $m = (x,y) \cap B$, B_m is not a regular ring and, thus, by Theorem 9, arrive at the desired conclusion.

Let $B = k[x^n, xy^j, x^2y^{\overline{2j}}, \ldots, x^{n-1}y^{\overline{(n-1)j}}, y^n]$. From the set $\{x^n, xy^j, \ldots, y^n\}$ pick a minimal generating set for B as a k algebra. $B = k[x^n, y^n, xy^j, x^{i_1}y^{j_1} \ldots x^{i_r}y^{j_r}]$. Now map the polynomial ring $k[T_1, \ldots T_{r+3}]$ to B by φ, $\varphi(T_1) = x^n$, $\varphi(T_2) = y^n$, $\varphi(T_3) = xy^j$, $\varphi(T_{s+3}) = x^{i_s}y^{j_s}$. Let $P = \ker \varphi$. Clearly $P \subset (T_1, \ldots, T_{r+3}) = n$ and $B_m = k[T_1, \ldots, T_{r+3}]_n/P_n$ is regular if and only if P_n can be generated by a subset of a regular system of parameters for $nk[T_1, \ldots, T_{r+3}]_n$. We will show that this cannot happen by proving that $P \subset n^2$. Let $f = f(T_1, \ldots, T_{r+3}) \in P$ and write $f = \alpha_1T_1 + \ldots + \alpha_{r+3}T_{r+3} + \Sigma\beta_{ij}T_iT_j + \ldots$. Then $f(x^n, y^n, xy^j, \ldots) = 0$. By the minimality of the generating set for B, clearly $\alpha_i = 0$, $1 \leq i \leq r+3$. (In fact, we suspect that $P \subset m^{j+1}$, since it seems that the polynomial $T_3^n - T_1T_2^j \in P$ has minimal initial power in P.)

This proposition provides us with many algebras generated by monomials over which C is faithfully flat. Any $B = k[x^n, y^m]$ for positive integers n, m will do.

On the other hand, algebras of the kind $B = k[x^n, xy^j, \ldots, x^{n-1}y^{\overline{(n-1)}}, y^n]$ $0 < j < n$ and g.c.d. $(j, n) = 1$, which never satisfy C faithfully flat over B, are close to sharing several finiteness and factoriality properties enjoyed by algebras B over which C is faithfully flat.

Let $B = k[x^n, xy^j, \ldots, x^{n-1}y^{\overline{(n-1)}}, y^n]$, and let K be the quotient field of B. Clearly $\text{tr deg}_k B = \dim B = 2$ and $K \cap C = B$.

Anderson [3, p. 9] proved that any localization of B by a maximal ideal other than $m = (x,y) \cap B$ is regular. Moreover, in [2, p. 222], he proved that $\text{cl}(B) \simeq Z/nZ$; therefore, if P is a prime ideal of B of height 1 then $((P^n)^{-1})^{-1}$ is principal and, thus, P is contained in the radical of a principal ideal. Let $P_1 = (x^n, xy^j, \ldots, x^{n-1}y^{\overline{(n-1)j}})$ and $P_2 = (xy^j, \ldots, x^{n-1}y^{\overline{(n-1)j}}, y^n)$. $\text{ht}(P_1) = \text{ht}(P_2) = 1$ and (P_1), (P_2) generate the free abelian subgroup of all bihomogeneous prime divisorial ideals of B, denoted BDiv(B). In [2, p. 222], it is proved that $\text{cl}(B) \simeq \text{BDiv}(B)/\text{BPrin}(B)$. These two prime ideals satisfy $((P_1^n)^{-1})^{-1} = x^n B$ and $((P_2^n)^{-1})^{-1} = y^n B$. Since $x^n \in P_1$ and $y^n \in P_2$ we actually have $P_1 = \sqrt{x^n B}$ and $P_2 = \sqrt{y^n B}$. Thus, B is close to being prefactorial.

The author wishes to thank the editor, Prof. William Heinzer, and the referee for valuable suggestions that improved the presentation of this paper. In particular, thanks are due for pointing out Nagata Otsuka's theorem that led to the strengthening of the results in Section 1.

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Received: January 1987
Final Revised Version: March 1988