Insurance: Mathematics and Economics 76 (2017) 118–134

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Modeling partial Greeks of variable annuities with dependence

Guojun Gan *, Emiliano A. Valdez

Department of Mathematics, University of Connecticut, 341 Mansfield Road U-1009, Storrs, CT, 06269, USA

HIGHLIGHTS

- We study a computational problem arising from variable annuities.
- We propose copula models to study the dependence structure of partial dollar deltas.
- The dependence of partial dollar deltas is well captured by covariates.
- The findings provide guidance on modeling partial dollar deltas.

ARTICLE INFO

Article history:
Received September 2016
Received in revised form April 2017
Accepted 26 July 2017
Available online 2 August 2017

Keywords:
Variable annuity
Portfolio valuation
Metamodeling
Gamma distribution
Copula

ABSTRACT

Dynamic hedging used to mitigate the financial risks associated with large portfolios of variable annuities requires calculating partial dollar deltas on major market indices. Metamodeling approaches have been proposed in the past few years to address the computational issues related to the calculation of partial dollar deltas. In this paper, we investigate whether the additional complication of modeling the dependence between the partial dollar deltas improves the accuracy of the metamodeling approaches. We use several copulas to model the dependence structures of the partial dollar deltas and conduct numerical experiments to compare different metamodels. Despite the evidence of strong dependence in the estimated models, our numerical results show that modeling the dependence structures in the metamodels does not improve the accuracy of the estimations at the portfolio level. This is because the dependence between the partial dollar deltas is well captured by the covariates used in the marginal models. This finding suggests that we should focus more on marginal models than specifying the dependence structure explicitly.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction

A variable annuity (VA), as known in the US, is a long-term life insurance contract between an individual and an insurer. Variable annuities are referred to as unit-linked products and segregated funds in Europe and Canada, respectively. A typical variable annuity has two phases: the accumulation phase and the distribution phase. In the accumulation phase, policyholders accumulate assets by allocating the premiums to several subaccounts of the insurer's separate account. A subaccount is similar to a mutual fund. In the distribution phase, policyholders can withdraw money in a lump sum or as a series of payments over time. The amount of withdrawal depends on the performance of the investment options and the benefit riders on the variable annuity contract.

Common benefit riders on a variable annuity contract include the guaranteed minimum death benefit (GMDB), the guaranteed minimum accumulation benefit (GMAB), the guaranteed minimum income benefit (GMIB), and the guaranteed minimum withdrawal benefit (GMWB). A GMDB rider guarantees a specified lump sum upon the death of the policyholder regardless of the performance of the investment. The most basic form of a GMDB rider is the return of premiums. However, a GMDB rider can be made complicated by guaranteeing that the benefit increases at some specified interest rate or by resetting the benefit to the account value if the latter is higher. A GMAB rider guarantees a specified lump sum on a specific maturity date or anniversary regardless of the performance of the investment. A GMAB rider allows the policyholder to lock the gains by entering a new contract. A GMIB rider guarantees a stream of income for life, following a waiting period. A GMWB rider guarantees that the policyholder can withdraw a specified percentage of the benefits base for a specified number of years. If the account value is higher than the guaranteed benefit, the insurer does not use its own money to make guarantee payments. If the account value is less than the guaranteed benefit, the insurer has to use its own money in the general account to cover the shortfall. For more information about the market development and product

* Corresponding author.
E-mail addresses: Guojun.Gan@uconn.edu (G. Gan), emiliano.valdez@uconn.edu (E.A. Valdez).

http://dx.doi.org/10.1016/j.insmatheco.2017.07.006
0167-6687/© 2017 Elsevier B.V. All rights reserved.

features of variable annuities, readers are referred to The Geneva Association Report (2013) and Ledlie et al. (2008).

The financial risks associated with guarantees embedded in variable annuities cannot be adequately addressed by traditional actuarial approaches (Hardy, 2003). Hedging strategies are commonly used to mitigate some of these financial risks. However, most of the existing work focuses on the hedging of the guarantees embedded in one VA contract under some simplified assumptions. See, for example, Moller (1998), Windcliff et al. (2002), Kleinow and Willder (2007), Li and Szimayer (2011), Kolkiewicz and Liu (2012), Ng and Li (2013), Forsyth and Vetzal (2014) and Melnikov and Tong (2014), to just name a few. In practice, dynamic hedging is usually adopted by insurers and the hedging is done on the whole portfolio of VA contracts. Since the guarantees embedded in VA contracts sold by insurance companies are complex, insurers resort to Monte Carlo simulation to calculate the Greeks (i.e., sensitivities of the guarantee values on major market indices) required by dynamic hedging. However, Monte Carlo simulation is extremely time-consuming when applied to a large portfolio of VA contracts for the following reasons (Dardis, 2016): first, long-term projection is usually involved as VA contracts are long-term contracts; second, each contract has its peculiar characteristics and needs to be modeled appropriately. Being able to calculate the partial Greeks in a timely manner is essential to the success of a dynamic hedging program.

To address the computational problems associated with the hedging of large portfolios of VA contracts, metamodeling approaches (Barton, 1994; Friedman, 1996) have been proposed in the past few years. A metamodel, also known as a surrogate model, is a model of another model (Barton, 1994). Its name is originated from systems engineering. Metamodeling refers to the process of generating metamodels. A metamodeling approach consists of four major steps:

1. Select a small number of representative VA contracts.
2. Use Monte Carlo simulation to calculate the Greeks for the selected representative VA contracts.
3. Create a metamodel based on the representative VA contracts and their corresponding Greeks.
4. Use the metamodel to estimate the Greeks for the whole portfolio of VA contracts.

Here a metamodel can be thought of as a model that approximates the Monte Carlo simulation model. In general, metamodels are much simpler and more efficient than the Monte Carlo simulation model. Examples of metamodels include regression models (Gan, 2013; Gan and Lin, 2015; Gan, 2015a; Gan and Lin, in press; Gan and Valdez, in preparation) and neural network models (Hejazi and Jackson, 2016).

Gan (2013) applied a data clustering method, the k-prototype algorithm (Gan, 2011), to select representative VA contracts and used kriging as a metamodel form. Gan and Lin (2015) applied a metamodeling approach to speed up the valuation of VA portfolios under a stochastic-on-stochastic framework, which is typically used in quarterly financial reporting to reflect the effect of hedging. Gan (2015a) compared the k-prototype algorithm and Latin hypercube sampling for selecting representative VA contracts. Gan and Lin (in press) proposed a two-level metamodeling approach to estimate efficiently the Greeks of VA portfolios for dynamic hedging. Gan and Valdez (in preparation) proposed the GB2 (generalized beta of the second kind) regression model as a metamodel form to capture the positive skewness typically observed in the distribution of the fair market values of VA guarantees. Hejazi and Jackson (2016) proposed a neural network model as the metamodel form. Gan and Valdez (in preparation) compared several methods for selecting representative VA contracts and found that the TFCM (Truncated fuzzy c-means) algorithm, which is a data clustering algorithm that is efficient in dividing a large dataset into many clusters (Gan et al., in preparation), and conditional Latin hypercube sampling are the best methods in term of accuracy.

In practice, dynamic hedging of a VA portfolio requires calculating the partial deltas on major market indices such as large cap equity, small cap equity, international equity, and government bond. The metamodeling approaches mentioned above, however, do not consider the dependence between the partial deltas on different market indices. In this paper, we investigate whether modeling dependence of partial deltas can help improve the accuracy of the metamodeling approach. This paper is partly motivated by empirical evidences of dependence and comovements in market indices (Hu, 2006; Garcia and Tsafack, 2011). Since partial dollar deltas are sensitivities to market indices, we also expect similar comovements among these partial dollar deltas. To that end, we propose a metamodeling approach by using copulas to capture the inherent relationship between partial deltas.

The remaining part of this paper is organized as follows. In Section 2, we describe the VA dataset used to test the performance of various proposed models. In Section 3, we present the copula models in detail. In Section 4, we give some numerical results to demonstrate the copula models. Finally, we conclude the paper with some remarks in Section 5.

2. Data description and transformation

We used a Java program to generate 10,000 synthetic VA contracts. The parameters used to generate the synthetic contracts are given in Appendix A. Each synthetic VA contract is described by 50 variables. Some variables have identical values. For example, fees of an investment fund are identical for all contracts. The information of some variables is contained in other variables. For example, the rider fee information is captured by the product type. We exclude these variables when creating metamodels.

Table 1 shows some summary statistics of the selected explanatory variables. There are two categorical explanatory variables: `gender` and `prodType`. Table 1(a) shows the counts by category. From the table, we see that about 40% of the policyholders are female and each product type accounts for 20% of the contracts. Table 1(b) shows the summary statistics of the continuous explanatory variables with the original 10 investment funds. Table 1(c) shows the summary statistics of the account values of the five indices. We use the fund mapping given in Table 2 to convert the ten investment funds into the five indices. Instead of using the account values of the original ten investment funds, we use the account values of the five indices in our model. In particular, we use the following explanatory variables in our model: `gender`, `productType`, `gmdbAmt`, `gmwbAmt`, `gmmbAmt`, `AV1`, `AV2`, `AV3`, `AV4`, `AV5`, `age`, and `ttm`.

