

Unlocking Reserve Assumptions Using Retrospective Analysis

Jeyaraj Vadiveloo, Gao Niu, Emiliano A. Valdez, and Guojun Gan

Abstract In this paper, we define a retrospective accumulated net asset random variable and mathematically demonstrate that its expectation is the retrospective reserve which in turn is equivalent to the prospective reserve. We further explore various properties of this retrospective accumulated net asset random variable. In particular, we find and demonstrate that this retrospective random variable can be used as a tool for helping us extract historical information on the pattern and significance of deviation of actual experience from that assumed for reserving purposes. This information can subsequently guide us as to whether it becomes necessary to adjust prospective reserves and the procedure to do so. The paper concludes, as an illustration, with a model of a block of in force policies with actual experience different from reserving assumptions and a suggested methodology on how prospective reserves could be adjusted based on the realized retrospective accumulated net asset random variable.

Keywords Life insurance reserves • Prospective loss • Retrospective accumulated net asset • Emerging mortality experience • Unlocking assumptions

1 Introduction

Reserves for life insurance products are funds set aside to meet the insurer's future financial obligations and they appear as a liability item on the insurer's balance sheet. This item usually represents a very large proportion of the insurance company's total liability and it is the task of the appointed actuary, responsible for the calculation of these reserves, to ensure that they are calculated according to well-accepted actuarial principles, within the guidelines set by the purpose of its calculation (e.g., statutory, tax), and that sufficient assets are available to back these reserves. See Atkinson and Dallas (2000, Chap. 6, pp. 313–356).

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Under old accounting rules, reserve basis and assumptions have typically been “locked-in” at policy issue so that they remain unchanged over time. However, it has become increasingly recognized that this “locked-in” principle can no longer be applicable under today’s dynamic conditions. For example, under the Financial Accounting Standards (FAS) 97 and 120 for Generally Accepted Accounting Principles (GAAP), reserves can now be re-evaluated using what has been referred to as “dynamical unlocking” which allows for the replacement of original actuarial assumptions with a more realistic set of assumptions that accurately reflects historical experience when projecting for future years. See Financial Accounting Standards Board (1987).

The “locked-in” principle has also been historically applicable for statutory accounting, the basis that is used to value insurer’s reserves and obligations to meet regulatory requirements for ensuring company solvency. Under old valuation standards, it has even been considered more deficient because the calculation of reserves has been static and formula-based. However, the National Association of Insurance Commissioners (NAIC), the organization responsible for formulating these uniform standards, has introduced in 2009 a new Standard Valuation Law (SVL) called Principle-Based Reserving (PBR). Under this PBR approach, insurance companies are now permitted to compute reserves by examining a wide range of more realistic future conditions, provided justified, and that the unlocking of reserve assumptions are permitted, again provided justified. This new valuation approach reflects the fact that insurance companies have been introducing more complex products to a more sophisticated market and that economic conditions are constantly evolving. See Manning (1990) and Mazyck (2013).

What these developments mean to the actuary is the need to continually evaluate historical experience and make necessary adjustments to the assumptions and reserves accordingly. The purpose of this article is to examine the use of a retrospective random variable to provide a guidance for unlocking reserve assumptions. For purposes of this article, we ignore the effect of expenses on reserves and focus on what has historically been called net level premiums reserves. Extension of concepts introduced in this article to reflect expenses should be straightforward, and our intent is to introduce first the concept so that it can be well explained more intuitively.

It is well known that net level premium reserves can be calculated prospectively and retrospectively at any duration for a policy that is in force. All major actuarial textbooks covering the mathematics of life contingencies demonstrate the equivalence between these two approaches based on an expected basis. See, for example, Bowers et al. (1986, Chap. 7, pp. 213–214) and Dickson et al. (2013, Chap. 7, pp. 220–225). To illustrate, consider a fully discrete n -year term insurance policy issued to a life aged x with a death benefit of M and an annual level premium of P determined according to the actuarial equivalence principle. At policy duration t , the prospective loss random variable is defined to be the difference between the present value of future benefits at time t ($PVFB_t$) and the present value of future premiums at time t ($PVFP_t$):

$$L_t^P = PVFB_t - PVFP_t, \quad (1)$$

where for our policy, we have

$$\text{PVFB}_t = M v^{K_{x+t}+1} I(K_{x+t} < n - t) \text{ and } \text{PVFP}_t = P \ddot{a}_{\overline{\min(K_{x+t}+1, n-t)}|},$$

where K_{x+t} refers to the curtate future lifetime of $(x + t)$ and $I(\cdot)$ is an indicator function. The expected value of this prospective loss random variable is the prospective reserve defined by

$$E(L_t^P) = E(\text{PVFB}_t) - E(\text{PVFP}_t) = M A_{x+t:\overline{n-t}}^1 - P \ddot{a}_{x+t:\overline{n-t}} \quad (2)$$

and is referred to as the prospective net level premium reserve for this policy. Implicit in this formula is the assumption that the policyholder (x) has reached to survive t years. A straightforward algebraic manipulation of Eq. (2) leads us to the following equivalent expression of this reserve:

$$\text{Retrospective Reserve} = P \frac{\ddot{a}_{x:\overline{t}}}{{}_tE_x} - M \frac{A_{x:\overline{t}}^1}{{}_tE_x}, \quad (3)$$

where ${}_tE_x = v^t {}_t p_x$. Equation (3) is referred to as the retrospective net level premium reserve which gives the difference between the actuarial accumulated value of past premiums and the actuarial accumulated value of past benefits. Note that the mathematical equivalence of the retrospective and prospective reserve assumes that premiums at issue are determined based on the actuarial equivalence principle and that reserving assumptions equal pricing assumptions.

However, only the prospective reserve is defined as the expected value of a corresponding prospective loss random variable. Defining the corresponding retrospective accumulated net asset random variable that leads us to Eq. (3) has not appeared in the literature, and indeed, Dickson et al. (2013, Chap. 7, pp. 222–223) and Gerber (1976) recognize the difficulty of defining such a random variable. In this paper, we define a retrospective accumulated net asset random variable whose expectation leads us to the retrospective reserve and is therefore equal to the prospective reserve. We are also able to intuitively provide an interpretation to this loss random variable. We further explore various properties of the retrospective accumulated net asset random variable and how its realized value provides valuable information on how prospective reserves may be established.

