COMMUTATIVE ALGEBRA

Analytical Methods

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A DIFFERENTIAL CHARACTERIZATION OF FLAT IDEALS IN COMMUTATIVE RINGS

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A topic of interest in commutative algebra is to develop tools to determine when an ideal in a ring is flat. The methods generally used in the literature are homological in nature. This paper concerns results found by the author in [2] and [3], where differential methods were introduced to the study of flatness in commutative, Noetherian rings.

RINGS OF CHARACTERISTIC 0

Let A be a Noetherian commutative ring and let $d: A \to M$ be a derivation from A into an A module M. For two elements f and g of an ideal I of A let $\Delta_d(f, g) = fd(g) - gd(f)$. Then $\Delta_d(f, g) \in IM$ and, if I is a flat ideal of A, $\Delta_d(f, g) \in I^2M$. In [11], Sally and Vasconcelos asked whether for a polynomial ring A in several variable over a field, an ideal I satisfying $\Delta_d(f, g) \in I^2M$ for every derivation d and any two elements f and g in I, is a flat ideal. This leads to a formulation of two differential conditions for an ideal I of A.

D1. For each A module M, every derivation $d: A \to M$ and any two elements f and g of I, $\Delta_d(f, g) \in I^2M$.

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And the weaker condition:

D2. For every derivation $d: A \to A$ and only two elements f and g of I, $\Delta_d(f, g) \in I^2$

The question is how does this type of behavior of an ideal under derivations affect the properties of the ideal. More specific: In which rings and for which ideals condition (D1) will imply the flatness of the ideal, and for which rings condition (D2) will suffice. And, in case I is a non-flat ideal satisfying (D1) is I "flat enough", where by "flat enough" I mean: is a power of I flat over its endomorphism ring? i.e. is I prestable? The property was introduced by Lipman [6], and further investigated by Eakin and Sathaye [4], and Sally and Vasconcelos [12].

One notices that if $(\Omega_{A/Z}, D)$ denote the module of Kahler differentials of A viewed as an algebra over the integers, Z, and $D: A \to \Omega_{A/Z}$ the canonical derivation, the universal property of this pair implies that $\Delta_d(f, g) \in I^2M$, for every module M and every derivation $d: A \to M$ iff $\Delta_d(f, g) \in I^2\Omega_{A/Z}$, and the unique derivation D. Unfortunately $\Omega_{A/Z}$ and D are hard to describe explicitly enough even for relatively simple rings. Other modules will be used to obtain results as the completion of A, the completion of the integral closure of A, and A itself. The module $\Omega_{A/Z}$ and the derivation D is useful mainly in constructing counterexamples. All properties involved are local properties, thus we may assume (A, m) is a local ring with maximal ideal m. In this section we will also assume that characteristic A is 0.

The question was completely settled for regular local rings.

THEOREM 1.1 [2] Let (A, m) be a regular local ring with ch A/m = 0 and let I be an ideal of A satisfying (D1), then I is flat.

The proof involves replacing A by its m-adic completion $\hat{A} = k[[x_1,\ldots,x_n]]$, a power series in n = dim A variables over a filed k of characteristic 0, and looking at the derivations $\partial/\partial x_i$. One starts by settling the question for an ideal generated by two elements and proceeds by reducing the general case to this case. The complete proof appears in [2].

If (A, m) is regular of characteristic 0, but ch A/m = p > 0 condition (D1) does note suffice to imply flatness. As an example let $A = Z_p[x]$ - the polynomial ring in one variable over the integers localized at a prime p. The nonflat ideal $I = (p, x^p)$ satisfies (D1). This example points out to the need of enough derivations of the ring, hence the need to restrict ourselves to equal characteristic rings.

For a large subclass of the regular local rings the weaker condition (D2) suffices to imply flatness.

THEOREM 1.2 [2] Let (A, m) be a regular local ring of analytic type over a subfield k, let I be an ideal of A satisfying (D2), then I is flat.

By a regular local ring (A, m) of analytic type over a subfield k, I mean following Matsummara [8], a regular local ring satisfying:

1. A/m is algebraic over k and ch k = 0
2. rank $Der_k(A)$ = dim A.

This type of rings include power series, convergent power series, localizations of polynomial rings at maximal ideals.

The next step was to consider domains of dimension one.

Here condition (D2) was too weak to imply flatness as seen in $A = k[[t^2, t^3]]$ and I the ideal (t^2, t^3). Then $d(I) \subset I$ thus I satisfies (D2) but I is not flat.

