PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 3, July 1989

ON THE COHERENCE AND WEAK DIMENSION OF THE RINGS $R\langle x\rangle$ AND $R(x)$

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(Communicated by Louis J. Ratliff)

ABSTRACT. Let R be a commutative ring. We first derive necessary and sufficient conditions for the rings $R\langle x\rangle$ and $R(x)$ to be coherent. Next, for stably coherent rings of finite weak dimension exact relations are found between the weak dimension of R and that of $R\langle x\rangle$ and $R(x)$. These relations are used to determine necessary and sufficient conditions for $R\langle x\rangle$ and $R(x)$ to be Von Neumann regular or semihereditary.

1. INTRODUCTION

Let R be a commutative ring. R is called a *regular ring* if every finitely generated ideal of R has finite projective dimension. This notion, which has been extensively studied for Noetherian rings, was extended to coherent rings with a considerable degree of success, [8, 15, 16, 17, 29, 33]. For a coherent ring R, the regularity condition is closely related to the behaviour of the weak dimension of modules over R. In particular, a coherent ring of finite weak dimension is a regular ring, although not every coherent regular ring has finite weak dimension [15]. The class of coherent regular rings includes several of the classical non-Noetherian rings, like Von Neumann regular rings and semihereditary rings.

Let R be a ring and let S be an R algebra. The type of investigation carried out in this paper considers the following kind of questions: Under what conditions will the extension $R \to S$ ascend or descend coherence and regularity? In particular what is the exact relation between the weak dimension of R, and that of S; and what necessary and sufficient conditions will ascend and descend Von Neumann regularity and semihereditarity?

Received by the editors June 15, 1988 and, in revised form, September 22, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 13F20, 13D05; Secondary 13F05, 13C15, 13C11.
Key words and phrases. Regular rings, coherence, Von Neumann regular, semihereditary, depth, grade, regular sequence.
The research on this paper was partially supported by National Science Foundation Grant DMS 8507944.

The algebras S considered in this paper are two well-known localizations of the polynomial ring in one variable over R, $R\langle x\rangle$ and $R(x)$. For a polynomial $f \in R[x]$, denote by $c(f)$—the so-called *content of* f—the ideal of R generated by the coefficients of f. Let

$$U = \{f \in R[x], f \text{ is monic}\}$$

and

$$V = \{f \in R[x], c(f) = R\} = R[x] - \bigcup\{mR[x], m \in \operatorname{Max}(R)\}.$$

U and V are multiplicatively closed subsets of $R[x]$, and $R\langle x\rangle = R[x]_U$, $R(x) = R[x]_V$. Note that $R[x] \subset R\langle x\rangle \subset R(x)$, and $R(x)$ is a localization of $R\langle x\rangle$.

The ring $R(x)$ is a very useful ring construction in commutative algebra. As a faithfully flat extension of R, it shares many of the properties of R. In addition it satisfies several other useful properties, which facilitate proving many results on R via passage to $R(x)$. Ascent and descent properties of the extension $R \to R(x)$ have been investigated by a number of authors. In [1], Akiba investigates normality of $R(x)$. In [6], D. D. Anderson, D. F. Anderson and Markanda (and in [4, 5]) conduct a thorough study of the properties of $R(x)$. Between other results they touch on conditions related to semihereditarity, namely that of being a Prüfer ring, a strongly Prüfer ring, and an arithmetical ring. Arnold [7], Hinkle, Huckaba [18], Huckaba, Papick [20, 21] relate the ring $R(x)$ and several other ring constructions to Prüfer and Prüfer like conditions of the ring R and $R(x)$. Ferrand [14], McDonald, Waterhouse [26] investigate projective modules over $R(x)$. Ratliff [31] studies $R(x)$ with regard to certain chain conditions.

