FINITE CONDUCTOR PROPERTIES
OF R(X) AND R<X>

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Dedicated to Jim Huckaba on the occasion of his retirement

1. INTRODUCTION

Let R be a commutative ring, and let x be an indeterminate over R. For a polynomial f ε R[x], denote by c(f) - the so called *content of f* - the ideal of R generated by the coefficients of f. Let

$$R(x) = R[x]_U \text{ , where } U = \{f \ \varepsilon \ R[x], f \text{ is monic}\}$$

and

$$R<x> = R[x]_V \text{ , where } V= \{f \ \varepsilon \ R[x], c(f) = R\}$$

U and V are multiplicatively closed subsets of R[x], and V = R[x] - ∪mR[x], where m runs over all maximal ideals of R. In

addition R[x] ⊂ R<x> ⊂ R(x), R(x) is a localization of R<x>, and both R<x> and R(x) are faithfully flat R modules.

Ever since R<x> played a prominent role in Quillen's solution to Serre's conjecture [51], and its succeeding generalizations to non-Noetherian rings (Brewer and Costa [15], Lequain and Simis [41]), there had been a considerable amount of interest in the properties of R<x>. This interest expanded to include similarly constructed localizations of R[x]. Notable among these constructions is the ring R(x), which, through a variety of useful properties, provides a tool for proving results on R via passage to R(x).

The interest in the properties of R<x> and R(x) branched in many directions, as attested by the abundance of articles on R<x> and R(x) appearing in the literature: Akiba [1], D. D. Anderson [3, 4], D.D. Anderson, D.F. Anderson and Markanda [11], D.F. Anderson, Dobbs and Fontana [12], Arnold [13], Brewer and Heinzer [16], Cahen, Elkhayyari and Kabbaj [18], Ferrand [20], Fontana, J. Huckaba and Papick [21], Gilmer [23], Gilmer and Heitman [24], Glaz [25, 26, 27, 28, 29], Halter-Koch [33], Hinkle and J. Huckaba [34], J. Huckaba [36], J. Huckaba and Papick [37, 38], Kang [39, 40], LeRiche [42], Lucas [43, 44], McDonald and Waterhouse [48], Ratliff [52], and Yengui [54, 55]. These investigations usually involve determining conditions under which given properties ascend or descend (or conterexamples to ascent or descent) between R and R<x> or R(x). Several of these articles consider finite conductor and related properties.

The purpose of the present article is to provide an overview of the known results on ascent and descent of finite conductor properties between R and R<x> and R(x), with a focus on the work done by the author in Glaz [30, 31], in which the finite conductor and related notions were extended to rings with zero divisors. It is a combination of survey of literature, the author's published results, and some new results and open problems.

The finite conductor property of a domain R, that is the finite generation of the conductor ideals (I:J) for any two principal ideals I and J of R, came into prominence with the publication of McAdams work [47]. The definition of a finite conductor domain appears in Dobbs [19] and the definition of a finite conductor ring is due to Glaz [30]. The notion embodies in its various aspects both factoriality properties and finiteness conditions. The finite generation of conductor ideals (I:J) for certain finitely generated ideals I and J of R characterizes the finiteness condition of being a coherent ring, and also characterizes the factoriality condition of

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being a Greatest Common Divisor (GCD) domain. Hence this article will be involved with ascent and descent of both finiteness conditions such as: coherence, quasi-coherence and finite conductor properties of rings; and factoriality conditions such as: the GCD, G-GCD, UFD and other factoriality properties of rings. It will also be concerned with ascent and descent of homological and muliplicative properties which connect the finiteness and factoriality conditions such as: regularity, finiteness of weak and global dimensions and PVMDness.