The results obtained are under certain restrictions on the one dimensional ring. First assume A is a domain, to avoid the case where $\Delta_d(f, g)$ lies in I^2M from reasons of zero-divisors. We also assume (A, m) is a localization of a k-algebra, where k is an algebraically closed field of characteristic 0, $A/m \simeq k$, A is essentially of finite type and analytically irreducible (i.e. the completion of A is a domain). Since the interest lies mainly in affine domains, the only real restrictions are the requirement that k be algebraically closed and that A be analytically irreducible. We pass from A to the m-adic completion of A, $\hat{A}$, and have a finite type embedding. $\hat{A} \to k[[x]]$, where k[[x]] the power series ring in one variable over k, is equal to the completion of the integral closure of A localized at a maximal ideal. Thus A has a definite structure, namely, every element of A can be written as a k-linear combination of a fixed set of polynomials $h_1,\dots,h_k$ plus an element in the conductor ideal $x^N k[[x]]$, and $h_1,\dots,h_k$ can be chosen such that if $n_i = jn_1$ for some i and j then $h_i = h_1^j$, and $\deg h_i < N$. One considers an ideal I of A satisfying (D1). Pick first an element f of I of minimal initial power n, then an element g of I of initial power $m \neq n$ with $u = m - n$ not an initial power of any element of A. Several results have been obtained, the main ones being:

PROPOSITION 1.3 [2] Let (A, m) and k be as above. If all h_i are monomials and I an ideal of A satisfying (D1) then I is flat.

PROPOSITION 1.4 [2] Let (A, m) and k be as in the above discussion. Let J be an ideal of A satisfying (D1), and $I = J\hat{A}$. If $u < n_2$ or I contains all h_i for i s, fixed integer s then J is flat.

The most general result in this category is:

THEOREM 1.5 [2] Let (A, m) and k be as in the above discussion, and let I be a prime ideal of A satisfying (D1) then I is flat.

Although no proof has been found yet that every ideal satisfying (D1) is flat in a ring A with the above properties the question seems decidable. In [2] we prove that it will suffice to show that for an integer u which is not an initial power of any element of A $u - 1$ is not an initial power of any element in $M = Ad(A)$ where $d: A \to k[[x]]$ denotes the restriction of the derivation $\partial/\partial x$ of $k[[x]]$ to A. Every element of M can be written as a k-linear combination of $\{h_i, h_i d(h_j)\}_{i,j=1}^{k}$ and an element in $x^{N-1} k[[x]]$. We note that $n_k < N - 1$. Next we eliminate one of the elements of the pair (h_i, h_j) with $n_i + n_j = N - 1$. We continue this process of elimination to obtain a finite number of possibilities for a set of "generators" for A. In all cases checked, $u - 1$ cannot be an initial power of any element of M.

A different series of results obtained at this stage for affine domains of higher dimension was based on the assumption that for a one dimensional domain of the type described without the required restriction that k be algebraically closed and A be analytically irreducible, condition (D1) implies flatness. These results are:

THEOREM 1.6 [2] Let A be a Cohen Macauley affine domain over a field of characteristic 0, and let I be an ideal of A satisfying (D1), then I is flat.

THEOREM 1.7 Let A be an affine domain over a field of characteristic 0, and let I be an ideal of A satisfying (D1), then all minimal prime ideals over I have height one.

Prestability have been obtained under heavier restrictions:

PROPOSITION 1.8 Let A be an affine domain over a field of characteristic 0, dim A $\leq$ 2 and let I be an ideal of A satisfying (D1), then I is prestable.

Recently working jointly with William Brown we came up with two examples that the conditions k algebraically closed and A analytically irreducible are necessary to insure flatness of every ideal satisfying (D1) in one dimensional affine domains. We also proved several results in the more general case. I am not going to discuss the details of this work in this presentation. In view of those examples we can no longer conjecture that (D1) implies flatness without the necessary two restrictions on the one dimensional domain, therefore the last three results are no longer valid in the generality stated above. However, the techniques involved in proving this results may be useful in proving the following, probably true, conjecture: For a local analytically irreducible affine algebra over an algebraically closed field k, condition (D1) implies flatness. As a consequence of the validity of this conjecture we can obtain.

THEOREM 1.9 [3] Let A be an affine domain over an algebraically closed field of characteristic 0, and let I be an ideal of A satisfying (D1), then I is prestable.