In this paper, we develop a formal definition of a retrospective accumulated net asset random variable whose expected value is equal to the retrospective reserve, which in turn equals the prospective reserve. However, while both the accumulated net asset random variable and prospective loss random variable have equal expectations, the probability distributions of both random variables are entirely different. The paper will provide an intuitive explanation and additional insight as to what the retrospective accumulated net asset random variable is measuring and how its distribution differs from the prospective loss random variable over time. More importantly, the paper additionally explores how the retrospective accumulated net

asset random variable could provide information on a company's historical claim experience and how the prospective reserve at any duration t should be adjusted if actual experience over the past t years differs from reserving assumptions. The retrospective accumulated net asset random variable as defined in this paper can help an insurance company in developing a claims tracking and monitoring process and provide a systematic procedure of adjusting future reserves to reflect actual experience. This procedure can then be implemented to meet valuation standards according to Principle-Based Reserving.

This paper has been structured as follows. Section 2 develops the theoretical foundation for defining the retrospective accumulated net asset random variable. Here, we demonstrate how this definition differs from the more familiar prospective loss random variable, though we also show that the two are always equal in expectation. This equality in expectation hinges on the premium being determined according to the actuarial equivalence principle. Section 3 extends the discussion of the retrospective accumulated net asset random variable in the case where we have a portfolio of insurance policies. This further gives us a natural interpretation of the retrospective accumulated net asset random variable. Furthermore, in this section, we show how one can derive the mean and variance of the retrospective accumulated net asset random variable for a portfolio that may vary in the amounts of death benefits and issue ages. This is important because we demonstrate how the standard deviation of the retrospective may be used to unlock the assumption of mortality so that prospective reserves may be adjusted accordingly. The adjustment in our demonstration may be arbitrary, for the moment, but it allows us to systematically make the adjustment. We conclude in Sect. 4.

2 Formulation

2.1 *Defining the Retrospective Accumulated Net Asset Random Variable*

The retrospective accumulated net asset random variable is best understood with a simple illustration. Extension to the case of other forms of insurance will be rather straightforward and we will examine a few of these other cases.

Consider a fully discrete n -year term insurance policy issued to a life aged x with a death benefit of M and an annual level premium of P determined according to the actuarial equivalence principle. For those unfamiliar with the concept of fully discrete, this refers to the death benefit being paid at the end of the year of death and that level premiums are paid at the beginning of each year the policyholder is alive. See Bowers et al. (1986, Chap. 7, pp. 215–221) and Gerber (1997, Chap. 6, pp. 59–73).

For a policyholder age x , denote his curtate future lifetime random variable by K_x . For $K_x < t$, the policyholder dies before reaching age $x + t$ and in this case, we define the retrospective accumulated net asset random variable to be

$$L_t^R = \frac{1}{p_x} \left[P \ddot{a}_{\overline{K_x+1}|} (1+i)^t - M(1+i)^{t-K_x-1} \right], \quad (4)$$

where p_x is the probability that policyholder (x) survives for t years. The first term $P \ddot{a}_{\overline{K_x+1}|} (1+i)^t$ clearly refers to the accumulated value at time t of all past premiums paid before death while the second term $M(1+i)^{t-K_x-1}$ refers to the accumulated value of the death benefit, paid at the end of the year of death, at time t .

In the case where $K_x \geq t$, we define the retrospective accumulated net asset random variable to be simply a constant equal to

$$L_t^R = \frac{P \ddot{a}_{\overline{t}|} (1+i)^t}{p_x}. \quad (5)$$

We can express this retrospective accumulated net asset random variable more succinctly as

$$\begin{aligned} L_t^R &= \frac{1}{p_x} \left[P(1+i)^t \left(\ddot{a}_{\overline{K_x+1}|} \cdot I(K_x < t) - \ddot{a}_{\overline{t}|} \cdot I(K_x \geq t) \right) - M(1+i)^{t-K_x-1} \cdot I(K_x < t) \right] \\ &= \frac{1}{p_x} \left[P(1+i)^t \ddot{a}_{\overline{\min(K_x+1, t)}|} - M(1+i)^{t-K_x-1} \cdot I(K_x < t) \right] \end{aligned} \quad (6)$$

In the case where $K_x \geq n$, the policyholder would have survived the term of the policy and in which case, L_t^R would still be Eq. (5).

It is therefore straightforward to interpret the retrospective accumulated net asset random variable. In this case, it can be viewed as the share per survivor of the accumulated net assets per \$1 of insurance at duration t . A similar concept of an expected share per survivor within the context of group benefits has been considered in Ramsay (1993) and Arias Lopez and Garrido (2001). In contrast, the prospective loss random variable can be viewed as the share per survivor of the present value of net liabilities per \$1 of insurance at duration t . We will define the expectation of this retrospective accumulated net asset random variable, $E(L_t^R)$, as the retrospective reserve.

Using formulas from mathematics of life contingencies, it is straightforward to prove the equivalence between prospective and the retrospective reserve. Note that we can express Eq. (6) as

$$L_t^R = \frac{1}{v^t p_x} \left[P \ddot{a}_{\overline{\min(K_x+1, t)}|} - M v^{K_x+1} \cdot I(K_x < t) \right] \quad (7)$$

so that we write

$$\begin{aligned} E(L_t^R) &= \frac{1}{v^t p_x} \left\{ P E \left[\ddot{a}_{\min(K_x+1, t)} \right] - ME \left[v^{K_x+1} \cdot I(K_x < t) \right] \right\} \\ &= \frac{1}{{}_tE_x} \left(P \ddot{a}_{x:\overline{t}|} - MA_{x:\overline{t}|}^1 \right). \end{aligned}$$

According to the actuarial equivalence principle, we have $P \ddot{a}_{x:\overline{n}|} = MA_{x:\overline{n}|}^1$. It follows therefore that

$$\begin{aligned} E(L_t^R) &= \frac{1}{{}_tE_x} \left(P \ddot{a}_{x:\overline{t}|} - MA_{x:\overline{t}|}^1 - P \ddot{a}_{x:\overline{n}|} + MA_{x:\overline{n}|}^1 \right) \\ &= \frac{1}{{}_tE_x} \left[M \left(A_{x:\overline{n}|}^1 - A_{x:\overline{t}|}^1 \right) - P \left(\ddot{a}_{x:\overline{n}|} - \ddot{a}_{x:\overline{t}|} \right) \right] \\ &= MA_{x+t:\overline{n-t}|}^1 - P \ddot{a}_{x+t:\overline{n-t}|} = E(L_t^P). \end{aligned}$$

Notice that although the expectations are equal at any duration t , the probability distributions of the two random variables are not. Indeed at policy issue, that is, at $t = 0$, it is easy to see that $L_0^R = 0$ although

$$L_0^P = B v^{K_x+1} I(K_x < n) - P \ddot{a}_{\min(K_x+1, n)}$$

and is not necessarily always equal to zero. However, by the equivalence principle, it follows directly that $E(L_0^P) = 0$. Because at policy issue there should be no net assets accumulated, we easily see that $L_0^R = 0$. Indeed, this alone shows that the two random variables are different in distribution.

