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A DIFFERENTIAL CHARACTERIZATION OF FLAT IDEALS IN RINGS OF CHARACTERISTIC p>0

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1. Introduction

In this paper we introduce a new, non homological, approach to flatness in commutative Noetherian rings of characteristic $p>0$. The problem was first raised by Sally and Vasconcelos [9], in terms of derivations of a ring into a module. More concretely, let A be a commutative ring, let I be an ideal of A and let $\underline{d}: A \to M$ be a derivation of A into an A module M. Sally and Vasconcelos proved that a flat ideal I satisfies the following property: (D) for every two elements f and g in I, $\underline{d}(f)g - f\underline{d}(g) \in I^2M$. In the same paper they speculated that the converse might be true for $A=M=k[x,y]$, the polynomial ring in two variables over a field k, with $\underline{d} = \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Condition (D) was used in [2], to find a differential criteria for flatness

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of ideals in rings of characteristic 0. The condition fails to imply flatness for rings of characteristic $p>0$. As an example, let $A = k[x,y]$, a polynomial ring in two variables over a field k of characteristic $p>0$, and let $I = (x^p, y^p)$. Then for every derivation $\underline{d}: A \to M$, $\underline{d}(x^p) = \underline{d}(y^p) = 0$ hence $\underline{d}(f)g - f\underline{d}(g) \in I^2M$ for any two elements f and g in I but I is not flat.

To study the problem in rings of characteristic $p>0$ we introduce Hasse-Schmidt differentiations of the ring A. Let A be a commutative ring of characteristic $p>0$, let B be a ring containing A, $\underline{t}$ and indeterminate over B and $E: A \to B[[\underline{t}]]$ a differentiation of A into B, i.e. a ring homomorphism satisfying $a \equiv E(a) \pmod{\underline{t}}$ for every element a of A. There is a one to one correspondence between differentiations of A and sequences of the type $D = \{D_0, D_1, \ldots\}$, where $D_i : A \to B$ are additive homomorphisms with D_0 = inclusion, and $D_n(ab) = \Sigma_{i+j=n} D_i(a)D_j(b)$ for every a and b in A and every n, given by $E(a) = \Sigma_{n \geq 0} D_n(a)t^n \in B[[\underline{t}]]$. If I is an ideal of A, $D_j(I^r) \subseteq I^{r-j}B$ for $0 < j < r$ therefore each term D_j of a differentiation D is uniformly continuous from the I-adic topology of A to the IB-adic topology of B, so that D can be uniquely extended to a differentiation from the I-adic completion of A to the IB-adic completion of B. If B = A, D is called a differentiation of A, which is said to be iterative if $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$ for every i and j.

Note that the term D_1 of a differentiation D is a derivation, and in certain important cases a derivation D_1 of A can be extended to a differentiation of A, although this is not always the case. For a differentiation $E(-)=\sum_{n\geq 0} D_n(-)\,t^i$ denote by $S(-) = \Sigma_{n\geq 1} D_n(-)t^i$, and by S the set $\{S(a)\,|\,a\in A\} \subseteq B[[\underline{t}]]$

A useful result on extendability of derivations to differentiations is give by Matsumura [7].

1.1 Theorem. (Matsumura [7]). Let (A,m) be a complete regular local ring with maximal ideal m, dimension n and characteristic $p>0$. Let $x_1,\ldots,x_r \in \underline{m}$ and $D_1,\ldots,D_r \in \mathrm{Der}(A)$ be such that $\det(D_i x_j)_{i,\ j=1}^{r} \notin \underline{m}$, $[D_i,D_j] \in \Sigma_{\alpha=1}^{r} AD_\alpha$ and $D_i^p \in \Sigma_{\alpha=1}^{r} AD_\alpha$, then there exist a subfield K of A and elements $x_{r+1},\ldots,x_n \in \underline{m}$ such that:

(i) $D_1,\ldots,D_r$ vanish on K and on $x_{r+1},\ldots,x_n$.

(ii) $x_1,\ldots,x_n$ form a regular system of parameters for A.

(iii) K is a coeffieient field of A, so that $A = K[[x_1,..x_n]]$ and $\Sigma_{i=1}^{r} AD_i = \Sigma_{j=1}^{r} A(\frac{\partial}{\partial x_j})$.