In Section 2 we survey the ascent and descent results of finiteness conditions such as coherence, quasi-coherence and finite conductor properties found in Glaz [25], or following from results in Glaz [30, 31]. We also introduce several special finite conductor rings such as coherent regular and classical rings of small weak and global dimensions. Sections 3 explores G-GCD rings, a class of finite conductor rings defined in Glaz [30], which includes UFDs, GCD domains and rings, G-GCD domains defined by the Andersons [6], and coherent regular rings. The exploration surveys some of the results found in Glaz [30, 31], and presents two new characterizations of G-GCD rings. Section 4 presents the work of Arnold [13], Glaz [25, 27], and LeRiche [42] concerning ascent and descent of coherent regularity and "small" weak and global dimensions, and touches on the general relations between the weak and global dimensions of R and of R<x> and R(x) obtained in Glaz [25, 27]. Section 5 surveys ascent and descent results of factoriality properties obtained by D.D. Anderson, D.F. Anderson and Markanda [11], LeRiche [42] and as corollaries of the results obtained in Glaz [30, 31]. It concludes with a brief discussion on some generalizations of the UFD condition to rings with zero divisors introduced by D.D. Anderson and Markanda [7, 8]. It also presents a new generalization of the UFD notion to rings with zero divisors and a possible area of investigation related to these generalizations. Section 6 introduces the notion of a PVMD and its relation to the finite conductor property. Also, in this section another localization of R[x] is introduced, namely the ring $R\{x\} = R[x]_W$, where $W = \{ f \, \varepsilon \, R[x] / \, (c(f)^{-1})^{-1} = R \}$. We bring several results of Kang [39, 40], Houston, Malik and Mott [35], J. Huckaba and Papick [37, 38], Matsuda [46], and Zafrullah [56, 57], relating between the PVMD properties of R and $R\{x\}$, and touch on the extension of the PVMD notion to rings with zero divisors.

2. ASCENT AND DESCENT OF FINITE CONDUCTOR AND RELATED FINITENESS PROPERTIES

Let R be a ring. For two ideals I and J of R denote by
$(I:J) = \{r \; \varepsilon \; R / \; rJ \subset I\}$. If $I = aR$ and $J = bR$ we write $(a:b)$ for $(I:J)$.

A ring R is called *a finite conductor ring* if $(0:c)$ and $aR \cap bR$ are finitely generated ideals of R for any elements a, b, and c of R.

A ring R is called *a quasi-coherent ring* if $(0:c)$ and $a_1R \cap ... \cap a_nR$ are finitely generated ideals of R for any elements $a_1, ..., a_n$, and c of R.

Finite conductor and quasi-coherent domains were defined, respectively, in Dobbs [19], and Barucci, D.F.Anderson and Dobbs [14]. The present extension of both definitions to rings with zero divisors is due to Glaz [30]. For comparison purposes we include here one of the many equivalent definitions of a coherent ring, which highlights the close relation between the three notions.

A ring R is called *a coherent ring* if $(0:c)$ and $I \cap J$ are finitely generated ideals of R for any element c of R and any two finitely generated ideals I and J of R.

Clearly coherent rings are quasi-coherent rings, and quasi-coherent rings are finite conductor rings. Glaz [30, 31] provides many examples of finite conductor and quasi coherent rings and domains which are not coherent, but no examples are known of finite conductor rings which are not quasi-coherent. Indeed, Gabelli and Houston [22] speculate that this two properties might coincide for a domain R.
The conditions under which R<x> or R(x) are coherent rings were determined by Glaz [25].

THEOREM 2.1 (Glaz [25]) *Let R be a ring. The following conditions are equivalent:*
1. R[x] is a coherent ring.
2. R<x> is a coherent ring.
3. R(x) is a coherent ring.

The proof of Theorem 2.1 relies on our ability to "translate" the

finiteness conditions defining coherence into homological conditions, namely, a ring R is coherent if and only if direct product of flat R modules is a flat R module (see Glaz [26, Chapter 2] for equivalent definitions of the coherence property). Unfortunately, although Glaz [30] provides a number of equivalent characterizations for the finite conductor and quasi coherence properties, none are "homological enough" to allow a similar approach.   Nevertheless it is proved in Glaz [30, 31] that localizations of finite conductor (respectively quasi-coherent) rings are finite conductor (respectively quasi-coherent) rings, and both properties descend under faithfully flat extensions. We can then conclude that:

THEOREM 2.2. *Let R be a ring. Then:*
1. *If R[x] is a finite conductor (respectively quasi-coherent) ring, so are R<x> and R(x).*
2. *If R<x> or R(x) is a finite conductor (respectively quasi-coherent) ring, so is R.*

From Theorem 2.2 and from [30, Theorem 4.1], it follows that if R is an integrally closed coherent domain then both R<x> and R(x) are finite conductor (actually quasi-coherent) rings, regardless of the possible non coherence of R[x].