In contrast, we see that at policy maturity $t = n$, the prospective loss is $L_n^P = 0$ since there is no more future net liabilities. However, the retrospective accumulated net asset random variable at policy maturity is

$$L_n^R = \frac{1}{{}_nE_x} \left[P \ddot{a}_{\min(K_x+1, n)} - B v^{K_x+1} I(K_x < n) \right]$$

which also is not necessarily equal to zero although it has zero expectation again because of the equivalence principle.

2.2 Understanding Differences Between the Prospective Loss and the Retrospective Accumulated Net Asset

To further understand the difference between these two random variables, consider a fully discrete 25-year term insurance policy issued to age $x = 40$ and assume mortality follows the Gompertz law with

$$\mu_{40+t} = B \cdot c^{40+t}, \text{ for } t \geq 0,$$

where $B = 0.0000429$ and $c = 1.1070839$. We examine the differences between the prospective loss and retrospective accumulated net asset random variables at the end of year 10. For illustration purpose, we assume that the annual effective interest rate is 5% and the death benefit, payable at the end of the year of death, is \$100,000.

First, note that the prospective random variable is based on the future lifetime of the policyholder from duration t . This refers to the loss that is conditional on survival of the policyholder at time t and we are looking at the difference between the present value of future benefits yet to be paid and future premiums yet to be collected. In contrast, the retrospective accumulated net asset random variable is based on the future lifetime of the policyholder from issue and this is because we must look back at what happened to the difference in the accumulation of premiums and benefits paid in the past prior to duration t . This explains why, as earlier stated, the prospective loss random variable can be viewed as the share per survivor of the present value of net liabilities per \$1 of insurance at duration t while the retrospective accumulated net asset random variable as the share per survivor of the accumulated net assets per \$1 of insurance at duration t .

We can further visualize this difference with the help of Fig. 1 where we compare the realized prospective loss and retrospective accumulated net asset at duration t given the policyholder dies at a point in time. For the prospective loss, because the random variable is conditional on survival at time t , we consider death at each year after reaching age $x + t$. For the retrospective accumulated net asset random variable, we consider death at each year after issue age x but up to age $x + t$. Despite this difference in the future lifetime random variables, we see that earlier deaths for the prospective case generates larger positive net liabilities than later deaths

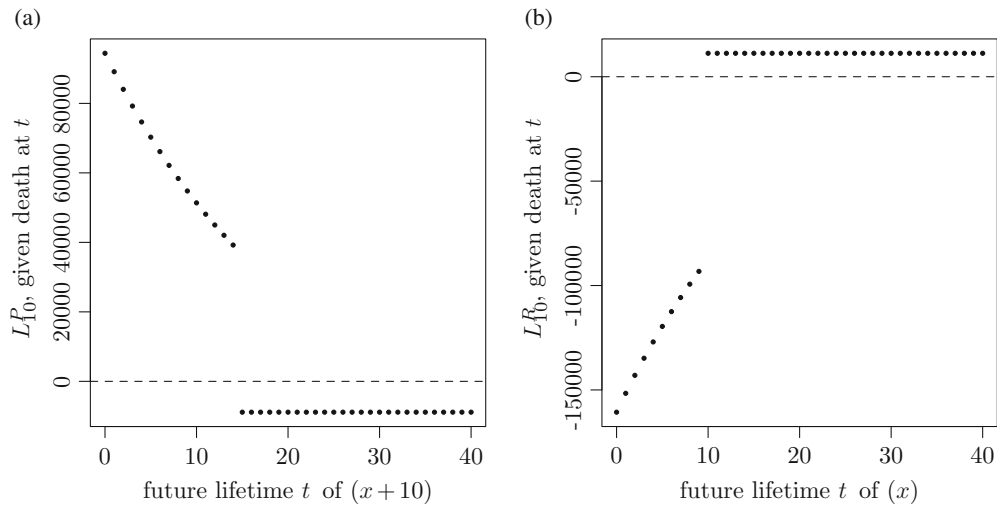


Fig. 1 Comparison of realized prospective loss and retrospective accumulated net asset at duration 10. (a) Prospective. (b) Retrospective

and this pattern is quite apparent in our example. For the retrospective case, earlier deaths generate fewer accumulated assets than later deaths. This can be intuitively explained by the fact that for early deaths, collected premiums will be fewer and that the death benefit is accumulated for a longer period from death to the duration in consideration; in this case, the duration is 10 years.

It is also interesting to note that for the prospective case, the random variable is constant after the term of the policy. This is because the prospective loss will have simply consisted of the present value, at duration 10 years, of future premiums collected up to the term of the policy since the death benefit portion will have always been zero. In contrast for the retrospective case, the random variable is constant for deaths after duration 10. This is because the retrospective accumulated net asset will have simply consisted of the share of the survivors of the accumulated value, at duration 10, of all premiums collected from issue till duration 10.

Finally, it is well worth examining the comparison between the shape of the distributions between the prospective loss and retrospective accumulated net asset. In Fig. 2, using the same set of assumptions to develop Fig. 1 and the Monte Carlo simulation, we compare the histograms between these two loss random variables. Observe the noticeably high proportion of a negative net liability in the prospective case and the noticeably high proportion of a positive net asset accumulation in the retrospective case. In the prospective case, this negative net liability is attributable to those survivors by the end of the policy term and beyond. In the retrospective case, this positive net asset accumulation is attributable to those survivors at duration 10 and beyond.