The ring $R\langle x\rangle$ received a considerable amount of attention due to its role in Quillen's solution to Serre's Conjecture [30, 19]; and the non-Noetherian extensions of this conjecture [10, 23]. Ascent and descent properties of the extension $R \to R\langle x\rangle$ have been investigated by a number of authors. In [6, 20, 21] the authors conduct investigations of $R\langle x\rangle$ analogous and intertwining with the ones of $R(x)$. Brewer, Heinzer [11], determine conditions for $R\langle x\rangle$ to be a Hilbert ring. Le Riche [24] provides an in-depth study of many of the properties of $R\langle x\rangle$. Between other results he determines necessary and sufficient conditions for $R\langle x\rangle$ to be a semihereditary ring.

In this paper we first derive necessary and sufficient conditions for $R\langle x\rangle$ and $R(x)$ to be coherent rings. This is done in Theorem 1. We next explore the relations between the weak dimension of R and that of $R\langle x\rangle$ and $R(x)$. In Theorem 2, using the notion of non-Noetherian grade, we pinpoint exact relations between these weak dimensions, provided that R is a stably coherent ring of finite weak dimension. As corollaries, we determine necessary and sufficient conditions for $R\langle x\rangle$ and $R(x)$ to be Von Neumann regular, for $R(x)$ to be semihereditary; and recapture Le Riche [24] necessary and sufficient conditions for $R\langle x\rangle$ to be semihereditary.

2. Main results

Using a device of Grüson, as in [12], we prove the following.

Theorem 1. *Let R be a ring and let x be an indeterminate over R, then the following conditions are equivalent.*

(1) *$R[x]$ is a coherent ring.*
(2) *$R\langle x\rangle$ is a coherent ring.*
(3) *$R(x)$ is a coherent ring.*

Proof. Clearly we need only prove that (3) implies (1). Let T be an arbitrary set, and consider the exact sequence $0 \to R[x]^T \xrightarrow{\phi} R(x)^T \to \operatorname{coker}\phi \to 0$, where ϕ is the natural map. According to [13], it suffices to show that $R[x]^T$ is a flat $R[x]$ module. Let I be a finitely generated ideal of R, then $IR(x) \cap R[x] = IR[x]$, therefore $R[x]$ is a pure R submodule of $R(x)$ [32, Theorem 3.44], and thus $R[x]^T$ is a pure R submodule of $R(x)^T$. Since $R(x)$ is a coherent ring $R(x)^T$ is a flat $R(x)$ module, therefore both a flat R module and a flat $R[x]$ module. We conclude by [9, p. 18] that $\operatorname{coker}\phi$ is a flat R module. Thus $\operatorname{w.dim}_{R[x]}\operatorname{coker}\phi \leq 1$ [22, Theorem 3, p. 172], and therefore $R[x]^T$ is a flat $R[x]$ module.

Recall that a ring R is called a *stably coherent ring*, if for every positive integer n the polynomial ring in n variables over R is a coherent ring along with R. The class of stably coherent rings includes a wide variety of rings. To name a few: Noetherian rings, Von Neumann regular rings, semihereditary rings, coherent rings of global dimension two, and several others, e.g., [34, 17]. If R is a stably coherent ring, then $R\langle x\rangle$ and $R(x)$ are coherent rings. Theorem 1, proves that if $R(x)$ or $(R\langle x\rangle)$ is a coherent ring so is $R[x]$. It is still an open question whether the coherence of $R[x]$ suffices to imply the stably coherence of R.

We next explore the homological properties of $R\langle x\rangle$ and $R(x)$ as exhibited in the behaviour of their weak dimensions. Regularity itself is easily disposed of as follows.

Proposition 1. *Let R be a ring for which $R[x]$ is a coherent ring, then the following conditions are equivalent:*

(1) *R is a regular ring.*
(2) *$R\langle x\rangle$ is a regular ring.*
(3) *$R(x)$ is a regular ring.*

Proof. To prove (1) $\to$ (2) use [16, Proposition 2.5]. To prove (3) $\to$ (1) use [15, Lemma 2].