Table 3(a) shows the summary statistics of the partial dollar deltas on five market indices. Since the guarantee embedded in a VA contract is similar to a put option, the partial dollar deltas are negative. The partial dollar deltas are calculated by the finite difference method. For $h = 1, 2, \ldots, 5$, the partial dollar delta of a VA contract on the hth market index is calculated as

$$\mathrm{Delta}(h) = \frac{FMV(AV_1, \ldots, AV_{h-1}, 1.01AV_h, AV_{h+1}, \ldots, AV_5)}{0.02} - \frac{FMV(AV_1, \ldots, AV_{h-1}, 0.99AV_h, AV_{h+1}, \ldots, AV_5)}{0.02}, \quad (1)$$

where AV_h is the partial account value on the hth index and $FMV(\cdot)$ denotes the fair market value calculated by Monte Carlo simulation. The shock size we used is 1% of the partial account value. We used a Java program to calculate the partial dollar deltas of the VA portfolio (Gan, 2015b).

Fig. 1(a) shows the histograms of the partial dollar deltas. From the figure, we see that the distributions of partial dollar deltas are

Table 1
Description and summary of the explanatory variables.

(a) Categorical variables

Variable	Description	Count by category
gender	Gender of the policyholder	F:4071, M:5929
productType	Product type of the VA contract	DBRP:2028, DBRU:2018, MB:1959, WB:1991, WBSU:2004

Variable	Description	Min	Mean	Max
(b) Continuous variables with original funds				
gmdbAmt	GMDB amount	0	137 908.45	984 963.95
gmwbAmt	GMWB amount	0	8339.56	75 065.14
gmwbBalance	GMWB balance	0	102 582.22	1 072 359.14
gmmbAmt	GMMB amount	0	54 715.121	499 925.4
FundValue1	Account value of fund 1	0	38 570.78	871 681.9
FundValue2	Account value of fund 2	0	44 998.42	1 032 433.7
FundValue3	Account value of fund 3	0	27 638.71	802 550.9
FundValue4	Account value of fund 4	0	29 567.86	587 646.6
FundValue5	Account value of fund 5	0	29 498.07	575 576.9
FundValue6	Account value of fund 6	0	39 351.66	898 768.1
FundValue7	Account value of fund 7	0	30 986.59	631 404.1
FundValue8	Account value of fund 8	0	31 762.53	716 196.8
FundValue9	Account value of fund 9	0	27 028.39	765 476
FundValue10	Account value of fund 10	0	28 853.71	708 491.1
age	Age of the policyholder	35	49.46	64
ttm	Time to maturity in years	1	14.48	28
(c) Account values of the five indices				
AV1	Account value of index 1	0	99 327.07	871 681.9
AV2	Account value of index 2	0	74 618.34	1 032 433.7
AV3	Account value of index 3	0	67 822.61	802 550.9
AV4	Account value of index 4	0	51 219.86	587 646.6
AV5	Account value of index 5	0	35 268.81	575 576.9

Table 2
The fund mapping. Each row is a mapping from an investment fund to five indices.

Fund	US large	US small	Intl equity	Govt bond	Money market
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0
5	0	0	0	0	1
6	0.6	0.4	0	0	0
7	0.5	0	0.5	0	0
8	0.5	0	0	0.5	0
9	0	0.3	0.7	0	0
10	0.2	0.2	0.2	0.2	0.2

skewed to the left and have long tails. Fig. 1(b) shows the scatter plots of the partial dollar deltas. The relationships between the partial dollar deltas are not linear. Table 3(b) shows the summary statistics of the ten shocked fair market values that are used to calculate the partial dollar deltas. Fig. 2 shows the histograms of the shocked fair market values. The histograms show that the distributions of the shocked fair market values are all positively skewed.

Since the partial dollar deltas have long tails, we scale the partial dollar deltas to reduce the tails as follows. Let Δ_{ih} be the partial dollar delta of the ith VA policy on the hth index. Let w_i be the weight used to scale Δ_{ih} and the ith VA policy. We determine w_i such that the scaled values of the scalable variables of the ith VA policy match the average values of these scalable variables of the portfolio as close as possible. The scalable variables include gmdbAmt, gmwbAmt, gmmbAmt, AV1, AV2, AV3, AV4, and AV5. Other

Table 3
Summary statistics of partial dollar deltas and the shocked fair market values. The five market indices are large cap, small cap, international equity, government bond, and money market. The numbers in parentheses are negative numbers.

Variable	Description	Min	Mean	Max
(a) Partial dollar deltas on market indices				
Delta1	On large cap	(205,141.33)	(13,215.66)	0
Delta2	On small cap	(193,899.27)	(8,670.87)	0
Delta3	On international equity	(386,730.84)	(9,616.43)	0
Delta4	On government bond	(286,365.30)	(8,994.71)	0
Delta5	On money market	(412,226.54)	(7,068.12)	0
(b) Shocked fair market values				
fmv1u	Index 1 shocked up 1%	(85,564.39)	(713.04)	241,891.25
fmv1d	Index 1 shocked down 1%	(85,564.39)	(448.73)	241,891.25
fmv2u	Index 2 shocked up 1%	(85,564.39)	(668.02)	241,891.25
fmv2d	Index 2 shocked down 1%	(85,564.39)	(494.60)	241,891.25
fmv3u	Index 3 shocked up 1%	(85,564.39)	(677.51)	239,496.34
fmv3d	Index 3 shocked down 1%	(85,564.39)	(485.18)	245,778.70
fmv4u	Index 4 shocked up 1%	(85,564.39)	(671.29)	241,891.25
fmv4d	Index 4 shocked down 1%	(85,564.39)	(491.40)	241,891.25
fmv5u	Index 5 shocked up 1%	(88,735.65)	(652.26)	241,891.25
fmv5d	Index 5 shocked down 1%	(82,369.22)	(510.90)	241,891.25

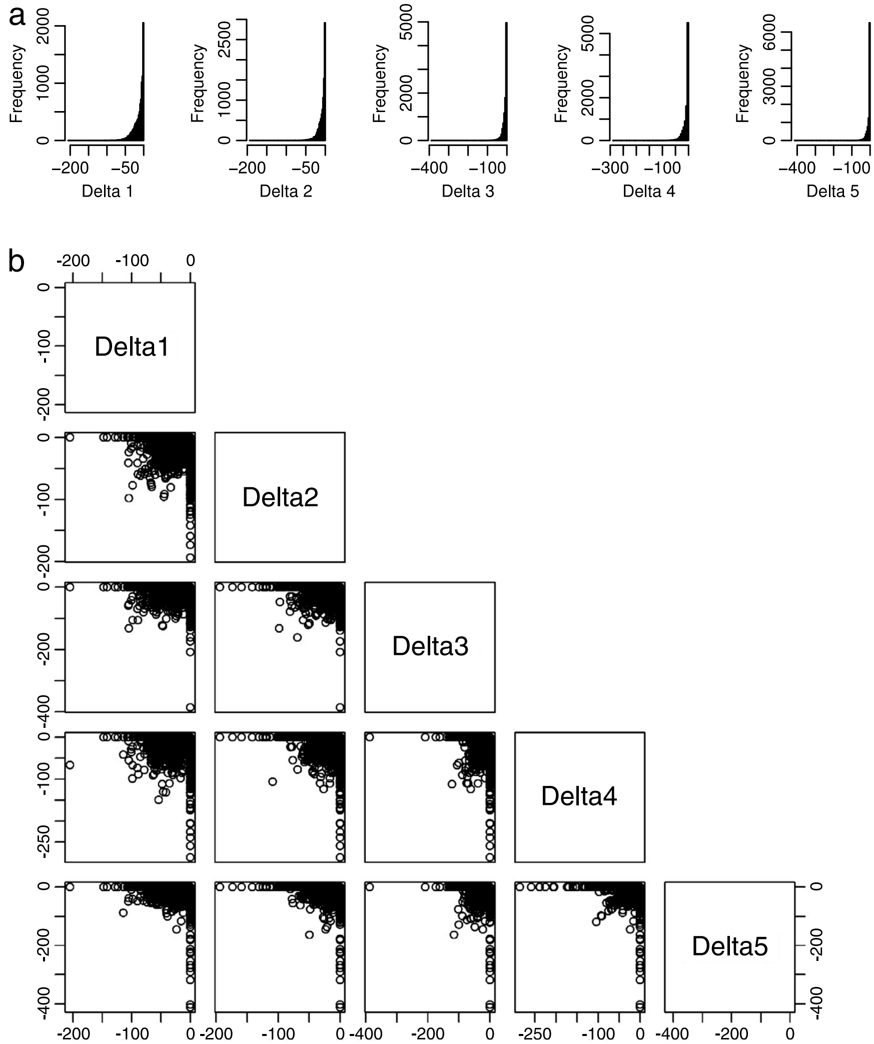

Fig. 1. Histograms and scatter plots of the partial dollar deltas. The numbers are in thousands.