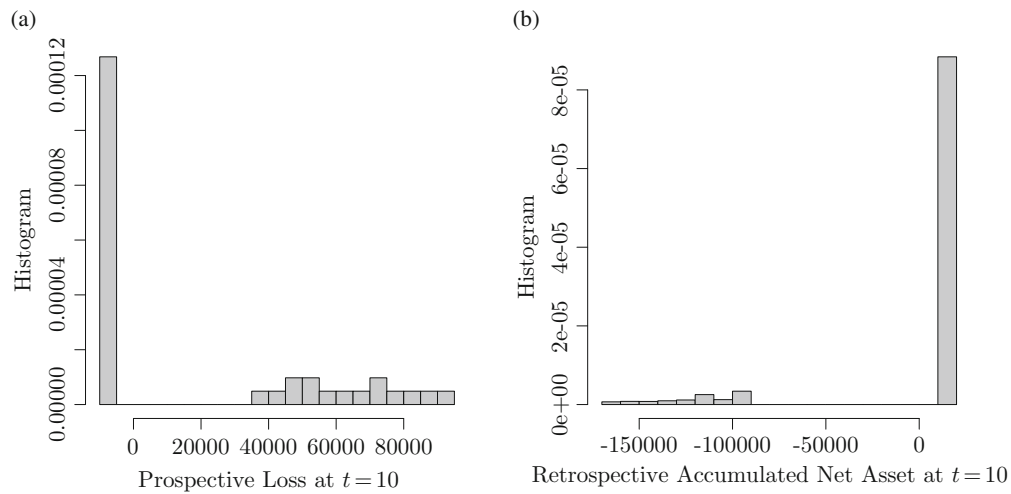


Fig. 2 Distribution of prospective loss and retrospective accumulated net asset at duration 10. (a) Prospective. (b) Retrospective

2.3 Numerical Illustration

To even further understand the retrospective accumulated net asset random variable, we consider a numerical illustration. For this purpose, we consider a fully discrete 20-year term insurance policy issued to age $x = 45$ with a death benefit of $M = \$1000$. For mortality assumption, we consider a table widely used in the industry for valuation purposes: the 2015 VBT Unismoke Age Nearest Birthday (ANB) mortality table. With interest rate equal to $i = 5\%$, we find that, using the equivalence principle, the net annual premium $P = 2.58$ per \$1000 of insurance.

Table 1 below shows the distribution of the retrospective accumulated net asset random variable at time $t = 10$ for the 11 possible realizations of the retrospective accumulated net asset random variable, L_{10}^R , for durations 1, 2, . . . 10, 11 and later. According to this calculation, we find that

$$E[L_{10}^R] = 17.19 \text{ and } SD[L_{10}^R] = 145.42$$

per \$1000 of insurance.

Table 2 shows the mean and standard deviation of both the retrospective and prospective loss random variables per \$1000 of insurance for the durations $t = 1, 2, \dots, 20$. Since the prospective loss random variable is a well-known random variable in the actuarial literature, we will assume the reader is familiar with its distribution for the simple insurance example we have illustrated. This table also demonstrates that for a given duration t , we can see that the expectations of the prospective loss and retrospective accumulated net asset are equal. However, the standard deviations for the same duration are not necessarily the same. In general, the standard deviation of the retrospective accumulated net asset random variable is smaller than the standard deviation of the prospective loss random variable in the early durations but it reverses in the later durations. Also, the standard deviation of the retrospective accumulated net asset random variable steadily increases as

Table 1 Distribution of the retrospective accumulated net asset random variable per \$1000 at duration 10, where $x = 45$, $n = 20$, $i = 5\%$, and gender = male

Duration t	Retrospective accumulated net asset	
	L_t^R	Probability
0	-1,569.77	0.0005
1	-1,490.75	0.0007
2	-1,415.49	0.0009
3	-1,343.82	0.0011
4	-1,275.56	0.0013
5	-1,210.55	0.0015
6	-1,148.63	0.0017
7	-1,089.66	0.0020
8	-1,033.51	0.0022
9	-980.02	0.0025
≥ 10	34.62	0.9856

Table 2 Mean and standard deviation of retrospective accumulated net asset and prospective loss random variables per \$1000, where $x = 45$, $n = 20$, $i = 5\%$, and gender = male

Duration t	Retrospective accumulated net asset RV		Prospective loss RV	
	Mean	Standard deviation	Mean	Standard deviation
1	2.24	21.68	2.24	138.35
2	4.37	34.96	4.37	142.91
3	6.39	47.71	6.39	147.07
4	8.31	60.38	8.31	150.90
5	10.16	73.05	10.16	154.51
6	11.91	86.09	11.91	157.81
7	13.51	99.79	13.51	160.69
8	14.94	114.21	14.94	163.08
9	16.18	129.36	16.18	164.95
10	17.19	145.42	17.19	166.14
11	17.92	162.43	17.92	166.53
12	18.30	180.59	18.30	165.85
13	18.26	200.01	18.26	163.81
14	17.76	220.68	17.76	160.18
15	16.71	242.77	16.71	154.42
16	14.95	266.51	14.95	145.65
17	12.45	291.92	12.45	132.81
18	9.18	318.95	9.18	114.15
19	5.06	347.68	5.06	85.00
20	0.00	378.27	0.00	0.00

duration increases, but this is not the case for the prospective loss random variable. Such pattern is to be expected as we have also demonstrated in our comparison in the previous section.

2.4 Extensions to Other Forms of Insurance

First, consider the case of a fully discrete whole life insurance policy. One can easily show the extension is straightforward because one can simply think of this as a term insurance with an infinite maturity. Premiums continue to be collected until death and policy expires at the end of the year of death of the policyholder.

In this case, we can express the retrospective accumulated net asset random variable in a similar fashion to Eq. (6). The only difference has to do with the value of the net annual premium. Using the equivalence principle, this leads us to

$$P \ddot{a}_x = MA_x \quad (8)$$

To demonstrate that the expectation of this retrospective accumulated net asset random variable is equal to that of the prospective loss random variable, we follow the same procedure as in the fully discrete term insurance.

$$\begin{aligned} E(L_t^R) &= \frac{1}{{}_tE_x} \left(P \ddot{a}_{x:\overline{t}|} - MA_{x:\overline{t}|}^1 - P \ddot{a}_x + MA_x \right) \\ &= MA_{x+t} - P \ddot{a}_{x+t} = E(L_t^P). \end{aligned}$$

In the case of a fully continuous whole life insurance, one can also easily develop the retrospective accumulated net asset random variable at duration t by defining it to be

$$L_t^R = \frac{1}{{}_t p_x} \left[\bar{P} (1+i)^t \bar{a}_{\overline{\min(T_x, t)}|} - M (1+i)^{t-T_x} \cdot I(T_x < t) \right] \quad (9)$$

where \bar{P} denotes the annual premium rate and T_x is the future lifetime of (x) . The corresponding prospective loss random variable in this case is defined to be

$$L_t^P = M v^{T_{x+t}} - \bar{P} \bar{a}_{\overline{T_{x+t}}|} \quad (10)$$

where T_{x+t} is the future lifetime of $(x+t)$.