RINGS OF CHARACTERISTIC p > 0

Another aspect of the same problem arises in rings of characteristic $p > 0$. Here condition (D1) fails to ensure flatness even in regular rings. For example, let k be a field of characteristic $p > 0$, let $A = k[x, y]_{(x,y)}$ and $I = (x^p, y^p)$, then $d(I) \subset IM$ for every derivation $d: A \to M$, thus I satisfies (D1) but is not flat.

To study the problem in this case a generalized form of Hasse-Schmidt differentials is introduced. For a ring B containing A and an indeterminate t over B, we define a differential $E: A \to B[[t]]$ to be a ring homomorphism satisfying $E(a) \equiv a (\text{mod } t)$ for every element a in A. There is a one to one correspondence between differentials E and sequences $\mathcal{D} = \{D_o, D_1, \ldots\}$ where $D_i: A \to B$ are additive homomorphisms with D_o = inclusion and $D_n(a\,b) = \Sigma_{i+j=n} D_i(a) D_j(b)$ for every a and b in A and every n, given by

$E(a) = \Sigma_{n\geq 0} D_n(a)t^n \in B[[t]]$. If $B = A$, $\mathcal{D} = E$ is called a differential of A which is said to be iterative if $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$ for every i and j. Note that the term D_1 of $\mathcal{D}$ is a derivation of A into B. For a differential $E(-) = \Sigma_{n\geq 0} D_n(-)t^n$ denote by $S(-) = \Sigma_{n\geq 1} D_n(-)t^n$, and by S the set $\{S(a)/a \in A\} \subseteq B[[t]]$.

For an ideal I of A we introduce the following conditions:

$(D1)_p$ For every ring B containing A, every differential $E: A \to B[[t]]$, and for any two elements f and g of I, $\Delta_E(f, g) = f\,E(g) - g\,E(f)$ lies in $I\,E(I)\,A\,S$, where E(I) denotes the image of I under E.

$(D2)_p$ For every iterative differential $E: A \to A[[t]]$, and for any two elements f and g of I, $\Delta_E(f, g)$ lies in $I\,E(I)\,A\,S$.

These conditions [3], are a natural extension of conditions (D1) and (D2) imposed in case the cahracteristic of the ring is 0. A flat ideal of A satisfies $(D1)_p$. We ask similar questions as asked for rings characteristic 0, and surprisingly obtain a more complete answer.

For regular local rings the following results had been proved:

THEOREM 2.1 [3] Let (A, m) be a regular local ring of characteristic $0 < p \neq 2$. Let I be an ideal of A satisfying $(D1)_p$ then I is a flat ideal.

THEOREM 2.2 [3] Let (A, m) be a regular local ring of characteristic $0 < p \neq 2$, satisfying that the canonical derivations E_i of the completion of A map A into A[[t]], and let I be an ideal of A satisfying $(D2)_p$, then I is flat.

The canonical derivations E_i of the completion of A are constructed as follows. $\hat{A} = k[[x_1,\ldots,x_n]]$ a power series in n = dim A variables over a field of characteristic $0 < p \neq 2$. The canonical derivations $\partial/\partial x_i$ of $\hat{A}$ admit an extension to differentials $E_i: A \to A[[t]]$, [9], satisfying: $E_i(x_j) = x_j + t\delta_{ij}$.

The next natural case to investigate is one dimensional domains. Let (A, m) be a one dimensional analytically irreducible domain which is a localization of a k-algebra, k being an algebraically closed field of characteristic p > 0, which is essentially of finite type. We then prove the following result.

THEOREM 2.3 [3] Let (A, m) and k be as above and let I be an ideal of A satisfying $(D1)_p$ then I is flat.

The techniques employed in proving the above result do not work for one dimensional domains which are not analytically irreducible. For example the ring $A = k[[x, y]]/(xy)$, k a field of characteristic $p > 0$ is a completion of a domain. $I = (x^2, y^2)$ is a nonflat ideal of A satisfying $(D1)_p$.

By analogy with the case where the characteristic of the ring is 0, if we extend result 2.3 to a conjecture for higher dimensional affine algebras over algebraically closed fields of characteristic $0 < p \neq 2$ which are analytically irreducible, we obtain that an ideal satisfying $(D1)_p$, in an affine domain over an algebraically closed field of characteristic $0 < p \neq 2$, is a prestable ideal.

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