As a consequence of this theorem it follows that the derivations $D_1^i = \frac{\partial}{\partial x_i}$ can be extended to a commutative set of iterative differentiations of A, namely $E_i : A \to A[[\underline{t}]]$ satisfying $E_i(x_j) = x_j + \delta_{ij}\underline{t}$.

For an ideal I of a commutative ring A we introduce the following conditions:

$(D1)_p$ For every ring B containing A every differentiation E

$E : A \to B[[\underline{t}]]$ and for any two elements f and g in I,

$\Delta_E(f,g) = fE(g) - gE(f)$ lies in I E(I) AS, where E(I) denotes the image of I under E.

$(D2)_p$ For every iterative differentiation $E: A \to A[[\underline{t}]]$ and for every two elements f and g in I, $\Delta_E(f,g)$ lies in IE(I)AS.

If I is a flat ideal of A, I satifies $(D1)_p$. The question investigated in this paper is to what extent the converse of this statement is true. We give a fairly complete answer for regular rings and one dimensional domains and some reflections on the general affine case.

Since flatness and conditions $(D1)_p$ and $(D2)_p$ are local properties, we will assume in the following that $(A,\underline{m})$ is a local ring with maximal ideal $\underline{m}$, and to eliminate trivial cases we will also ask that A be a domain. Over a Noetherian local domain an ideal is flat iff it is principal, hence this paper is investigating the possibility of characterizing principal ideals in terms of differentiations.

2. Regular rings

Let $(A,\underline{m})$ be a regular local ring of characteristic $p > 0$, maximal ideal $\underline{m}$ and dim A=n. The $\underline{m}$-adic completion of A, $\hat{A} = k[[x_1,\ldots,x_n]]$ where $A/\underline{m} = k$ and $x_1,\ldots,x_n$ is a regular system of parameters for A. Let f be a non-zero element in a ring of power series $k[[x_1,\dots,x_s]]$, over a field k of characteristic p. Define deg f = r where $f = f_r + F_r$, f_r is the homogeneous component of f of degree r and F_r the sum of the homogeneous components of f of degree higher the r. Put $\deg(0) = \infty$, and for a subset V of $k[[x_1,\dots,x_s]]$ put $\deg V = \inf\{\deg f, f\in V\}$. Let $E_i: \hat{A} \to \hat{A}[[\underline{t}]]$ be the differentiations of $\hat{A}$ described in the introduction, and I an ideal of $\hat{A}$.

<u>2.1 Lemma</u>. <u>Let</u> $\hat{A}$, I <u>and</u> E_i <u>be as above</u>. <u>There exists and element</u> f of I <u>such that</u> $E_i(f)$ <u>does not lie in</u> $I\hat{A}[[\underline{t}]]$ <u>for some</u> $1 \le i \le n$.

<u>Proof</u>. By the above discussion $\hat{A} = k[[x_1,\dots,x_n]]$, set $r = \deg I, r > 0$ and let f be an element of I of degree r. Assume $E_i(g)$ lies in $I\hat{A}[[\underline{t}]]$ for every element g in I and every i. Let $f=f_r + F_r$ and say x_1 appears in f_r with highest power s, $0 < s \le r$. We expand $f_r = M_{r-s}x_1^s + M_{r-(s-1)}x_1^{s-1} + \dots + M_0$; $M_{r-s} \neq 0$, and $r-s < r$. Calculating the coefficient of $\underline{t}^s$ in $E_1(f)$ we get an element $M_{r-s} + \dots$ in I, contradicting the minimality of r.

We conclude that for an ideal I, there exist an element f of I such that if $E_i = \{D_0^{(i)}, D_1^{(i)}, \dots\}$, where $D_0^{(i)}$= inclusion, $D_1^{(i)} = \frac{\partial}{\partial x_i}$, then $D_j^{(i)}(f)$ does not lie in $I\hat{A}$ for some $1 \le i \le n$ and some j.

2.2 Theorem. Let $(A,\underline{m})$ be a regular local ring with maximal ideal $\underline{m}$ and characteristic $p>2$. Let I be an ideal of A satisfying $(D1)_p$, then I is flat.