Analogous to the development of the fully discrete, we have the retrospective reserve, equal to the expectation of the retrospective accumulated net asset random variable, for a fully continuous whole life as follows

$$E(L_t^R) = \bar{P} \frac{\bar{a}_{x:\overline{t}|}}{{}_tE_x} - M \frac{\bar{A}_{x:\overline{t}|}^1}{{}_tE_x}, \quad (11)$$

and the prospective reserve, equal to the expected value of the prospective loss random variable, is

$$E(L_t^P) = M \bar{A}_{x+t} - \bar{P} \bar{a}_{x+t}. \quad (12)$$

According to the actuarial equivalence principle, we have $\bar{P} \bar{a}_x = M \bar{A}_x$. Following similar proof as in the fully discrete case, it is straightforward to show the two expectations are equal.

To close this section, it is interesting to consider the case of an n year pure endowment policy where a benefit of 1 is payable at maturity if the policyholder, age x , survives then. Here we assume that premiums are payable annually at the rate of \bar{P} and are determined according to the actuarial equivalence principle so that we have

$$\bar{P} = \frac{{}_nE_x}{\bar{a}_{x:\overline{n}}|}.$$

In this case, we write the retrospective accumulated net asset random variable at time $t < n$ as

$$L_t^R = \bar{P} \frac{1}{{}_t p_x} \bar{a}_{\overline{t}|} (1+i)^t,$$

for $T_x < t$ and

$$L_t^R = \bar{P} \frac{1}{{}_t p_x} \bar{a}_{\overline{t}|} (1+i)^t,$$

for $T_x \geq t$.

As clearly interpreted in this paper, this random refers to the “the share per survivor of the accumulated net assets per \$1 of insurance at duration t ”. For those people who died before duration t , they would have paid total premiums up to their time of death. For those who have survived to duration t , they would have paid total premiums up to time t . In either case, no pure endowment benefit has yet been paid since $t < n$. Hence, the interpretation as stated. This same random variable can be succinctly written as

$$L_t^R = \bar{P} \frac{1}{{}_t E_x} \bar{a}_{\overline{\min(T_x, t)}|}. \quad (13)$$

3 Reserve Adjustment Based on the Retrospective Accumulated Net Asset Random Variable for a Portfolio

Consider a portfolio of m independent policies all issued with possible varying death benefit amounts and issue ages. Denote the benefit amount, typically called face amount in practice, for the i th policy by M_i and the aggregate retrospective accumulated net asset variable at duration t for this portfolio by $L_{\text{agg},t}^R$. It is not difficult to see that if $L_{i,t}^R$ is the retrospective accumulated net asset random variable per dollar of death benefit, then the i th policy retrospective accumulated net asset random variable can be expressed as $M_i \times L_{i,t}^R$ so that the aggregate retrospective accumulated net asset random variable for the portfolio can be expressed as

$$L_{\text{agg},t}^R = \sum_{i=1}^m M_i \times L_{i,t}^R$$

Dividing this by the total face amount of $\sum_{i=1}^m M_i$, we get the aggregate retrospective accumulated net asset per dollar of insurance:

$$L_{\text{agg},1,t}^R = \frac{L_{\text{agg},t}^R}{\sum_{i=1}^m M_i} = \sum_{i=1}^m \frac{M_i}{\sum_{i=1}^m M_i} \times L_{i,t}^R = \sum_{i=1}^m p_i \times L_{i,t}^R,$$

where

$$p_i = \frac{M_i}{\sum_{i=1}^m M_i} \text{ for } i = 1, 2 \dots m.$$

Assuming independent future lifetimes of all individual policyholders within the portfolio, then aggregate mean per dollar of insurance is

$$E(L_{\text{agg},1,t}^R) = \sum_{i=1}^m p_i \times E(L_{i,t}^R) \quad (14)$$

and aggregate variance per squared dollar of insurance is

$$\text{Var}(L_{\text{agg},1,t}^R) = \sum_{i=1}^m p_i^2 \times \text{Var}(L_{i,t}^R). \quad (15)$$

These results simply demonstrate that the mean and the standard deviation of the retrospective accumulated net asset random variable per dollar of insurance of any portfolio of policies that were issued in the same year, can be analytically determined from the mean and standard deviation of the retrospective accumulated net asset random variable per dollar of insurance of the individual policies. These results have been heavily applied in the illustration of our portfolio development and reserve adjustment in the subsequent subsections.

3.1 Interpretation of the Retrospective Accumulated Net Asset Random Variable

The retrospective accumulated net asset random variable can be best interpreted by modeling a portfolio of policies with the same issue age x . Assuming that the only decrement is death, then at duration t , there are two values that could be generated from the model:

- (a) accumulated net assets (i.e. accumulated premiums less accumulated death benefits) at $x + t$ based on the actual mortality experience of the portfolio in the first t durations, and
- (b) expected number of policies remaining in force in duration t .

Then the realized retrospective accumulated net asset random variable is the ratio of (a) to (b) above, and it represents the share per survivor of the realized net assets at duration t . The distribution of the retrospective accumulated net asset random variable can be obtained by generating all possible realizations of this ratio (a)/(b). It is apparent that this cannot be done analytically, but the distribution of the retrospective accumulated net asset random variable can be obtained via simulation.

Table 3 shows the mean, standard deviation and various quantiles of interest of the retrospective accumulated net asset random variable per \$1000 of face amount at various durations for a portfolio of 100 term insurance policies at each duration, issued at age 45 for face amount \$100,000. For this purpose, we generated mortality patterns according to the 2015 VBT Unismoke Age Nearest Birthday (ANB) mortality table. The quantiles we are showing in Table 4 are mean $\pm 0.1*SD$, mean $\pm 0.2*SD$, mean $\pm 0.5*SD$, mean $\pm SD$ and mean $\pm 3*SD$, where SD refers to the standard deviation.