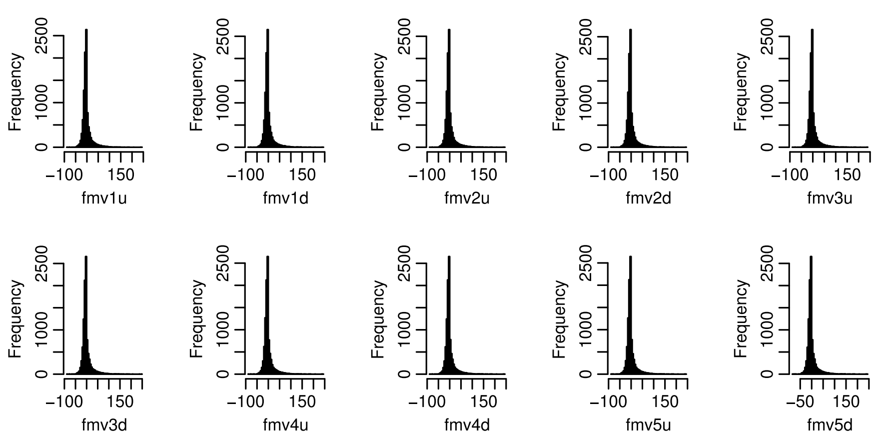

Fig. 2. Histograms of the shocked fair market values. The numbers are in thousands.

variables (e.g., `gender`, `productType`, `age`, and `ttm`) are not scalable. Mathematically, we determine w_i by minimizing the following objective function

$$\sum_{j=1}^{J}(w_i A_{ij} - \bar{A}_j)^2, \tag{2}$$

where J is the number of scalable variables, A_{ij} is the value of the jth scalable variable of the ith VA policy, and $\bar{A}_j$ is the average value of the jth scalable variable of the portfolio, i.e.,

$$\bar{A}_j = \frac{1}{n}\sum_{i=1}^{n} A_{ij}.$$

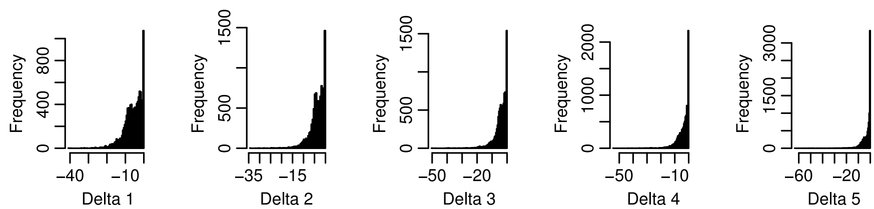

Fig. 3. Histograms of the scaled partial dollar deltas. The numbers are in thousands.

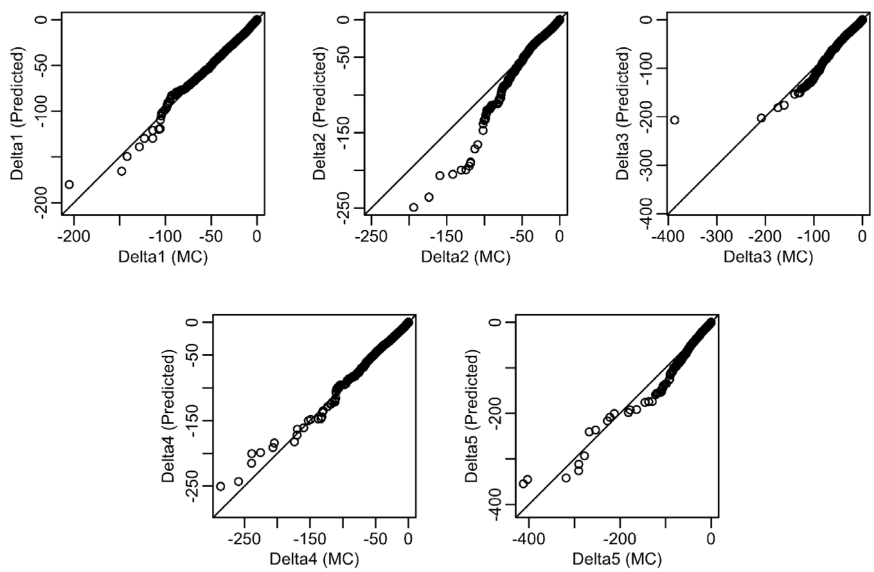

Fig. 4. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the marginal models when $s = 160$.

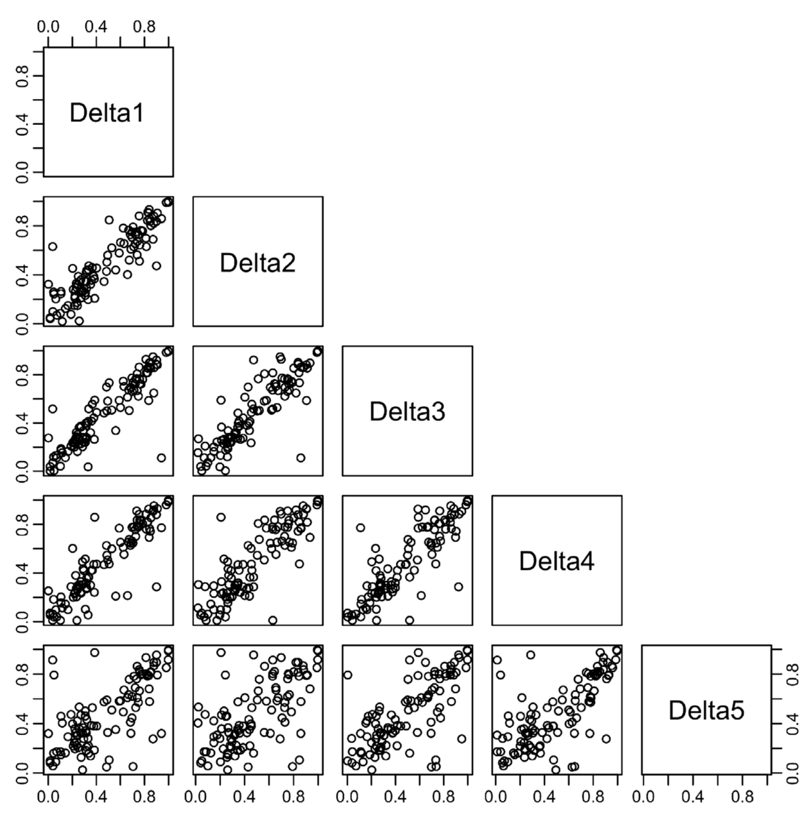

Fig. 5. The scatter plots of uniform marginals when $s = 160$.

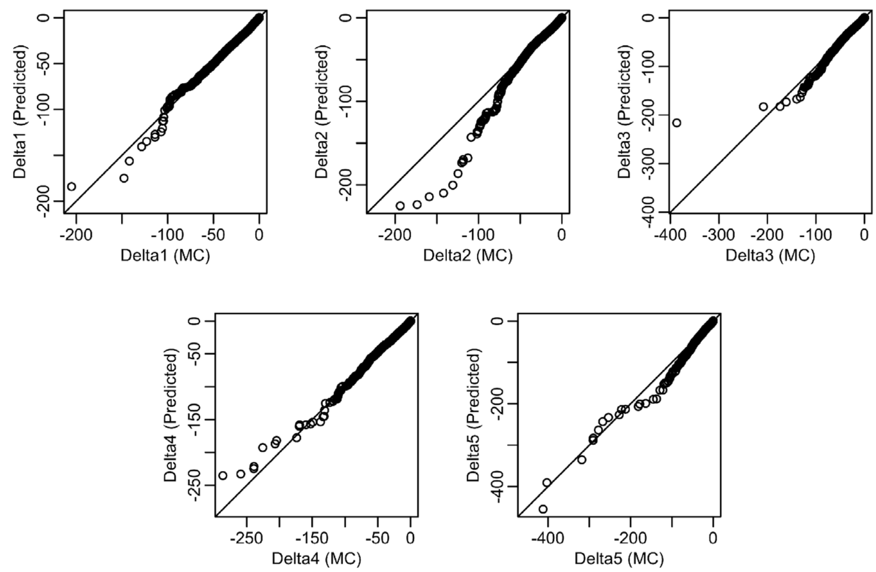

Fig. 6. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the independence copula based model when $s = 160$.

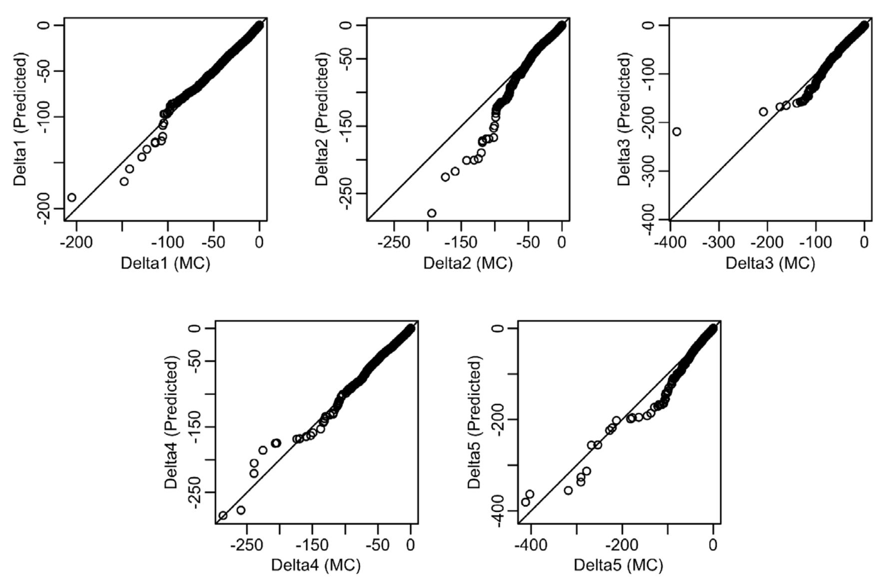

Fig. 7. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the Gaussian copula based model when $s = 160$.

The optimization problem in Eq. (2) can be solved analytically. The weight w_i is given by

$$w_i = \frac{\sum_{j=1}^{J} A_{ij}\bar{A}_j}{\sum_{j=1}^{J} A_{ij}^2}. \tag{3}$$

Once we determine the weights $w_1, w_2, \ldots, w_n$, we scale the partial dollar deltas and the VA policies together as follows:

$$S_{ih} = w_i \Delta_{ih} \tag{4}$$

and

$$B_{ij} = w_i A_{ij}, \quad j = 1, 2, \ldots, J. \tag{5}$$

Fig. 3 shows the histograms of the scaled partial dollar deltas. Comparing Figs. 1(a) and 3, we see that the extreme values of the scaled partial dollar deltas are reduced significantly. However, some skewness is still present.