On the other hand, Theorem 2.1 yields a large class of rings which ascend the finite conductor and quasi-coherent properties to R<x> and R(x), namely all rings for which R[x] is coherent. Glaz [26, Chapter 7] provides an extensive treatment of the topic of stable coherence, and we refer the reader there for references and proofs of all the results stated below.

Recall that a ring R is called *a stably coherent ring* if for every positive integer n, the polynomial rings in n variables over R is a coherent ring.

It is not known wether the coherence of the polynomial ring in one variable over a ring R implies its stable coherence, but all rings for which it is known that R[x] is coherent are stably coherent rings. These rings include all the classical coherent rings of "small" weak and global dimension. For the reader's convenience we include a few basic facts about these rings.

A ring R is called *a semisimple ring* if every R module is projective. Equivalently, R is semisimple if gl. dim R = 0. Such rings are Noetherian, actually they are finite direct products of fields, and as such stably coherent. A semisimple domain is a field.

A ring R is called *a Von Neumann regular ring* if every R module is flat. Equivalently R is Von Neumann regular if w. dim R = 0. Such rings are stably coherent. Von Neumann regular domains are fields.

A ring R is called *a hereditary ring* if every ideal of R is projective. Equivalently, R is hereditary if gl. dim R = 1. Such rings are stably coherent. A hereditary domain is a Dedekind domain.

A ring R is called *a semihereditary ring* if every finitely generated ideal of R is projective. Equivalently, R is semihereditary if R is coherent and w. dim R = 1. Such rings are stably coherent. A semihereditary domain is a Prufer domain.

In addition to coherent rings of weak and global dimension less or equal to 1, coherent rings of global dimension 2 are also stably coherent, but coherent rings of weak dimension 2 need not be such even when they are domains. Soublin [53] and Alfonsi [2] provide examples of coherent rings and domains (respectively) of weak dimension 2 over which the polynomial ring in one variable is not coherent. All rings mentioned above fall into the category of rings called coherent regular rings.

Recall that a ring R is called *a regular ring* if every finitely generated ideal of R has finite projective dimension.

The classical notion of regularity and the present definition coincide for Noetherian rings. Every coherent ring of finite weak dimension is a coherent regular ring although the converse is not necessarily true even when the ring is local. In the following Sections we will elaborate on the relation between the finite conductor properties and regularity, and say more about ascent and descent of finite conductor properties when some regularity conditions are present.

3. G-GCD AND REGULARITY PROPERTIES

The notion of a GCD domain highlights the factoriality aspect of the finite conductor condition.

Recall that a domain R is *a GCD domain* if any two elements of R have a greatest common divisor. Equivalently, R is a GCD domain if and only if $aR \cap bR$ is a principal ideal of R for any two elements a and b of R. Thus a GCD domain is a finite conductor, and quasi coherent, domain. The Andersons [6] extended the notion of GCD domains as follows:

A domain R is called *a G-GCD domain* (a generalized GCD domain) if the intersection of any two invertible ideals of R is an invertible ideal of R.

Clearly G-GCD domains are quasi-coherent as well. Glaz [30, 31] extended both notions to rings with zero divisors.

A ring R is called *a GCD ring* if principal ideals of R are projective and the intersection of any two principal ideals of R is a principal ideal of R.

A ring R is called *a G-GCD ring* if principal ideals of R are projective and the intersection of any two finitely generated flat ideals of R is a finitely generated flat ideal of R.

GCD rings are G-GCD rings and those in turn are quasi-coherent rings. Glaz [30, 31] explores in some depth the nature of G-GCD rings. In particular it is noted that in the presence of the natural requirement that principal ideals of R are projective, the definition of a G-GCD ring given above is the only possible generalization of the concept of a G-GCD domain to rings with zero divisors. To make this statement more precise we need to recall a few facts:

A regular ideal is an ideal containing a nonzero divisor (so called *regular element*). One can define invertibility of ideals in rings with zero divisors analogously with the definition for domains replacing the field of fractions of R with the total ring of fractions of R. In this case an ideal is invertible if and only if it is regular and projective.