Proof. We pass to the completion of A, $\hat{A} = k[[x_1,..,x_n]]$ $\operatorname{ch} k = p > 2$. Let I be a nonflat ideal of A satisfying $(D1)_p$. The ideal $I\hat{A}$ satisfies the following property: for two elements f and g in I, $\Delta_{E_i}(f,g) \in IE_i(I)\underline{t}\hat{A}[[\underline{t}]]$. We will make use of this property to conclude that $I\hat{A}$ and hence I, is nonflat. Thus we replace A by $\hat{A}$ and I by $I\hat{A}$ in the proof below.

We first prove the conclusion of the theorem in case I is generated by two elements. In this case, we may assume the generators of I have no common factor, hence form a regular sequence in A and thus $\overset{2}{\Lambda}(I/I^2)$, the second exterior power of I/I^2, is an A/I-free module [4 page 142,remark]. By lemma 2.1 it suffices to show that $D_i^{(j)}(h)$ annihilates $\overset{2}{\Lambda}I$, for every h in I, $1 \le i \le n$ and for every j.

Fix $E = E_i$, $E = \{D_0, D_1, \ldots\}$; let f and g be elements of I. $\Delta_E(f,g)$ lies in $IE(I)\underline{t}A[[\underline{t}]]$, hence there exist elements u_j in $A[[\underline{t}]]$, ℓ_j and h_j in I such that:

$$\Sigma_i(fD_i(g) - gD_i(f))\underline{t}^i = \Sigma_j\ u_j E(\ell_j) h_j \underline{t}$$

$$= \Sigma_j\ u_j h_j \underline{t}(\ \Sigma_i D_i(\ell_j)\underline{t}^i)$$

Comparing coefficients of $\underline{t}^i$ on both sides one gets:

(1) $fD_i(g) - gD_i(f) = \Sigma_j u_j' h_j(\Sigma_{k<i} D_k(\ell_j)), u_j' \in \overset{2}{\Lambda} A$

We will show $D_i(h)$ annihilates ΛI using induction on i. For i=0, D_0= identity map, thus:
$h(f\wedge g) = hf\wedge g = h\wedge fg = hg\wedge f = h(g\wedge f)$ and $2h(f\wedge g) = 0$. Since ch A>2 we have $h(f\wedge g) = 0$.

Assume $D_k(h)$ annihilates $\overset{2}{\Lambda} I$ for h in I and $1 \leq k < i$.
$D_i(h)(f\wedge g) = D_i(h)f\wedge g = hD_i(f)\wedge g = h\wedge D_i(g)f = hD_i(g)\wedge f = D_i(h)(g\wedge f)$
where the equalities above are justified by the induction hypothesis and equality (1). Hence $D_i(h)(f\wedge g) = 0$ as ch A>2.

We now proceed with the general case. Let deg $I = r > 0$ and let $f = f_r + F_r$ be an element of I of degree r. The degree of $IE(I)\underline{t}A[[\underline{t}]]$ as a subset of $k[[x_1,\dots,x_n,\underline{t}]]$ is $2r + 1$.

Let $g = g_r + G_r$ be an element of I of degree r. Then:

(2) $\Delta_{E_i}(f,g) = \Delta_{E_i}(f_r,g_r) + \Delta_{E_i}(f_r,G_r) + \Delta_{E_i}(g_r,F_r) + \Delta_{E_i}(F_r.G_r)$

Since $\Delta_{E_i}(f,g)$ lies in $IE(I)\underline{t}A[[\underline{t}]]$ we have $\Delta_{E_i}(f_r,g_r) = 0$ and thus by the case where the ideal is generated by two elements g_r is a multiple of f_r.

Let $g = g_s + G_s$, $s > r$ be an element of I. If f_r divides g_s for every such s, $I \subseteq (f_r + F_r) + \underline{m}^i$ for every i, hence by Krull intersection theorem [6 page 69, Corollary 2],

$I = (f_r + F_r)$. So assume there exists $s > 0$ such that f_r does not divide g_s, and pick s minimal integer, with this property.

We proceed to list a possible generating set for I. The generators of I of degree r are of the form $f_r + F_r'$. If $h = h_u + H_u$ is a generator of I of degree $u < s$, f_r divides h_u hence by subtracting suitable multiples of f from h, we can replace h by a generator of degree higher or equal to s. We then have a finite number of generators of degree higher or equal to s.