The rational behind the above scaling is that the partial dollar delta is approximately proportional to the values of the scalable variables. For example, if the values of the scalable variables double and everything else holds constant, then we expect the partial dollar deltas to also double. We use all scalable variables to determine the weights in order to obtain robust weights. If only one scalable variable is used, then the weights of some VA policies can be close to zero or huge.

Table 4
Number of zeros in partial dollar deltas.

	Delta1	Delta2	Delta3	Delta4	Delta5
Number of zeros	973	1219	1356	1862	2710
Percent of zeros	9.73%	12.19%	13.56%	18.62%	27.10%

Once we have the scaled account values of the five indices, we normalize all numerical explanatory variables (i.e., `gmdbAmt`, `gmwbAmt`, `gmmbAmt`, `AV1`, `AV2`, `AV3`, `AV4`, `AV5`, `age`, and `ttm`) to the interval [0, 1] and convert the categorical variables to dummy binary variables. Then we represent the covariates of the ith VA policy in the portfolio by a $(d + 1)$-dimensional numerical vector

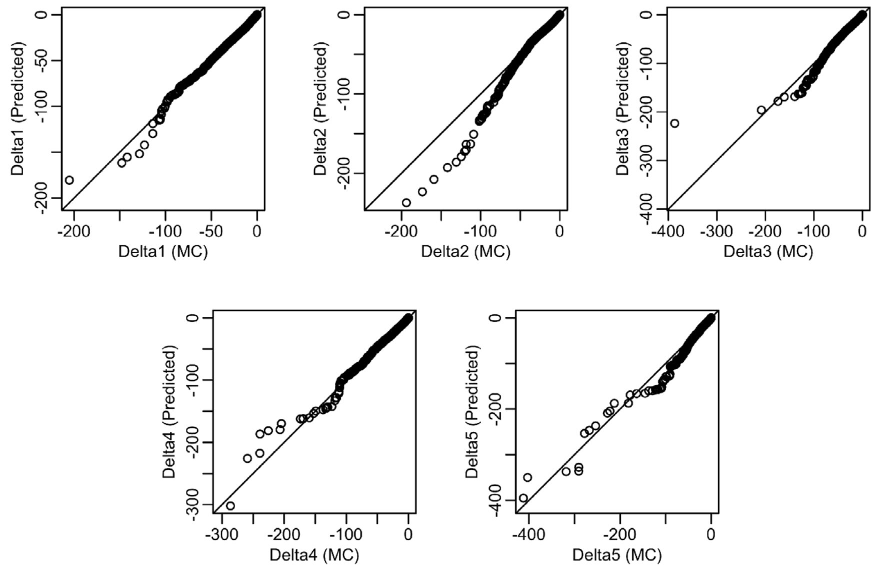

Fig. 8. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the t copula based model when $s = 160$.

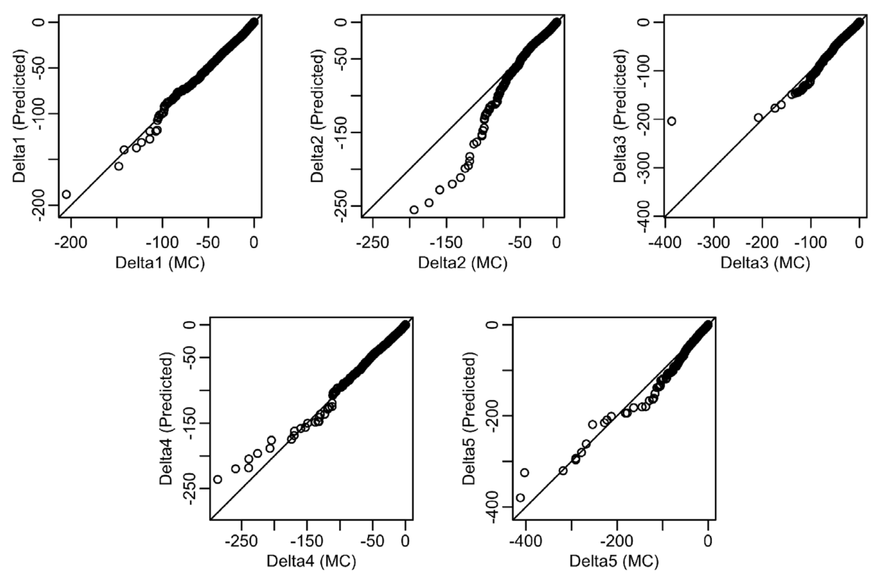

Fig. 9. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the Gumbel copula based model when $s = 160$.

$\mathbf{x}_i = (1, x_{i1}, x_{i2}, \ldots, x_{id})'$, where $d = 16$ is the number of explanatory variables including the binary dummy variables. We include 1 in the vector to allow an intercept term in the regression model.

3. Specification of the copula model

In this section, we describe the marginal models and the copula models in detail.

3.1. Marginal models

If the funds invested by a policyholder are not related to an index, then the policy's partial dollar delta on the index is zero. In this case, the policy has zero sensitivity on the index. Table 4 shows the proportion of zeros in the partial dollar deltas for the portfolio. From the table, we see that the partial dollar deltas on the 5th index contain 27.1% zeros. We ignore the zeros when fitting marginal models. In particular, we use gamma distributions to model the scaled negative partial dollar deltas because gamma distributions are medium-tailed distributions.

To describe the gamma regression model, we let Y_h denote the negative partial dollar delta on the hth index for $h = 1, 2, \ldots, 5$. Since the partial dollar deltas are non-positive, we have $Y_h \geq 0$ for $h = 1, 2, \ldots, 5$. We define the probability density function for positive Y_h as

$$f(x; k_h, \theta_h) = \frac{x^{k_h - 1} \exp\left(-\frac{x}{\theta_h}\right)}{\theta^{k_h} \Gamma(k_h)}, \tag{6}$$

where k_h and θ_h are parameters. The corresponding cumulative density function is defined as

$$F_h(x) = F(x; k_h, \theta_h) = \frac{\gamma\left(k_h, \frac{x}{\theta_h}\right)}{\Gamma(k_h)}, \tag{7}$$

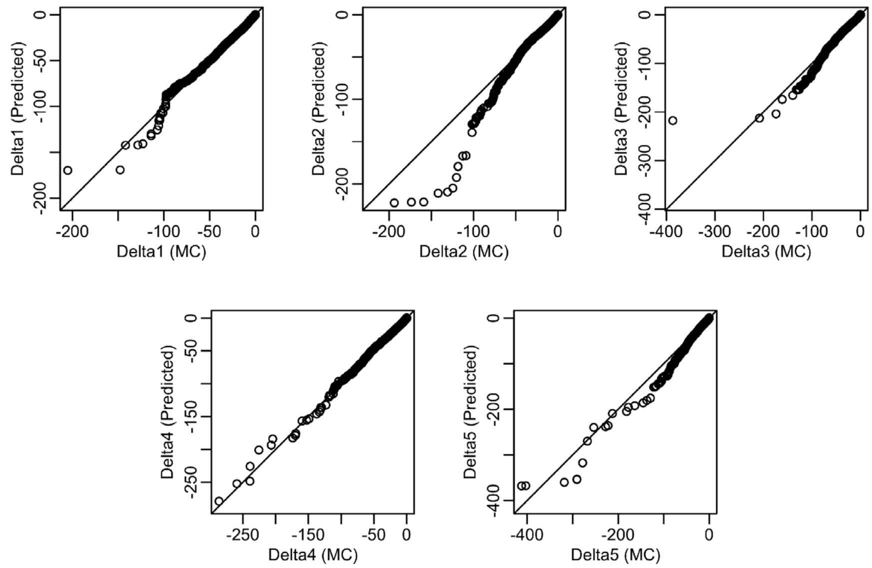

Fig. 10. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the Clayton copula based model when $s = 160$.

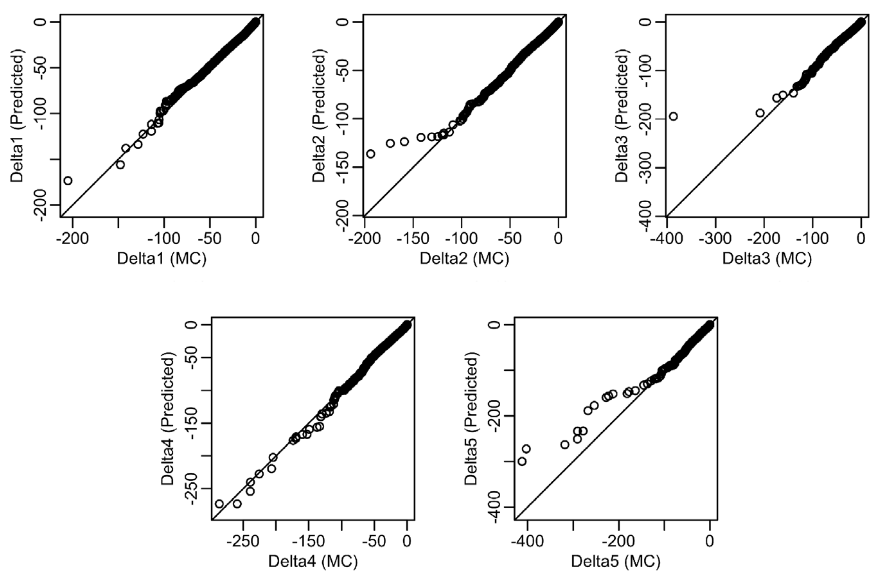

Fig. 11. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the marginal models when $s = 320$.

where $\gamma(\cdot, \cdot)$ is the lower incomplete gamma function defined by

$$\gamma(a, b) = \int_0^b t^{a-1} e^{-t} \, \mathrm{d}\, t. \tag{8}$$

Since the partial dollar deltas contain many zeros (see Table 4), we fit the gamma regression model to positive data only. To fit the gamma model to the positive data, we use the following parameterization:

$$k_h \theta_{ih} = \exp\left(\mathbf{z}_i' \boldsymbol{\beta}_h\right), \tag{9}$$

where $\mathbf{z}_i$ is the numerical vector representing the ith representative VA policy, and k_h and $\boldsymbol{\beta}_h$ are parameters to be estimated.