We will embark on a brief discussion of non-Noetherian grade as defined by Alfonsi [2, 3], and its relation to the weak dimension for regular coherent rings.

As a definition of grade for a finitely presented module we will adopt its equivalent condition [3, Proposition 1.2]. Let R be a ring, let M be a finitely presented R module, and let N be an R module, then $\operatorname{grade}_R(M, N) \geq n$ if there exists a faithfully flat R algebra S, which may be taken to be a polynomial extension of R, and elements $f_1, \ldots, f_n \in (0 :_S M \otimes_R S)$ which form an $N \otimes_R S$ regular sequence. The largest such integer n is the $\operatorname{grade}_R(M, N)$. If no largest integer n exists put $\operatorname{grade}_R(M, N) = \infty$.

If M is a general R module, then $\operatorname{grade}_R(M, N) \geq n$ if for every $y \in M$, $(0 :_R y)$ contains a finitely generated ideal I_y satisfying $\operatorname{grade}_R(R/I_y, M) \geq n$.

It is clear that if M is a finitely presented R module and S is a faithfully flat R algebra then $\operatorname{grade}_R(M, N) = \operatorname{grade}_S(M \otimes S, N \otimes S)$. To show that this conclusion remains valid for any R module M, we first cite a Lemma proved in [2, Proposition 1.6].

Lemma 1. *Let R be a ring, let N be an R module, and let I and J be two finitely generated ideals of R, then*

(1) *If $I \subset J$ and $\operatorname{grade}_R(R/I, N) \geq n$ then $\operatorname{grade}_R(R/J, N) \geq n$.*
(2) *If $\operatorname{grade}_R(R/I, N) \geq n$ and $\operatorname{grade}_R(R/J, N) \geq n$ then $\operatorname{grade}_R(R/IJ, N) \geq n$.*

Lemma 2. *Let R be a ring, let M and N be two R modules, and let S be a faithfully flat R algebra, then $\operatorname{grade}_R(M, N) = \operatorname{grade}_S(M \otimes S, N \otimes S)$.*

Proof. Assume that $\operatorname{grade}_R(M, N) \geq n$. Let $y = \sum_{i=1}^k y_i \otimes b_i \in M \otimes S$, and let $I_{y_i} \subset (0 :_R y_i)$ be finitely generated ideals satisfying $\operatorname{grade}_R(R/I_{y_i}, N) \geq n$. Then $I = \prod I_{y_i}$ satisfies $\operatorname{grade}_R(R/I, N) \geq n$, and thus $\operatorname{grade}_S(S/IS, N \otimes S) \geq n$. But $IS \subset (0 :_S y)$, thus $\operatorname{grade}_S(M \otimes S, N \otimes S) \geq n$.

Assume that $\operatorname{grade}_S(M \otimes S, N \otimes S) \geq n$. Let $y \in M$ and let $I \subset (0 :_S y \otimes 1) = (O :_R y)S$ be a finitely generated ideal of S satisfying

$$\operatorname{grade}_S(S/I, N \otimes S) \geq n.$$

Let J be the finitely generated ideal contained in $(0 :_R y)$ satisfying $I \subset JS$. Then $\operatorname{grade}_R(R/J, N) = \operatorname{grade}_S(S/JS, N \otimes S) \geq n$, therefore $\operatorname{grade}_R(M, N) \geq n$.

Let (R, m) be a local ring with maximal ideal m, and let M be an R module, the *depth of* M is defined as: $\operatorname{depth}_R M = \operatorname{grade}_R(R/m, M)$.

Let R be a ring, the *small finitistic projective dimension of* R, is defined as follows: $\text{f. p. dim}\, R = \sup\{\text{proj. dim}\, M,\ M \text{ is an } R \text{ module admitting a resolution consisting of finitely generated projective } R \text{ modules, and } \text{proj. dim}\, M < \infty\}$.