We expand $\Delta_{E_i}(f,g)$ as in formula (2), and set it equal to a proper summation of generators of I and E(I) with coefficients in $\underline{t}A[[\underline{t}]]$. Comparing homogenous components of minimal degree on both sides of the described equality we conclude that $\Delta_{E_i}(f_r,g_s)$ lies on $F_r E(f_r)\underline{t}A[[\underline{t}]]$. Applying the result of the case where the ideal is generated by two elements we obtain an element $h = af_r + bg_s$, deg $h < r$, a and b in A which generates the ideal generated by f_r and g_s, and thus an element $h + aF_r + bG_s$ of degree strictly lower than r lying in I.

<u>2.3 Corollary. Let (A,m) be a regular local ring of characteristic $p>2$, satisfying that the canonical differentiations E_i of the completion of A map A into A[[t]], and I an ideal of A satisfying (D2)p, then I is flat.</u>

As a consequence of corollary 2.3 we obtain that for localization by maximal ideals of polynomial rings, convergent power series rings and complete regular local rings condition $(D2)_p$ implies flatness.

3. <u>One dimensional domains</u>

Let k be an algebraically closed field of characteristic p>0, and let (A,<u>m</u>) be a one dimensional local k-algebra essentially of finite type, with A/<u>m</u> ≃ k, which is an analytically irreducible domain. The <u>m</u>-adic completion, $\hat{A}$, of A can be finitelly embedded in k[[x]] - the power series in one variable over k. To see this pass the C, a localization by a maximal ideal of the integral closeure of A in its field of quotients. By Krull-Akizuki theorem [8, page 115]. C is a discrete valuation ring with residue field isomorphic to k, and hence its completion is a power series k[[x]], where x is the uniformizing parameter of C [6, page 210]. Since completion and taking integral closure commute [10, Chapter VIII, § 13] we obtain the desired embedding. Let x^N k[[x]] be the conductor ideal of k[[x]] into $\hat{A}$; then $\hat{A}/x^N k[[x]]$ is generated as a vector space over k by the classes of $\{1,h_1,\dots,h_k\}$ where h_i are power series in k[[x]] with n_i = deg h_i = initial power of x in h_i, strictly less than N and we may assume that $1 < n_1 < n_2 < \dots < n_k < N$. Thus every element of $\hat{A}$ can be written as a k-linear combination of $\{1,h_1,\dots,h_k\}$plus an element of $x^N k[[x]]$. For a subset V of $k[[x_1,\dots,x_e]]$ put $\nu(V) = \{n; n = \deg f,\ f \in V\}$ and note that $\nu(\hat{A}) = \{0, n_1, \dots, n_k, N, N+1, \dots\}$

Let J be a nonflat ideal of $\hat{A}$ wich satisfies $(D1)_p$, and let f be a nonzero element of J of minimal degree $n > 0$. Let g be an element of J of degree $m \neq n$ which is not a multiple of f. The Krull intersection theorem [6] allows us to pick g such that $u = m-n \notin \nu(\hat{A})$; let m be minimal satisfying this property. We then readjust a set of generators for J, by subtracting suitable multiples of f from necessary elements, to consist of f (necessarily in every minimal set of generators of J) and $\{G_i\}$ a finite number of elements of degree higher or equal to m. In particular we will be concerned with ideals $J = I\hat{A}$, where I is an ideal of A satisfying $(D1)_p$.

Let $E: \hat{A} \to k[[x]][[\underline{t}]]$ be the restriction of the differentiation of $k[[x]]$ satisfying $E(x) = x+t$, let $S = \{S(a) \mid a \in \hat{A}\}$. Consider $\Delta_E(f,g)$ an element of $JE(J)\hat{A}\hat{S}$ of degree $n+m = 2n + u$. Note that every element of $JE(J)\hat{A}\hat{S}$ can be written as a linear combination with coefficients in $\hat{A}\hat{S}$ of $fE(f)$, $fE(G_i)$, $G_iE(f)$ and $G_iE(G_j)$. Since $\hat{A}\hat{S}$ does not contain 1 we necessarily have $\Delta_E(f,g) = fE(f)\varepsilon + F$ where $\varepsilon \in \hat{A}\hat{S}$, $\deg \varepsilon = u$ and $\deg F > n+m$.

3.1 Theorem. Let k be an algebraically closed field of characteristic $p > 0$, and let (A,m) be a one dimensional local k-algebra, essentially of finite type with $A/m \simeq k$, which is

<u>an analytically irreducible domain, Let I be an ideal of A satisfying $(D1)_p$, then I if flat</u>.