We use the expectation to predict an arbitrary's VA policy's partial dollar delta on the hth index when the policy has a positive account value on the hth index. For an arbitrary VA policy $\mathbf{x}_i$ in the portfolio $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, we predict its partial dollar delta on the hth index as follows:

$$\widehat{y}_{ih} = \begin{cases} 0, & \text{if } AV_{ih} = 0, \\ -\dfrac{\exp(\mathbf{x}_i' \widehat{\boldsymbol{\beta}}_h)}{w_i}, & \text{if } AV_{ih} > 0, \end{cases} \tag{10}$$

where AV_{ih} is the account value of the ith VA policy in the portfolio on the hth index and $\widehat{\boldsymbol{\beta}}_h$ is the vector of estimated parameters. In the above equation, we divide the expectation by the weight w_i to convert back the dollar delta to the original scale. The weights of all VA policies in the portfolio are determined in Eq. (3). Note that the weights depend only on the explanatory variables.

3.2. Copula models

A copula is a multivariate probability distribution where the marginal probability distribution of each variable is uniform (Nelsen, 2006). Copulas are flexible in that a different family can be used for each marginal distribution. Due to their flexibilities,

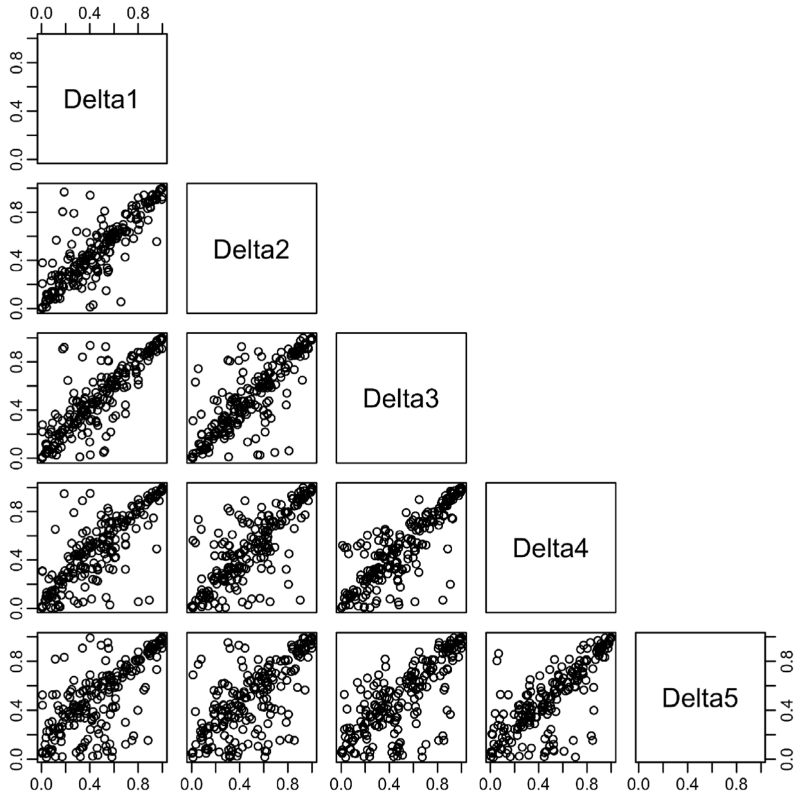

Fig. 12. The scatter plots of the uniform marginals when $s = 320$.

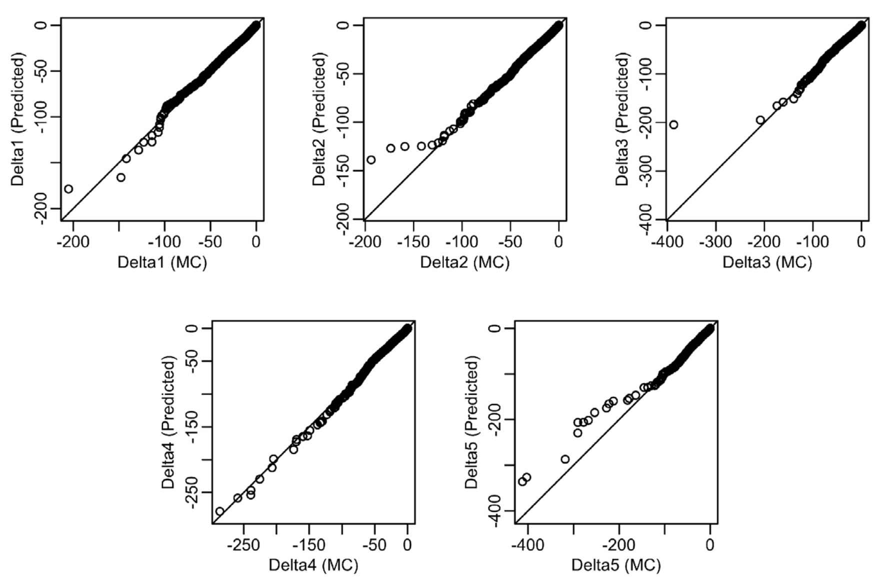

Fig. 13. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the independence copula based model when $s = 320$.

copulas are commonly used to model the dependence between variables (Frees and Valdez, 1998).

To define a copula, let $U_1, U_2, \ldots U_p$ be p uniform random variables. A copula of the random vector $(U_1, U_2, \ldots, U_p)$ is defined as the joint cumulative distribution function:

$$C(u_1, u_2, \ldots, u_p) = P(U_1 \le u_1, U_2 \le u_2, \ldots, U_p \le u_p).$$

There are several different families of copulas. In this paper, we consider the following five copulas to model the dependence of partial dollar deltas: the independence copula (i.e., the product copula), the Gaussian copula, the t copula, the Gumbel copula, and the Clayton copula. These copulas are described below.

The independence (or product) copula has the form

$$C(u_1, u_2, \ldots, u_p) = u_1 \times u_2 \times \cdots \times u_p$$

and is the copula that is clearly associated with independent random variables.

The copula generated by a multivariate Gaussian distribution with covariance matrix Σ is given by

$$C(u_1, u_2, \ldots, u_p) = H\left(\Phi^{-1}(u_1), \Phi^{-1}t(u_2), \ldots, \Phi^{-1}(u_p)\right)$$

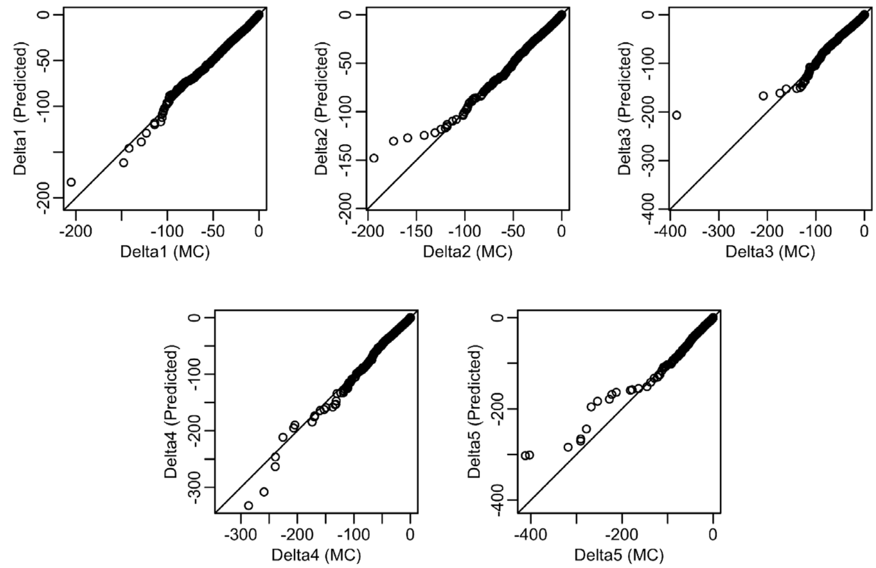

Fig. 14. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the Gaussian copula based model when $s = 320$.