THEOREM 3.1 (Glaz [30]). *Let R be a ring whose principal ideals are projective. The following conditions are equivalent:*
1. R is a G-GCD ring.
2. The intersection of any two finitely generated projective ideals of R is a finitely generated projective ideal of R.
3. The intersection of any two invertible ideals of R is an invertible ideal of R.
4. The intersection of any two principal ideals of R is a finitely generated projective ideal of R.
5. The intersection of any two principal regular ideals of R is an invertible ideal of R.

G-GCD rings have a number of interesting properties that we capture in the next theorem. Denote by Q(R) the total ring of fractions of a ring R.

THEOREM 3.2 *Let R be a ring. The following conditions are equivalent:*
1. R is a G-GCD ring.
2. R is a finite conductor ring satisfying that R_P is a GCD domain for all prime (maximal) ideals P of R.
3. R_P is a GCD domain for all prime (maximal) ideals P of R, Q(R) is a Von Neumann regular ring, and $aR \cap bR$ is a finitely generated ideal of R for any two nonzero divisors a and b of R.

We sketch the proof of Theorem 3.2: 1 implies 2 by [30, Proposition 3.1]. To show that 2 implies 3, we need only show that under the conditions of 2, Q(R) is a Von Neumann regular ring. From [26, Theorem 4.2.10] it follows that it suffices to show that principal ideals of R are projective. Since R_P is a domain for all P, principal ideals of R are flat. Since annihilators of elements are finitely generated, principal ideals are finitely presented and therefore projective. To show that 3 implies 1, we first argue that 3 implies that principal ideals of R are projective. To see this note that since R_P is a domain principal ideals of R are flat. Since Q(R) is Von Neumann regular we conclude from [26, Theorem 4.2.10] that principal ideals of R are projective. Now according to Theorem 3.1 we need only show that if a and b are nonzero divisors of R, then the ideal I = $aR \cap bR$ is a projective R module. Since R_P is a GCD domain I is locally flat. It follows that I is flat and by the hypothesis finitely generated. According to [30, Theorem 3.3] it is a projective ideal.

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In addition it is shown in Glaz [30] that a G-GCD ring R is integrally closed in its total ring of fractions. Examples of G-GCD rings, besides GCD rings, G-GCD domains and GCD domains and UFDs include all finite conductor rings R of w. dim R = 2 , Glaz [30], and:

THEOREM 3.3. (Glaz [30]) *Every coherent regular ring is a G-GCD ring.*

In particular all the classical coherent rings of finite weak and global dimension, stably coherent or not, mentioned in Section 2 are by this theorem G-GCD rings. Examples are provided in Glaz [30, 31] that show that in general this is not true, that is, there are coherent rings which are not G-GCD rings and vice versa.

4. ASCENT AND DESCENT OF "SMALL" WEAK AND GLOBAL DIMENSIONS AND OF COHERENT REGULARITY.

We first consider ascent and descent of coherent regularity.

THEOREM 4.1 (Glaz [25]) *Let R be a ring for which R[x] is coherent. The following conditions are equivalent:*
1. R is a regular ring.
2. R<x> is a regular ring.
3. R(x) is a regular ring.

It follows that stably coherent regular rings R ascend coherent regularity, and hence their G-GCD property to R<x> and R(x). In particular all classical "small" weak and global dimension rings mentioned in Section 2 ascend the G-GCD property to R<x> and R(x). But we actually know more then that:

THEOREM 4.2. (Glaz [25]) *Let R be a ring. The following conditions are equivalent.*
1. R is Von Neumann regular.
2. R<x> is Von Neumann regular.
3. R(x) is Von Neumann regular.

THEOREM 4.3 (Glaz [25]) *Let R be a ring. Then:*
1. R<x> is a semihereditary ring if and only if R is a semihereditary ring
 of Krull dim R $\leq$ 1.
2. R(x) is a semihereditary ring if and only if R is a semihereditary ring.