Figure 3 provides an interesting visualization of how the mean and standard deviation of the retrospective accumulated net asset random variable emerge over a period of duration 20. A few observations can be made here. First, for a term insurance policy, the retrospective reserve starts small and follows a parabolic pattern. At maturity, the retrospective reserve is equal to zero. Finally, it is interesting to note that standard deviation increases with duration, thus the widening of the confidence band. This increase with duration can be explained by the fact that we become increasingly uncertain of the retrospective accumulated net asset for later durations. In this article, we suggest to use such confidence bands to make the necessary adjustment to prospective reserves. This increasing standard deviation over time implies that as we accumulate enough experience over time, enough information will become available to give us greater confidence of making the necessary adjustment.

Table 4 shows the same results for the prospective loss random variable per \$1000 of face amount by analyzing the future present value of net liabilities per policy at duration t based on 100 in force policies at duration t that were issued t years ago with all policies at issue age 45.

In comparing Tables 3 and 4, we can make the following inferences:

- The retrospective accumulated net asset random variable always satisfies the condition that

$$E(\text{retrospective accumulated net asset random variable}) = E(\text{prospective loss random variable})$$

- Since all policies have the same face amount, the retrospective (and prospective) reserve per \$1000 is equal to the reserve for a single \$1000 face amount policy. However, the standard deviation per \$1000 equals the corresponding SD for a single \$1000 face amount policy divided by the square root of the number of policies in the portfolio (i.e., 10 in this example). This conforms to our earlier results on how the mean and standard deviation of the retrospective accumulated

Table 4 Mean, standard deviation and quantiles of prospective loss random variable per \$1000 for a portfolio, where $x = 45$, $n = 20$, $M = \$100,000$, $i = 5\%$, gender = male, and number of policies = 100

Duration	Mean	SD	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean
			-0.1*SD	+0.1*SD	-0.2*SD	+0.2*SD	-0.5*SD	+0.5*SD	-0.5*SD	+0.5*SD	-SD	+SD	-3*SD	+3*SD					
1	2.24	13.83	0.86	3.63	-0.52	5.01	-4.67	9.16	-4.67	9.16	-11.59	16.08	-4.49	43.75					
2	4.37	14.29	2.94	5.80	1.51	7.23	-2.77	11.52	-2.77	11.52	-9.92	18.66	-8.74	47.24					
3	6.39	14.71	4.92	7.86	3.45	9.33	-0.97	13.74	-0.97	13.74	-8.32	21.09	-12.78	50.51					
4	8.31	15.09	6.80	9.82	5.29	11.33	0.76	15.85	0.76	15.85	-6.78	23.40	-16.62	53.58					
5	10.16	15.45	8.62	11.71	7.07	13.25	2.43	17.89	2.43	17.89	-5.29	25.61	-20.32	56.51					
6	11.91	15.78	10.33	13.49	8.75	15.06	4.02	19.80	4.02	19.80	-3.87	27.69	-23.82	59.25					
7	13.51	16.07	11.90	15.12	10.30	16.72	5.48	21.54	5.48	21.54	-2.56	29.58	-27.02	61.72					
8	14.94	16.31	13.31	16.57	11.68	18.20	6.78	23.09	6.78	23.09	-1.37	31.25	-29.87	63.86					
9	16.18	16.50	14.53	17.83	12.88	19.48	7.94	24.43	7.94	24.43	-0.31	32.68	-32.37	65.67					
10	17.19	16.61	15.53	18.85	13.87	20.51	8.88	25.50	8.88	25.50	0.57	33.80	-34.38	67.03					
11	17.92	16.65	16.26	19.59	14.59	21.25	9.60	26.25	9.60	26.25	1.27	34.58	-35.84	67.88					
12	18.30	16.58	16.64	19.96	14.98	21.62	10.01	26.59	10.01	26.59	1.72	34.89	-36.60	68.05					
13	18.26	16.38	16.62	19.90	14.98	21.53	10.07	26.45	10.07	26.45	1.88	34.64	-36.51	67.40					
14	17.76	16.02	16.16	19.36	14.55	20.96	9.75	25.77	9.75	25.77	1.74	33.77	-35.51	65.81					
15	16.71	15.44	15.16	18.25	13.62	19.79	8.99	24.43	8.99	24.43	1.26	32.15	-33.41	63.03					
16	14.95	14.56	13.50	16.41	12.04	17.87	7.67	22.24	7.67	22.24	0.39	29.52	-29.91	58.65					
17	12.45	13.28	11.12	13.78	9.79	15.11	5.81	19.09	5.81	19.09	-0.83	25.73	-24.90	52.29					
18	9.18	11.41	8.03	10.32	6.89	11.46	3.47	14.88	3.47	14.88	-2.24	20.59	-18.35	43.42					
19	5.06	8.50	4.21	5.91	3.36	6.76	0.81	9.31	0.81	9.31	-3.44	13.56	-10.13	30.56					
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00					

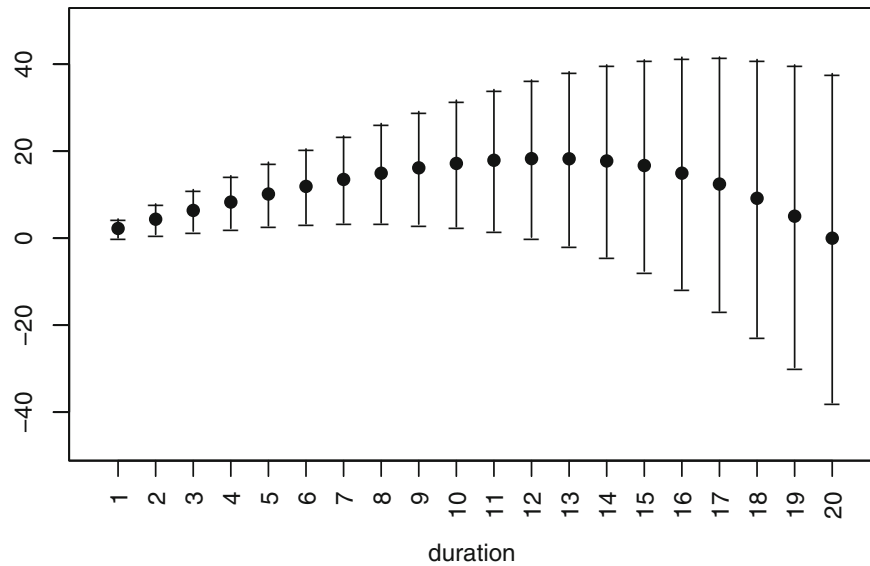


Fig. 3 Mean \pm one standard deviation of the retrospective accumulated net asset random variable

net asset random variable for a portfolio of policies may be conveniently calculated.