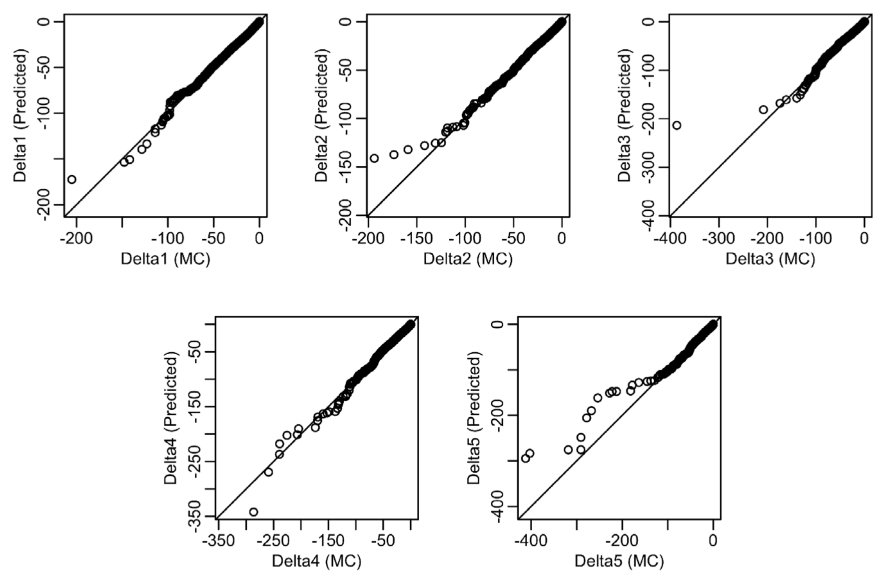

Fig. 15. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the t copula based model when $s = 320$.

where H is the joint distribution function of a standard multivariate normal random vector expressed as

$$H(x_1, x_2, \ldots, x_p) = \int_{-\infty}^{x_p}\int_{-\infty}^{x_{p-1}}\cdots\int_{-\infty}^{x_1}\frac{1}{\sqrt{(2\pi)^p|\Sigma|}} \times \exp\left(-\frac{1}{2}\mathbf{z}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{z}\right)d\mathbf{z}$$

and $\Phi^{-1}(\cdot)$ is the inverse of a standard normal distribution. It is critical to note that Gaussian copulas have zero tail dependence. A family of copulas that have non-zero tail dependence is the copula generated by a multivariate t-distribution expressed as

$$C(u_1, u_2, \ldots, u_p) = T_\nu\left(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_p)\right),$$

where ν refers to the degrees of freedom, Σ is the covariance matrix, T is the joint distribution function of a standard multivariate t random vector expressed as

$$T_\nu(x_1, \ldots, x_p) = \frac{\Gamma\left(\frac{\nu+p}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi|\Sigma|)^{1/2}} \times \int_{-\infty}^{x_p}\cdots\int_{-\infty}^{x_1}\left(1+\frac{1}{\nu}\mathbf{z}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{z}\right)^{-(\nu+p)/2}d\mathbf{z}$$

and $t_\nu^{-1}(\cdot)$ is the inverse of a standard t-distribution. See Cherubini et al. (2004).

Finally, two copulas that belong to a class of Archimedean copulas are the Gumbel and Clayton copulas. The Gumbel copula has the form defined by

$$C(u_1, u_2, \ldots, u_p) = \exp\left\{-\left[\sum_{k=1}^{p}(-\log u_k)^\alpha\right]^{1/\alpha}\right\},$$

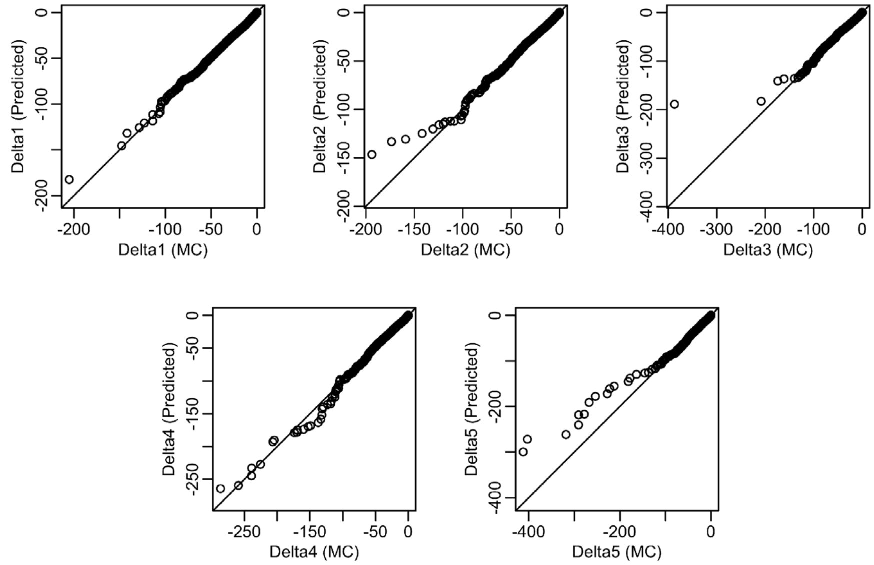

Fig. 16. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the Gumbel copula based model when $s = 320$.

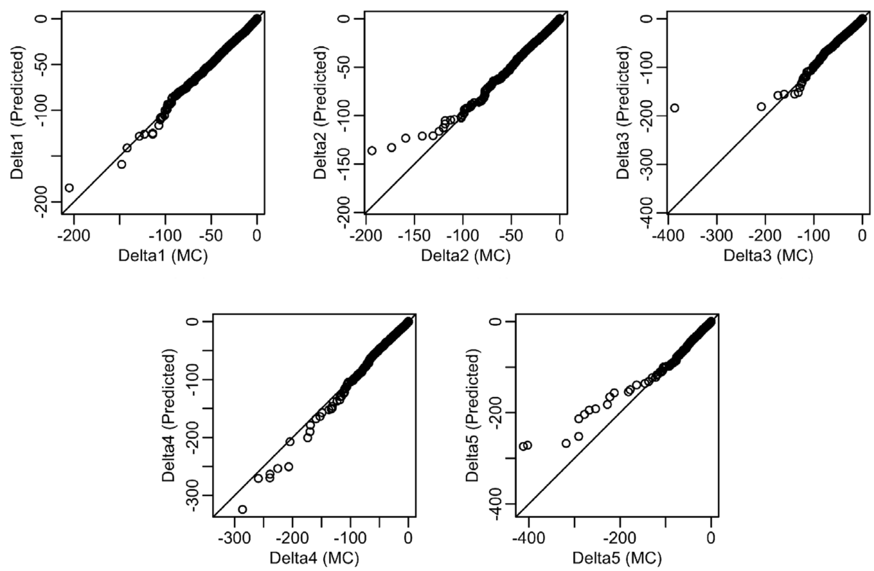

Fig. 17. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the Clayton copula based model when $s = 320$.

while the Clayton copula has the form defined by

$$C(u_1, u_2, \ldots, u_p) = \left[\sum_{k=1}^{p} u_k^{-\alpha} - p + 1\right]^{-1/\alpha}.$$

See Genest and Mackay (1986) for additional details.

Clearly, a copula is the function that links the univariate marginals to their full multivariate distribution. It provides for a flexible and realistic model of allowing for the dependence structure while at the same time, separating the effects of peculiar characteristics of the marginal distributions such as thickness of tails.

For our purpose, to model the dependence between the positive partial dollar deltas, we define copulas as follows:

$$C(u_1, u_2, \ldots, u_5) = P\left(Y_1 \le F_1^{-1}(u_1), Y_2 \le F_2^{-1}(u_2), \ldots, Y_5 \le F_5^{-1}(u_5)\right), \tag{11}$$

where F_h is the cumulative density function defined in Eq. (7). Once we select a particular copula, we can fit the copula to the positive subset of the data matrix $(u)_{s\times 5}$, where s is the number of representative VA policies and the (i, h)-entry is defined as

$$u_{ih} = F_h(v_{ih}). \tag{12}$$

In the above equation, v_{ih} denotes the negative partial dollar delta of the ith representative VA policy on the hth index.

Once a copula is fitted, we can use the copula to predict the partial dollar deltas of an arbitrary VA policy in the portfolio by using the Monte Carlo simulation method. Suppose that we want to use the copula to estimate the partial dollar delta of the ith VA policy $\mathbf{x}_i$ in the portfolio on the hth index, we use Monte Carlo simulation as follows:

1. Draw a sample of size N from the copula. Let the sample be denoted by the matrix $(R)_{N\times 5}$.

Table 5
Estimated dependence parameters of the fitted copulas when $s = 160$.

	Normal	t	Gumbel	Clayton
Estimated parameters	$\rho = 0.791$	$\rho = 0.876$, $df = 3.425$	$\alpha = 2.207$	$\alpha = 1.090$

Table 6
Cramer–von Mises test statistics of copula models when $s = 160$.

	Normal	t	Gumbel	Clayton
S_n	0.226	0.071	0.384	1.797

Table 7
Validation measures of different models when $s = 160$.

	Delta1	Delta2	Delta3	Delta4	Delta5
(a) Marginal models					
PE	0.038	0.074	0.042	0.022	0.143
MSE	60.161	56.484	58.735	46.271	76.771
CCC	0.857	0.815	0.849	0.888	0.847
(b) Independence copula					
PE	0.037	0.073	0.043	0.020	0.144
MSE	62.750	55.478	60.163	49.707	88.742
CCC	0.852	0.816	0.846	0.880	0.826
(c) Gaussian copula					
PE	0.037	0.074	0.041	0.022	0.145
MSE	64.229	58.435	60.814	50.711	91.070
CCC	0.849	0.810	0.844	0.879	0.824
(d) t copula					
PE	0.037	0.073	0.043	0.020	0.140
MSE	64.287	54.867	62.375	48.971	85.597
CCC	0.849	0.818	0.842	0.881	0.830
(e) Gumbel copula					
PE	0.037	0.072	0.039	0.019	0.134
MSE	62.813	59.283	60.056	49.028	81.043
CCC	0.852	0.807	0.845	0.881	0.836
(f) Clayton copula					
PE	0.037	0.072	0.040	0.020	0.145
MSE	62.911	56.952	61.434	51.492	85.773
CCC	0.852	0.813	0.844	0.877	0.834

Table 9
Cramer–von Mises test statistics of copula models when $s = 320$.