   Theorem 4.3 (1) was also proved by LeRiche [42]. The case where
R is a semihereditary domain, that is a Prufer domain, was proved by
Arnold [13] for R(x), and by LeRiche [42] for R<x>. Theorems 4.2 and
4.3 were corollaries of a more general investigation carried out by Glaz
[25] into the exact relation between the weak dimensions of R, R<x> and
R(x). It was found in [25] that in the presence of stably coherence and
finite weak dimension of R, w. dim R(x) = w. dim R, but w.dim R<x> can
be either equal w.dim R or w.dim R + 1, depending on the depth of certain
localizations of R. [29] contains a survey of results on this topic.
   Contrary to the complete picture obtained in the case of finite weak
dimension, the case of finite global dimension is only partially understood.
The Noetherianess of rings R of gl. dim R = 0, makes it an easy case.

THEOREM 4.4 (Glaz [25]) *Let R be a ring. The following conditions are*
equivalent:
1. R is a semisimple ring.
2. R<x> is a semisimple ring.
3. R(x) is a semisimple ring.

   The case of a hereditary ring that is a ring of global dimension one,
was very difficult to solve, but it eventually yielded:

THEOREM 4.5 (Glaz [27]) *Let R be a ring. The following conditions are*
equivalent:
1. R is a hereditary ring.
2. R<x> is a hereditary ring.
3. R(x) is a hereditary ring.

   The hereditary domain case, that is the Dedekind domain case,
which follows as a corollary from Theorem 4.5, was also obtained by
Arnold [13] for R(x), and LeRiche [42] for R<x>. In the same paper [27]
Glaz carries the investigation into the relation between the global
dimensions of R, R<x> and R(x) a little further. It is shown that if R is
$\aleph_0$-*Noetherian*, that is every ideal of R is countably generated, then the

global dimensions of R<x> and R(x) are bounded below by gl. dim R and above by gl. dim R + 1. This allows Glaz [27] to construct an example of a valuation domain V of gl.dim V = 2, such that gl. dim V(x) = 2, but gl. dim V<x> = 3. No further results are known concerning the relations between the global dimensions of the rings involved. For a more detailed exposition and a more extensive bibliography on this topic we refer the reader to [29].

5. ASCENT AND DESCENT OF FACTORIALITY AND G-GCD PROPERTIES

In this section we consider ascent and descent of factoriality properties of finite conductor rings between R and R<x> and R(x). In case R is a domain, the situation is pretty clear.

A domain R is called *a π-domain* if every nonzero principal ideal of R is a product of prime ideals (necessarily invertible). Equivalently, R is a π-domain if R is a locally factorial Krull domain (D. D. Anderson [5])

THEOREM 5.1 (LeRiche [42], D.D. Anderson, D.F. Anderson and Markanda [11]). *Let R be a domain, then:*
1. R<x> is a UFD if and only if R is a UFD.
2. R(x) is a UFD if and only if R is a π-domain.

THEOREM 5.2 (LeRiche [42], D.D.Anderson, D.F.Anderson and Markanda [11]) *Let R be a domain. Then:*
1. R<x> is a GCD domain if and only if R is a GCD domain.
2. R<x> is a G-GCD domain if and only if R is a G-GCD domain.
3. R(x) is a GCD domain (equivalently, R(x) is a G-GCD domain) if and only if R is a G-GCD domain.

The last theorem leads us to believe that the following conjecture is true:

CONJECTURE: *Let R be a ring. The following conditions are equivalent:*
1. R is a G-GCD ring.
2. R<x> is a G-GCD ring.
3. R(x) is a G-GCD ring.

In addition to the validity of this conjecture for the domain case, it follows from Glaz, [30, Theorem 4.3], that the conjecture holds for a large class of G-GCD rings which are not domains, namely:

THEOREM 5.3 *Let R be a coherent regular ring. Then both R<x> and R(x) are G-GCD rings.*

We would like to emphasize the fact that the last result holds even in the absence of coherence of the rings R(x) or R<x> , by displaying examples of coherent rings and domains R such that R(x) and R<x> are G-GCD rings but not a coherent rings.

EXAMPLE. Let S_i = Q[[t,u]] be countably many copies of the power series ring in two variables t and u over the rational numbers Q, and let S = ΠS_i. It is shown in Soublin [53] that S is a coherent ring of w.dim S = 2, and that S[x] is not coherent. It follows from Glaz [30], that S(x) and S<x> are both non coherent G-GCD rings.