Lemma 3. *Let R be a local coherent regular ring then* $\operatorname{depth} R = \text{w. dim}\, R$.

Proof. By [3, Corollary 2.7] we have $\operatorname{depth} R = \text{f. p. dim}\, R$. Since R is a coherent ring any finitely presented R module M satisfies $\text{w. dim}\, M = \text{proj. dim}\, M$

[28, Lemma 1.2], and admits a resolution consisting of finitely generated free modules. Since R is a coherent regular ring any finitely generated ideal of R has finite projective dimension, hence any finitely presented cyclic R module has finite projective dimension. It follows by induction on the number of generators of a finitely presented R module M, that $\text{proj.dim}\, M < \infty$. We conclude that $\text{f.p.dim}\, R = \text{w.dim}\, R$, and the claim follows.

Lemma 4. *Let R be a ring for which $R[x]$ is a coherent ring then*

(1) $\text{w.dim}\, R \leq \text{w.dim}\, R\langle x\rangle \leq \text{w.dim}\, R + 1$.
(2) $\text{w.dim}\, R \leq \text{w.dim}\, R(x) \leq \text{w.dim}\, R + 1$.

Proof. The left-hand side inequalities follow from the fact that $R\langle x\rangle$ and $R(x)$ are faithfully flat R modules [27, Proposition 1.34]. The right-hand side inequalities follow from the fact that $\text{w.dim}\, R[x] = \text{w.dim}\, R + 1$ [34, Theorem 0.14].

Theorem 2. *Let R be a stably coherent ring of* $\text{w.dim}\, R = n < \infty$, *then*

(1) $\text{w.dim}\, R(x) = \text{w.dim}\, R$.
(2a) *If for every prime ideal p of R which is not maximal we have depth $R_p < n$, then* $\text{w.dim}\, R\langle x\rangle = \text{w.dim}\, R$.
(2b) *Otherwise* $\text{w.dim}\, R\langle x\rangle = \text{w.dim}\, R + 1$.

Proof.

(1) There is a $1:1$ correspondence between maximal ideals of R and maximal ideals of $R(x)$, given by $m \leftrightarrow mR(x)$, and satisfying $R(x)_{mR(x)} = R_m(x)$. Consider the faithfully flat local homomorphism

$$(R_m, mR_m) \to (R_m(x), mR_m(x)).$$

By Lemma 2, $\text{depth}\, R_m = \text{depth}\, R_m(x)$. By Lemma 3,

$$\text{w.dim}\, R_m = \text{w.dim}\, R_m(x).$$

Taking supremum over all the maximal ideals m of R, on both sides we obtain $\text{w.dim}\, R = \text{w.dim}\, R(x)$.

(2) Note that for every prime ideal p of R, depth $R_p = \text{w.dim}\, R_p \leq \text{w.dim}\, R = n$.

(2a) Assume that $\text{depth}\, R_p < n$ for all nonmaximal ideals p of R. By Lemma 4, our claim will be complete if we show that for every maximal ideal M of $R\langle x\rangle$ we have $\text{w.dim}\, R\langle x\rangle_M \leq n$.

Let $M = PR\langle x\rangle$, where P is a prime ideal of $R[x]$ not containing a monic polynomial, then $R\langle x\rangle_M = (R[x]_U)_{PR[x]_U} = R[x]_P$.

Let $p = P \cap R$.

If $P = pR[x]$, then $R[x]_P = R[x]_{pR[x]} = R_p[x]_{pR_p[x]} = R_p(x)$. By (1) we obtain $\text{w.dim}\, R[x]_P = \text{w.dim}\, R_p(x) = \text{w.dim}\, R_p \leq n$.