	Normal	t	Gumbel	Clayton
S_n	0.341	0.112	0.745	3.290

Table 10
Validation measures of different models when $s = 320$.

	Delta1	Delta2	Delta3	Delta4	Delta5
(a) Marginal models					
PE	0.015	0.012	−0.007	0.014	0.020
MSE	57.786	36.524	53.451	48.895	67.549
CCC	0.859	0.855	0.848	0.885	0.836
(b) Independence copula					
PE	0.014	0.011	−0.006	0.012	0.022
MSE	61.307	38.343	56.185	54.275	78.643
CCC	0.852	0.848	0.841	0.873	0.812
(c) Gaussian copula					
PE	0.015	0.011	−0.008	0.014	0.022
MSE	63.280	38.412	56.547	56.179	79.707
CCC	0.847	0.848	0.840	0.870	0.812
(d) t copula					
PE	0.015	0.012	−0.006	0.011	0.018
MSE	63.731	39.582	57.302	52.900	77.348
CCC	0.846	0.844	0.839	0.875	0.813
(e) Gumbel copula					
PE	0.015	0.008	−0.010	0.010	0.012
MSE	61.812	38.138	55.988	53.017	75.110
CCC	0.850	0.848	0.840	0.875	0.815
(f) Clayton copula					
PE	0.017	0.012	−0.006	0.018	0.020
MSE	63.406	39.321	54.903	55.547	70.778
CCC	0.848	0.844	0.844	0.873	0.828

2. Calculate a sample of Y_h by applying the inverse marginal cumulative density function, that is,

$$\hat{y}_{ih}^{(j)} = \begin{cases} 0, & \text{if } AV_{ih} = 0, \\ -\dfrac{1}{w_i} F_{ih}^{-1}(R_{jh}), & \text{if } AV_{ih} > 0, \end{cases} \quad j = 1, 2, \ldots, N, \tag{13}$$

where w_i is the weight defined in Eq. (3) and

$$F_{ih}^{-1}(q) = \gamma^{-1}\left(\widehat{k}_h, q\Gamma(\widehat{k}_h)\right) \frac{\exp\left(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_h\right)}{\widehat{k}_h}. \tag{14}$$

In the above equation, γ^{-1} is the inverse of the lower incomplete gamma function defined in Eq. (8), $\mathbf{x}_i$ is the numerical vector representing the ith VA policy, and $\widehat{k}_h$ and $\widehat{\boldsymbol{\beta}}_h$ are estimated parameters.
3. Estimate the partial dollar delta by the average:

$$\hat{y}_{ih} = \frac{1}{N} \sum_{j=1}^{N} \hat{y}_{ih}^{(j)}. \tag{15}$$

4. Numerical results

In this section, we present some numerical results of the copula approach. We also present the accuracy of the estimation by the marginal models only without explicitly specifying the dependence structure.

4.1. Validation measures

To compare the accuracy of different results, we use the following three validation measures: the percentage error at the portfolio level, the mean squared error (Frees, 2009), and the concordance correlation coefficient (Lin, 1989).

Let y_{ih} denote the partial dollar delta of the ith VA policy in the portfolio on the hth index that is calculated by the Monte Carlo simulation method described in Gan (2015b). Let $\hat{y}_{ih}$ denote the corresponding partial dollar delta estimated by a metamodel. Then the percentage error (PE) at the portfolio level is defined as

$$PE_h = \frac{\sum_{i=1}^{n}(\hat{y}_{ih} - y_{ih})}{\sum_{i=1}^{n} y_{ih}}, \tag{16}$$

Table 8
Estimated dependence parameters of the fitted copulas when $s = 320$.

	Normal	t	Gumbel	Clayton
Estimated parameters	$\rho = 0.783$	$\rho = 0.850$, $df = 3.340$	$\alpha = 2.038$	$\alpha = 0.979$

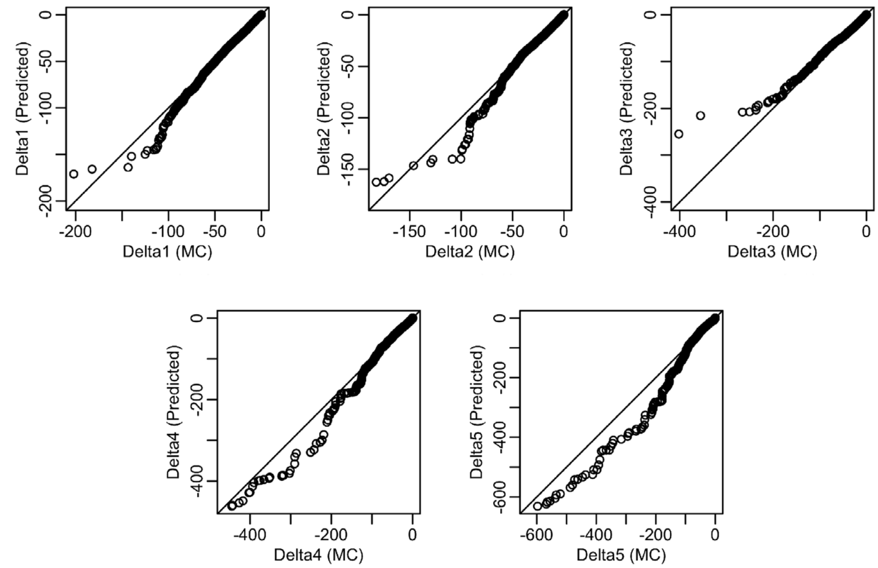

Fig. B.18. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the marginal models for the second portfolio when $s = 320$.

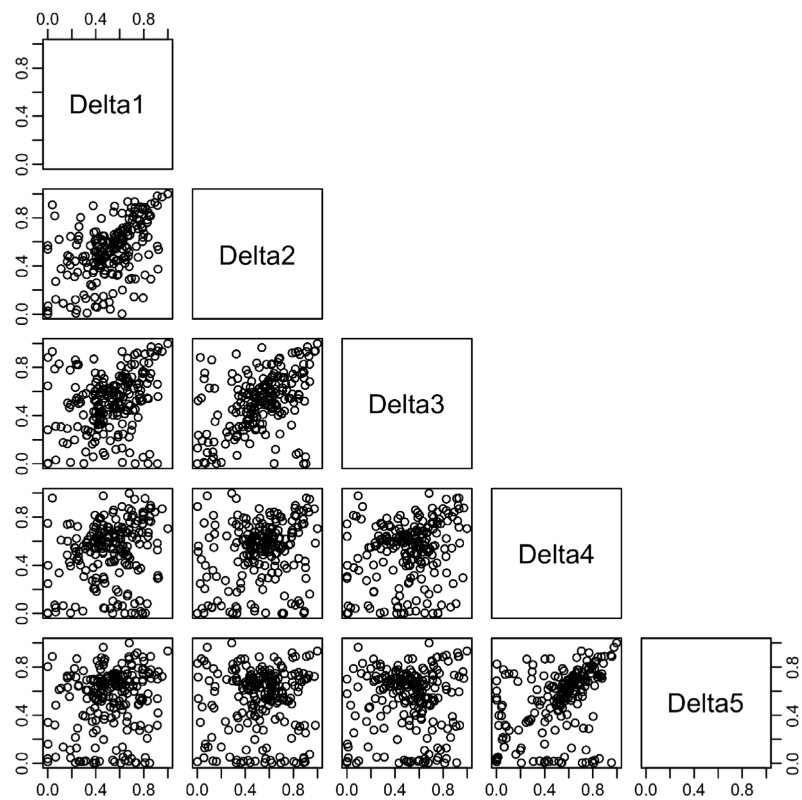

Fig. B.19. The scatter plots of the uniform marginals when $s = 320$.

where n is the number of VA policies in the portfolio. Between two results, the one producing a PE that is closer to zero is better.

The mean squared error (MSE) is calculated as

$$MSE_h = \frac{1}{n}\sum_{i=1}^{n}(\hat{y}_{ih} - y_{ih})^2. \tag{17}$$

Between two results, the one that produces a lower MSE is better.

The concordance correlation coefficient (CCC) is used to measure the agreement between two variables. It is defined as

$$CCC_h = \frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 + (\mu_1 - \mu_2)^2}, \tag{18}$$

where ρ is the correlation between $(y_{1h}, y_{2h}, \ldots, y_{nh})$ and $(\hat{y}_{1h}, \hat{y}_{2h}, \ldots, \hat{y}_{nh})$, σ_1 and μ_1 are the standard deviation and the mean of $(y_{1h}, y_{2h}, \ldots, y_{nh})$, respectively, and σ_1 and μ_1 are the standard deviation and the mean of $(\hat{y}_{1h}, \hat{y}_{2h}, \ldots, \hat{y}_{nh})$, respectively. Between two results, the model that produces a higher CCC

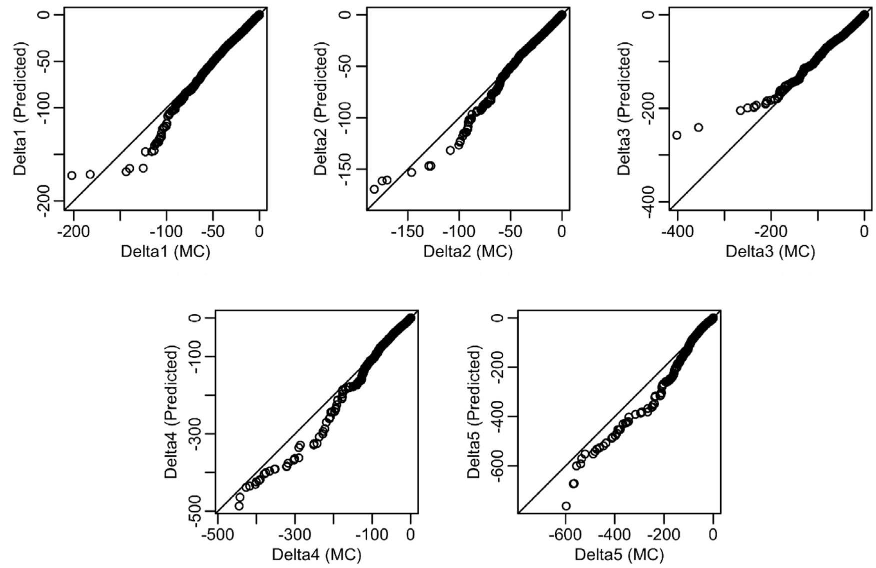

Fig. B.20. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the independence copula based model for the second portfolio when $s = 320$.