<u>Proof:</u> We may replace A by $\hat{A}$ and I by $J = I\hat{A}$. To prove the theorem suffices to prove that $\nu(A) = \nu(AS)$.

Note that for an element of A, deg a = deg E(a) = deg S(a), and that every element of AS can be written as a k-linear combination of $\{S(h_i), h_iS(h_j), 1 \le i, j \le k$ and an element in $k[[x]][[\underline{t}]]$ of degree $\ge N.\}$

We have that $S(h_i) = (x + t)^{n_i} - x^{n_i} + M_i = (t^{n_i} + n_i t^{n_i - 1} x + \ldots) + M_i$ where deg $M_i > n_i$ and $h_i\ S(h_j) = x^{n_i}(x + t)^{n_j} \le x^{n_i} x^{n_j} + \ldots + M_{ij} = (x^{n_i} t^{n_j} + \ldots) + M_{ij}$ where deg $M_{ij} > n_i + n_j$ hence no cancellation of terms of minimal degree can occur in a k-linear combination of $S(h_i)$ and $h_iS(h_j)$ and $\nu(A) = \nu(AS)$.

An important case where a domain satisfy the condition of theorem 3.1 is an analytically irreducible, local affine domain over an algebraically closed field of characteristic $p > 0$.

The following example will show that the technique employed in analytically irreducible one dimensional domains does not work for the general one dimensional case. This example was inspired by an example constructed by the author jointly with William Brown in a different context.

Let $A = k[[x,y]]/(xy)$ where k is a field of characteristic $p > 0$. A is the completion of a domain. Let I be the ideal of A generated by x^2 and y^2, $E : A \to B[[\underline{t}]]$ a differentiation of A and $f = ax^2 + by^2$, $g = a_1x^2 + b_1y^2$ two elements of I. Then: $E(f)g - E(g)f = x^2E(x^2)[a_1E(a) - aE(a_1] + y^2E(y^2)[b_1E(b) - bE(b_1)] + x^2 E(y^2)\alpha + y^2E(x^2)\beta$ where $\alpha, \beta \in B[[\underline{t}]]$, hence $E(f)g - E(g)f \in IE(I)AS$ provided $x^2E(y^2) = y^2E(x^2) = 0$. We prove that by induction on n where $E(-) = \Sigma D_n(-)t^i$. For $n = 0$ D_o = inclusion and $x^2y^2 = 0$. Assume $x^2D_i(y^2) = 0$ for $i < n$. $D_n(y^2) = 2D_n(y)\, y + \sum_{\substack{i+j=n \\ i,j<n}} D_i(y)D_j(y)$ hence $x^2D(y^2)=0$ and $x^2E(y^2)=0$. By symmetry $y^2E(x^2)=0$. This example strongly suggests that it is necessary to ask that the domain be analytically irreducible in order to obtain $(D1)_p$ implies flatness. However it is very likely the following conjecture is true:

<u>Conjecture:</u> Let $(A,\underline{m})$ be a local analytically irreducible affine domain over an algebraically closed field of characteristic $p>0$. An ideal I of A satisfying $(D1)_p$ is flat.

Following the terminology of [3] and [5] we call an ideal I of a domain A stable if there exist an x in I such that $xI=I^2$. An ideal in a local ring is prestable if some power of I is stable. In general the ideal I is stable (prestable) if IA_p is stable (prestable) for each prime $p \subset A$. In [3, page 446] it is proved that if in addition A is a Noetherian domain the stability (prestability) of I is equivalent to I (some power of I) being projective - equivalently flat - over its endomorphism ring. We adopt the later description as a definition of stability (prestability) of an ideal. The validity of the above conjecture ensures an "almost" flatness property for I in the general affine case, namely:

3.2 Proposition: Let (A,m) be a local affine domain over an algebraically closed field of characteristic $p > 0$. Let I be an ideal of A satisfying $(D1)_p$, then I is prestable.

Proof: The integral closure $\bar{A}$ of A is a semilocal affine domain, satisfying the hypothesis of the conjecture for every localization by a maximal ideal. We conclude that $I\bar{A}$ is prestable (flat) and by Lemma E of [3], I is prestable.

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