If $P \supsetneqq pR[x]$ we have two possibilities. If p is a maximal ideal of R, then P contains a monic polynomial and need not be taken into account. If p is not a maximal ideal of R, then $R_p[x] = R[x]_{(R-p)}$, thus $R[x]_P = R_p[x]_{PR_p}[x]$, and $\text{w.dim}\, R[x]_P = \text{w.dim}\, R_p[x]_{PR_p[x]} \le \text{w.dim}\, R_p[x] = \text{w.dim}\, R_p + 1 = \text{depth}\, R_p + 1 \le n$.

(2b) Let p be a nonmaximal ideal of R satisfying $\text{depth}\, R_p = n$. If $n = 0$, then R is a Von Neumann regular ring, hence every ideal of R is maximal, and this case falls in the category of (2a). Thus $n \ge 1$. By Lemma 4, it suffices to construct a prime ideal Q in $R[x]$ satisfying $Q \cap R = p$, Q contains no monic polynomial, $pR[x] \subsetneqq Q$ and $\text{depth}\, R[x]_Q \ge n + 1$.

Let $p \subsetneqq m$ for a maximal ideal m of R and let $a \in m - p$. Set $Q = pR[x] + (ax + 1)R[x]$. It is clear that Q contains no monic polynomial.

To show that Q is a prime ideal we follow an argument given by Le Riche [24]. We note that it suffices to show that $Q/pR[x]$ is a prime ideal of

$$R[x]/pR[x] = R/p[x].$$

Let F be the field of quotients of R/p, then the image of $Q/pR[x]$ in $F[x]$ is generated by the irreducible polynomials $(a+p)x + (1+p)$, and is therefore a prime ideal. Thus $Q/pR[x]$ is a prime ideal.

To show that $Q \cap R = p$, let $r \in Q \cap R$. Write $r = f(x) + g(x)(ax + 1)$, $f(x) = b_k x^k + \cdots + b_0$, $g(x) = c_k x^k + \cdots + c_0$, $b_i \in p$, $0 \le i \le k$, $c_i \in R$, $0 \le i \le k$. We substitute these expressions of $f(x)$ and $g(x)$ in the quality describing r, and compare coefficients of powers of x on both sides. Since $a \notin p$ but $b_i \in p$, $0 \le i \le k$ we obtain that $c_i \in p$, $0 \le i \le k$ and thus $r = b_0 + c_0 \in p$.

We will now show that depth $R[x]_Q \ge n + 1$. Since $n = \text{depth}\, R_p = \text{grade}_{R_p}(R_p/pR_p, R_p)$ up to a polynomial extension of R_p, we may assume that there are elements $a_1, \ldots, a_n \in p$ such that $a_1, \ldots, a_n \in pR_p$ form an R_p regular sequence. Now $R[x]_Q = R_p[x]_{QR_p[x]}$, therefore it suffices to show that $a_1, \ldots, a_n$, $ax + 1 \in QR_p[x]$ form an $R_p[x]_{QR_p[x]}$ regular sequence.

Since $a_1, \ldots, a_n$ is an R_p regular sequence, it is an $R_p[x]$ regular sequence, and as $a_1, \ldots, a_n \in QR_p[x]$, it stays an $R_p[x]_{QR_p[x]}$ regular sequence.

Now, let $(f(x)/g(x))(ax + 1) = (f_1(x)/g(x))a_1 + \cdots + (f_n(x)/g(x))a_n$ with $g(x)$, $f(x)$, $f_i(x) \in R_p[x]$ and $g(x) \in R_p[x] - QR_p[x]$. Since R is a coherent ring of finite weak dimension, R_p, and hence $R_p[x]$ is a domain [34, Corollary 5.16], thus

$f(x)(ax + 1) = f_1(x)a_1 + \cdots + f_n(x)a_n$. Let $f(x) = b_k x^k + \cdots + b_0$, $f_1(x) = c_k^i x^k + \cdots + c_0^i$ with b_j, $c_j^i \in R_p$, $0 \le j \le k$, $1 \le i \le n$. Substituting these expressions of $f(x)$ and $f_i(x)$ in the above equality, and comparing coefficients of powers of x on both sides we obtain that $b_j \in (a_1, \ldots, a_n)R_p$ for $0 \le j \le k$ and thus $f(x)/g(x) \in (a_1, \ldots, a_n)R_p[x]_{QR_p[x]}$. We conclude that $a_1, \ldots, a_n$, $ax + 1$ form an $R_p[x]_{QR_p[x]}$ regular sequence.