- There are variations in the standard deviations of the retrospective accumulated net asset and prospective loss random variables by duration.
- There are variations in the quantiles of the retrospective and prospective loss random variables by duration.

This leads us to the next couple of questions. Based on how we have defined the retrospective accumulated net asset random variable, what does it really mean from an insurance company's perspective? Furthermore, what can we learn from the volatility of the retrospective accumulated net asset random variable in setting the prospective reserves from an insurer's perspective.

3.2 Implications of the Retrospective Accumulated Net Asset Random Variable for Insurers

The retrospective reserve in the actuarial literature has been viewed as algebraically equivalent to the prospective reserve in expectation and a convenient alternative to determining policy reserves for certain product designs. By creating a retrospective accumulated net asset random variable, we hope to help increase the importance of the retrospective reserve as the mean of the distribution of the accumulated net assets per \$1000 of insurance. This is a useful random variable for insurers to analyze in evaluating historical claims experience and determining how to set, or reset, prospective reserves.

Specifically, if the realized retrospective accumulated net asset random variable lies outside some pre-established confidence band for the retrospective accumulated net asset random variable, then the prospective reserve could be adjusted to reflect the fact that actual historical experience is significantly different from reserving assumptions. This could become the regulatory basis for adjusting future reserves in accordance with Principles Based Reserving. This can also form the basis of a claims tracking and monitoring process for an insurer.

In the illustration that follows, we consider a portfolio of term life insurance policies. For purpose of setting up the mortality pattern, we consider the same valuation table we have previously used: the 2015 VBT Unismoke Age Nearest Birthday (ANB) mortality table.

In order for an insurance company to implement a process by which prospective reserves are adjusted for an in force block of policies in a systematic manner to reflect the realized retrospective accumulated net asset random variable, the following steps have to be done:

1. The in force block has to be broken down into issue year groupings and by plan of insurance.
2. For a given issue year and plan of insurance, the historical premiums and death claims paid have to be accumulated to the valuation date to determine the realized retrospective reserve per \$1000 of face amount.
3. The realized retrospective reserve determined in Step (2) above will have to be compared to the retrospective accumulated net asset random variable per \$1000 of face amount confidence band at a pre-established level of confidence (e.g., mean \pm SD). Note that both the mean and standard deviation of the confidence band vary by policy duration and we can use the result to determine the portfolio confidence band.
4. To recognize the fact that as duration from issue date to valuation date increases, the retrospective accumulated net asset random variable is based on more credible historical experience, the confidence bands could vary by duration. For example, the later durations (i.e., earlier issues) could use a tighter confidence band while earlier durations (i.e., later issues) could use a wider confidence band.
5. A possible (and certainly hypothetical) rule for adjusting the prospective reserves for this issue year block and plan of insurance could be as follows:
 - If the realized retrospective accumulated net asset random variable falls within the pre-established confidence band around the mean, then no adjustment is made to the prospective reserve.
 - If the realized retrospective accumulated net asset random variable exceeds the upper confidence band by \$1 per \$1000 of insurance, then the prospective reserve for the issue year block can be reduced by \$1 per \$1000 of insurance.
 - If the realized retrospective accumulated net asset random variable is below the lower confidence band by \$1 per \$1000 of insurance, then the prospective reserve for the issue year block should be increased by \$1 per \$1000 of insurance.

An area of further research that has not been explored in detail in this paper is developing a more systematic process of determining the width of the confidence interval (CI) by duration for the retrospective accumulated net asset random variable. One possible approach is to make some type of a credibility adjustment similar in concept to credibility concepts of adjusting expected claims based on past claims experience. There is certainly additional research work needed in this area. See, for example, a method used for variable annuity products in Longley-Cook et al. (2001). However, here we offer some possible approaches:

1. An overall consistency requirement is that the later the policy duration, the tighter the confidence interval has to be because of more credible historical experience.
2. Define the confidence interval width as $0.5 * (\text{upper CI} \pm \text{lower CI})$ and either:
 - keep the confidence width fixed for each duration which leads to tighter confidence intervals as duration increases since the standard deviation of the retrospective reserve increases by duration, or
 - linearly reduce the confidence width to zero from duration 1 to the end of the coverage period.
3. Any other reasonable method could be explored.

The following is an illustration of how the prospective reserves could be adjusted for a hypothetical in force block of 20-year, fully discrete term insurance policies issued over the past 10 years. For this hypothetical illustration, we assume the following:

1. For each issue year, 100 policies are issued and they are randomly issued over issue ages 35–55 and face amounts \$100,000–\$500,000.
2. Policy premiums are calculated based on the actuarial equivalence principle.
3. For durations 1–5 (i.e., more recent issues), actual historical mortality is assumed to be 25% lower than reserving assumptions.
4. For durations 6–10 (i.e., earlier issues), actual historical mortality is assumed to be 25% higher than reserving assumptions.
5. Prospective reserves are adjusted based on deviations of the realized retrospective accumulated net asset random variable from the confidence interval of the retrospective accumulated net asset random variable. The confidence interval is based on $0.10 * SD$ for policies in duration 10 at the valuation date, $0.20 * SD$ for policies in duration 9, etc. and $1 * SD$ for policies in duration 1 at the valuation date as illustrated in Table 5. Note that issue year 1 represents policies in duration 10, issue year 10 represents policies in duration 1, and so forth.
6. Assume the only decrement is mortality and that the prospective reserve is being calculated at end of duration 10.

Table 6 shows how the prospective reserve per \$1000 is adjusted by duration to reflect actual mortality experience based on our pre-established confidence interval methodology as illustrated in Table 5.