Moreover, according to Alfonsi [2] there exists a localization of S, S_P which is a coherent domain of weak dimension two with S_P[x] not coherent. A similar argument yields that the domain S_P has the property that S_P<x> and S_P(x) are non coherent GCD domains.

We conclude this section with a brief discussion regarding possible generalizations of the notion of a UFD to rings with zero-divisors. The notion of a UFD has been generalized successfully to rings with zero divisors in a number of ways. The principle behind most generalizations is to start from one of the equivalent definitions of a UFD and attempt to find the most natural extension of this definition to rings with zero divisors. As a UFD can be equivalently defined in a number of ways we end up with a host of , non equivalent, generalizations. To make the situation even more interesting, many of the equivalent definitions of a UFD display two conditions that need to be satisfied in order that the ring be a UFD. Thus generalizations, to rings with zero divisors and domains alike, become possible by asking that one of the conditions be satisfied but not the other. It is not our intention to survey all generalizations of the UFD notion. We just point out several such interesting generalizations to rings with zero divisors that can be found in: D.D.Anderson and R. Markanda [7, 8], D.D. Anderson, D.F. Anderson and R. Markanda [11], and add a generalization

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of our own. Neither the relations between these generalizations, nor the possibilities of ascent and descent of these generalizations between R and R<x> and R(x), have yet been fully explored.

A ring R is called *a π-ring* if every principal ideal is a product of prime ideals.

A ring R is called *a UFR (a Unique Factorization Ring)* if every principal ideal of R is a product of principal prime ideals.

A ring R is called *a factorial ring* if every regular element of R is a product of (regular) principal primes. Equivalently, R is a factorial ring if its regular elements satisfy the two factorization conditions on elements required for a domain to be a UFD.

A UFR is factorial, but not conversely. Interested readers can find more about these and related generalizations in the references cited above, and in: D.D Anderson and S. Valdes-Leon [9, 10].

We extend a definition of UFDs found in Bourbaki [17], and define *a UF-ring (Unique Factorization ring)* to be a GCD ring satisfying the ascending chain condition on principal ideals. That is, a ring R is a UF-ring if it satisfies the following three conditions:
(i) Principal ideals of R are projective
(ii) The intersection of any two principal ideals of R is principal ideal of R.
(iii) R satisfies ACC on principal ideals.

A UF-ring is factorial. A UF-ring is a GCD ring and therefore a G-GCD ring. It is a reduced ring whose localizations at prime ideals are UFDs, and whose total ring of quotients is a Von Neumann regular ring.
The requirement that principal ideals of R be projective controls the behavior of the zero divisors of R. Assuming that principal ideals of R are projective, it will be interesting to determine to what extent conditions (ii) and (iii) are equivalent to any of the other generalizations of the UFD condition described above. D.D. Anderson and Markanda [8] carried out a similar investigation with a number of related factoriality properties. The zero divisor controlling condition they imposed is the Marot property introduced by Marot in [45].

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A ring R is said to *satisfy the Marot property* , or be *a Marot ring*, if every regular ideal of R is generated by regular elements.

If R is a Marot ring and I is a regular (fractional) ideal of R, then $(I^{-1})^{-1}$ is equal to the intersection of the principal fractional ideals containing I. This allowed D.D Anderson and Markanda [8] to obtain the equivalence of several of the factoriality conditions defined on rings with zero divisors. In particular for a Marot ring R, R is factorial if and only if R satisfies the UF-ring conditions (ii) and (iii) for regular principal ideals.

Regarding R<x> and R(x) neither the factorial nor the UF-ring conditions have yet been explored. For the UFR and π-ring conditions we have:

THEOREM 5.4 (D.D. Anderson, D.F. Anderson and Markanda [11]). *Let R be a commutative ring. Then:*
1. R<x> is a UFR if and only if R is a UFR.
2. R<x> is a π-ring if and only if R is a π-ring.
3. R(x) is a UFR (equivalently, R(x) is a π-ring) if and only if R is a π-ring.

6. THE PVMD CONDITION AND THE RING R{x}

In this section we bring together two concepts that touch on both themes of this article: the notion of a PVMD-- a condition on a domain that touches on the finite conductor condition; and R{x}-- another localization of R[x] by a multiplicative set of polynomials defined through a property of their content ideals.