Corollary 1. *Let R be a Noetherian regular ring, then* $\text{w.dim}\, R\langle x\rangle = \text{w.dim}\, R$.

Proof. If $\text{w.dim}\, R = \infty$, by Lemma 4 we are done. Otherwise $\text{w.dim}\, R = n < \infty$. By [25, p. 156], for every prime ideal p of R we have $\text{depth}\, R_p = \text{w.dim}\, R_p = \text{gl.dim}\, R_p = \text{Krull dim}\, R_p = htp$. Thus Krull $\dim R = n$ and the only prime ideals p of height n are maximal ideals. By Theorem 2 (2a) the conclusion follows.

Corollary 2. *Let R be a ring. The following conditions are equivalent.*

(1) *R is a Von Neumann regular ring.*
(2) *$R\langle x\rangle$ is a Von Neumann regular ring.*
(3) *$R(x)$ is a Von Neumann regular ring.*

Proof. A ring A is a Von Neumann regular ring if and only if $\text{w.dim}\, A = 0$. To show $(1) \to (2)$ use the fact that every ideal of R is maximal and Theorem 2 (2a). To show $(3) \to (1)$ use Theorem 1 and Lemma 4.

Note that Corollary 2 also easily follows from the fact that R is a Von Neumann regular ring if and only if Krull $\dim R = 0$ and R is reduced. Also, since Krull $\dim R = 0$, here, $R\langle x\rangle = R(x)$.

Since a ring A is a semisimple ring if and only if A is a Von Neumann regular Noetherian ring we obtain that R is a semisimple ring if and only if $R\langle x\rangle$ is a semisimple ring if and only if $R(x)$ is a semisimple ring.

Corollary 3. *Let R be a ring, the following conditions are equivalent.*

(1) *R is a semihereditary ring.*
(2) *$R(x)$ is a semihereditary ring.*

Proof. A ring A is a semihereditary ring if and only if A is a coherent ring of $\text{w.dim}\, A \leq 1$ [28, Proposition 2.2]. Now use Theorem 2 (1) and Lemma 4.

We now recapture Le Riche's results [24].

Corollary 4. *Let R be a ring. The following conditions are equivalent.*

(1) *R is a semihereditary ring of* Krull $\dim R \leq 1$.
(2) *$R\langle x\rangle$ is a semihereditary ring.*

Proof. $(1) \to (2)$. Let p be a nonmaximal ideal of R; then p is minimal. Since R_p is a domain [34, Corollary 5.16], it is a field, thus $\text{depth}\, R_p = 0$. By Theorem 1 and Theorem 2 (2a), $R\langle x\rangle$ is semihereditary.

$(2) \to (1)$. If $R\langle x\rangle$ is a semihereditary ring using Lemma 4, and the faithful flatness of $R\langle x\rangle$ over R we conclude that R is a semihereditary ring. If $\text{w.dim}\, R = 0$ then $\text{Krull}\dim R = 0$. If $\text{w.dim}\, R = 1$ then $\text{w.dim}\, R = \text{w.dim}\, R\langle x\rangle$ and by Theorem 2 (2), $\text{depth}\, R_p = 0$ for every nonmaximal prime ideal p of R. Since R_p is a domain, we conclude that R_p is a field, and thus p is minimal and $\text{Krull}\dim R \leq 1$.

ACKNOWLEDGMENT

The author wishes to thank the referee for helpful comments.

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