Table 5 Retrospective accumulated net asset random variable confidence band example

Duration	Issue year	Retrospective accum net asset RV		Retrospective accum net asset RV confidence band	
		Mean	SD	Lower bound	Upper bound
10	1	17.60	15.83	Mean - 0.1 * SD	Mean + 0.1 * SD
9	2	16.60	14.53	Mean - 0.2 * SD	Mean + 0.2 * SD
8	3	15.86	12.35	Mean - 0.3 * SD	Mean + 0.3 * SD
7	4	13.10	10.33	Mean - 0.4 * SD	Mean + 0.4 * SD
6	5	13.05	9.54	Mean - 0.5 * SD	Mean + 0.5 * SD
5	6	11.20	7.96	Mean - 0.6 * SD	Mean + 0.6 * SD
4	7	7.26	5.85	Mean - 0.7 * SD	Mean + 0.7 * SD
3	8	6.95	5.04	Mean - 0.8 * SD	Mean + 0.8 * SD
2	9	4.32	3.33	Mean - 0.9 * SD	Mean + 0.9 * SD
1	10	2.33	2.04	Mean - 1 * SD	Mean + 1 * SD

19.19

19.50

19.57

17.24

17.82

15.97

11.35

10.98

7.32

4.37

Table 6 Prospective reserve adjustment example

Duration	Issue year	Realized retro loss RV		Expected prosp loss		Adjusted prosp loss		Realized prosp loss	
		Mean	Deviation	Mean (reserve)		Mean (reserve)		Mean (reserve)	
10	1	13.30	-2.72	17.60		20.33		26.91	
9	2	13.09	-0.60	16.60		17.20		26.04	
8	3	13.00	Within interval	15.86		15.86		25.68	
7	4	11.07	Within interval	13.10		13.10		22.10	
6	5	11.31	Within interval	13.05		13.05		23.05	
5	6	12.44	Within interval	11.20		11.20		1.12	
4	7	7.94	Within interval	7.26		7.26		-0.65	
3	8	7.47	Within interval	6.95		6.95		-2.54	
2	9	4.55	Within interval	4.32		4.32		-4.01	
1	10	2.42	Within interval	2.33		2.33		-6.13	
	Aggregate	9.66		10.80		11.11		11.04	

Based on these tables, we make the following observations:

- The realized retrospective accumulated net asset random variable per \$1000 of insurance is simply the mean of the retrospective accumulated net asset random variable and modifying the annual mortality based on the actual historical mortality assumptions (3) and (4) above.
- The realized retrospective accumulated net asset random variable is then compared to the theoretical mean and standard deviation of the retrospective accumulated net asset random variable based on the original reserving assumptions to determine the adjustment to the prospective reserves per \$1000.

We can additionally make the following observations. First, since the standard deviation of the retrospective accumulated net asset random variable varies by duration, the impact of actual mortality experience varying from reserving assumptions has to be analyzed by issue year. Second, the overall realized prospective reserve is \$11.04 per \$1000 of face amount. This represents the mean of the prospective loss random variable using the actual mortality assumptions of 25% lower mortality for more recent issues in durations 1–5 and 25% higher mortality for earlier issues in durations 6–10. Third, the overall realized retrospective reserve is \$9.66. Based on our approach of varying confidence interval to adjusting the prospective reserves, the overall adjusted prospective reserve per \$1000 of insurance is \$11.11, while the overall expected prospective reserve is \$10.80. This represents an overall increase in prospective reserves of 30 cents for every \$1000 of insurance. Finally, as shown in Table 7, for the in force block in year 10 after annual sales of 100 policies per year, there are approximately 993 remaining policies with an aggregate face amount of \$297,226,683. Then the adjusted prospective reserve results in an increase of \$93,471 in aggregate prospective reserves. This translates to a \$22,808 higher than the overall mean of the prospective loss random variable based on actual mortality experience (i.e., realized prospective reserve). This implies a slight degree of conservatism in our methodology for adjusting aggregate prospective reserves.

Table 7 Difference between the adjusted and expected prospective reserves

Remaining policies	993
Remaining policies face amount	297,226,683
Expected retrospective reserve	10.80
Expected prospective reserve	10.80
Adjusted prospective reserve	11.11
Realized prospective reserve	11.04
Expected aggregate prospective reserve	3,210,105
Adjusted aggregate prospective reserve	3,303,576
Realized aggregate prospective reserve	3,280,768
Per \$1000 difference between expected and realized prospective reserves	−0.24
Per \$1000 difference between adjusted and realized prospective reserves	0.08
Aggregate difference between expected and realized prospective reserves	(70,662)
Aggregate difference between adjusted and realized prospective reserves	22,808

4 Concluding Remarks

The implications of this paper are important for a few reasons:

1. This paper expands the actuarial literature on unlocking reserve assumptions based on the retrospective accumulated net asset random variable, a concept that is similar to the prospective loss random variable that is used to calculate reserves. Similar retrospective concept has appeared in Arias Lopez and Garrido (2001) and Ramsay (1993).
2. The retrospective accumulated net asset random variable as defined in this article has practical implications in developing a claims tracking and monitoring process for a company and in adjusting prospective reserves in a systematic manner that would satisfy Principle Based Reserving (PBR) standards. The PBR approach is being gradually adopted by the National Association of Insurance Commissioners (NAIC) for calculating more realistic reserves. See Mazyck (2013).
3. The methodology recommended in this article is timely because PBR regulation allows insurance companies to use their own experience to value life insurance reserves. The approach suggested here can also be viewed as a methodical way of tracking and monitoring insurance claims experience. See Vadiveloo et al. (2014).

The paper has focused on the retrospective accumulated net asset random variable for a term insurance product. Clearly, our findings can be extended to other insurance products like endowment insurance, whole life insurance, disability income, long term care, life annuities, and pension plan products. For disability income and long-term care, the retrospective accumulated net asset random variable provides historical information on how actual incidence and termination rates vary from expected and whether they are significant enough to adjust the prospective reserves for the business. For annuities and pension products, the retrospective accumulated net asset random variable provides insights into the longevity risk for these products and how prospective reserves may be adjusted to reflect actual longevity experience that is significantly deviating from expected.

With this paper, future students of mathematics of life contingencies may learn about the importance of a retrospective accumulated net asset random variable in assisting insurance companies provide information on historical claims experience and how prospective reserves may be adjusted to reflect this emerging actual experience. This may also help trigger their appreciation of the concept of the retrospective reserve, rather than simply mathematically demonstrating the equivalence between the retrospective and prospective reserves.

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