For a domain R and a nonzero (fractional) ideal I of R, I_v is defined as $(I^{-1})^{-1}$, and $I_t = \cup J_v$, the union being taken over all finitely generated subideals J of I. A fractional ideal with the property that $I = I_v$ (respectively, $I = I_t$) is said to be divisorial or a v-ideal (respectively, a t-ideal). A prime t-ideal is called a t-prime. Ideals maximal with respect to being t-ideals are t-prime, and are refered to as maximal t-primes. A v-ideal I is said to be of finite type if $I = J_v$ for some finitely generated ideal J.

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A domain R is called *a PVMD (a Prufer v-multiplication domain)* if the finite type v-ideals of R form a group under v-multiplication, that is, if for each finitely generated fractional ideal I of R there is a finitely generated ideal J of R such that $(IJ)_v = R$. Equivalently, a domain R is a PVMD if every finite type v-ideal I is t-invertible $(II^{-1})_t = R$. Another equivalent definition is, R is a PVMD if R_m is a valuation domain for each maximal t-ideal m of R (Griffin [32]).

PVMDs make constant appearance in the literature since they were defined in Griffin [32]. A selected list of references for the classical results listed below are: Gilmer [23], Zafrullah [56, 57, 58], Houston, Malik and Mott [35], J. Huckaba and Papick [37, 38], Papick [50], Kang [39, 40], Mott and Zafrullah [49]. The class of PVMDs includes Prufer domains, Krull domains, GCD domains and UFDs. Properties of interest include the fact that localizations by prime ideals of a PVMD and polynomial rings over a PVMD are PVMDs. In particular if R is a PVMD, then so are R<x> and R(x). Also PVMDs are integrally closed domains. In fact every PVMD is an intersection of valuation domains. Regarding the finite conductor properties of PVMDs we can say that not every PVMD is a finite conductor ring. For example a Krull domain which is not a finite conductor ring satisfies this requirement Glaz [31, Example 2.5]. Also not every finite conductor domain is a PVMD, as any finite conductor but not integrally closed domain shows. But there is a large class of finite conductor domains which are PVMDs.

THEOREM 6.1 (Zafrullah [56]) *An integrally closed finite conductor domain is a PVMD.*

Next we consider another localization of R[x]. Set

$$R\{x\} = R[x]_W \text{ , where } W = \{f \in R[x]/ (c(f)^{-1})^{-1} = R\}$$

Among the many interesting results in Kang [39, 40], there is an ascent-descent result of the PVMD property between R and R{x}.

THEOREM 6.2 (Kang [39, 40]) *Let R be a domain. The following conditions are equivalent:*
1. R is a PVMD.
2. R[x] is a PVMD.

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3. R{x} is a PVMD.
4. R{x} is a Prufer domain.
5. R{x} is a Bezout domain.
6. R is integrally closed and every prime ideal of R{x} is extended from R.
7. Every principal ideal of R{x} is extended from R.
8. Every ideal of R{x} is extended from R.

In Theorem 6.2, the equivalence of 1 and 2 can also be found in Houston, Malik, and Mott [35]; the equivalence of 1, 4 and 5 can also be found in Zafrullah [57]. J. Huckaba and Papick [38] raised a number of questions regarding R{x}, two of which were:
(1). If R{x} is a Prufer (Bezout) domain, are the prime ideals of R{x} extended from R?
(2). If R{x} is a Prufer domain, is R{x} a Bezout domain?
Both questions have an answer in the above theorem, and also in the works of Matsuda [46], Zafrullah [57].

J. Huckaba and Papick [37] extended the PVMD notion to rings with zero divisors. Their definition is essentially analogous to the definition given above for domains with all appropriate inverses of ideals taken in the total ring of fractions. They call such a ring *a PVMR* and prove a partial equivalent of the results of Theorem 6.2 for the PVMR condition provided that the ring R satisfies several additional properties, including the requirement that R be a Marot ring. A detailed exposition of results in this direction can be found in [36, Section 22]. In general the PVMR property seems to be difficult to work with. One wonders if other generalizations of the PVMD notion to rings with zero divisors are possible, and how would the various notions compare to each other and to the finite conductor property.

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