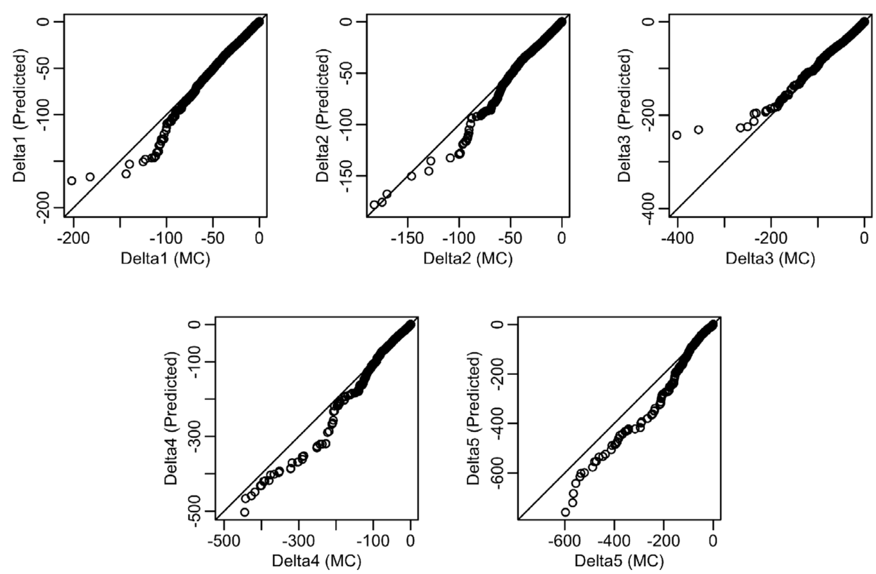

Fig. B.21. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the Gaussian copula based model for the second portfolio when $s = 320$.

is considered a better model. In particular, a value of 1 indicates perfect agreement between the two results.

4.2. Results

We use data clustering as the experimental design method. In particular, we use the TFCM algorithm (Gan et al., in preparation) to select representative VA policies from the portfolio. The TFCM algorithm is an efficient clustering algorithm for dividing a large dataset into many clusters. Regarding the number of representative VA policies, we follow the suggestion given in Loeppky et al. (2009) and start with 10 times the number of explanatory variables. Since we have 16 explanatory variables, we start with $s = 160$ representative VA policies. However, we also test the accuracy of the models with 320 representative VA policies.

We use the R function `glm` to fit the gamma model for the marginal distributions and functions from the R package `copula` to fit various copulas to the data. For the normal and t copulas, we use an exchangeable covariance structure. To calculate the lower incomplete gamma function and its inverse, we use the functions `Igamma` and `Igamma.inv` from the R package `zipfR`, respectively. To estimate the partial dollar deltas based on copula models, we generate samples of size $N = 10$ from the fitted copulas. Using a large N makes the estimation step slow. We found that setting $N = 10$ produces stable results.

Fig. 4 shows the QQ plots of the partial dollar deltas calculated by the Monte Carlo simulation model and those estimated by the marginal models without considering the dependence structure. We treat the partial dollar deltas calculated by Monte Carlo simulation as benchmarks for comparing different models. The QQ plots are based on the 10,000 partial dollar deltas of the whole portfolio. From Fig. 4, we see that gamma distributions fit the data well.

Fig. 5 shows the scatter plots of the positive subset of the matrix $(u)_{s\times 5}$ defined in Eq. (12). The scatter plots show that the uniform marginals have strong linear relationships.

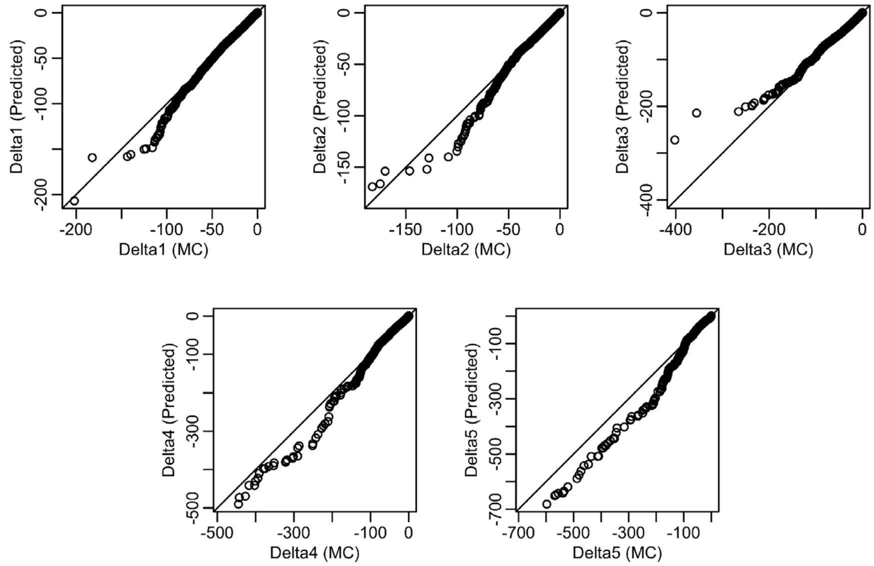

Fig. B.22. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the t copula based model for the second portfolio when $s = 320$.

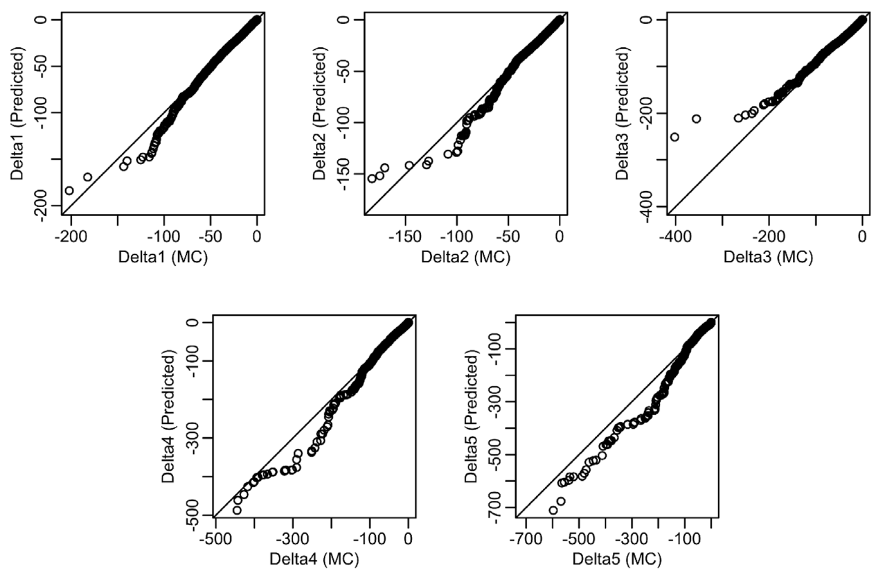

Fig. B.23. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the Gumbel copula based model for the second portfolio when $s = 320$.

Figs. 6, 7, 8, 9, and 10 shows the QQ plots of the partial dollar deltas calculated by the Monte Carlo simulation model and those estimated by the copula models for the whole portfolio of 10,000 VA policies. From these QQ plots, we see that the metamodels produce similar results.

Table 5 provides the estimated values of the corresponding dependence parameters for the different copulas considered in this paper where $s = 160$. According to these values, we see that there is strong dependence among the partial dollar deltas. To compare the goodness-of-fit of the various copulas, we use the Cramer–von Mises test statistics (Genest et al., 2009) as shown in Table 6. According to these results, the t copula provides the best fit among the four copulas. Since the independence copula does not have any parameters, the Cramer–von Mises test statistic is not available.

Table 7 shows the validation measures of the marginal models and the copula models. From the table, we see that all models produce similar results in terms of the percentage error. Incorporating dependence structure in the metamodels does not improve the accuracy of the estimation at the portfolio level. Looking at the MSE and the CCC values, we see that the estimations obtained by marginal models only are more accurate than those obtained by copula models. The results show that the dependence of the partial dollar deltas is captured by the covariates. Modeling the dependence structure explicitly does not improve the accuracy of the metamodels.

Fig. 11 shows the QQ plots of the partial dollar deltas calculate by the Monte Carlo simulation model and those estimated by the marginal models when the number of representative VA policies is set to 320. From the QQ plots, we wee that the gamma distributions fit the partial dollar deltas well.

Fig. 12 shows the scatter plots of the positive subset of the matrix $(u)_{s\times 5}$ defined in Eq. (12). The scatter plots show that the uniform marginals have strong linear relationships.

Fig. 13, 14, 15, 16, and 17 shows the QQ plots of the partial dollar deltas calculated by the Monte Carlo simulation model and those estimated by the copula models when $s = 320$. The QQ plots

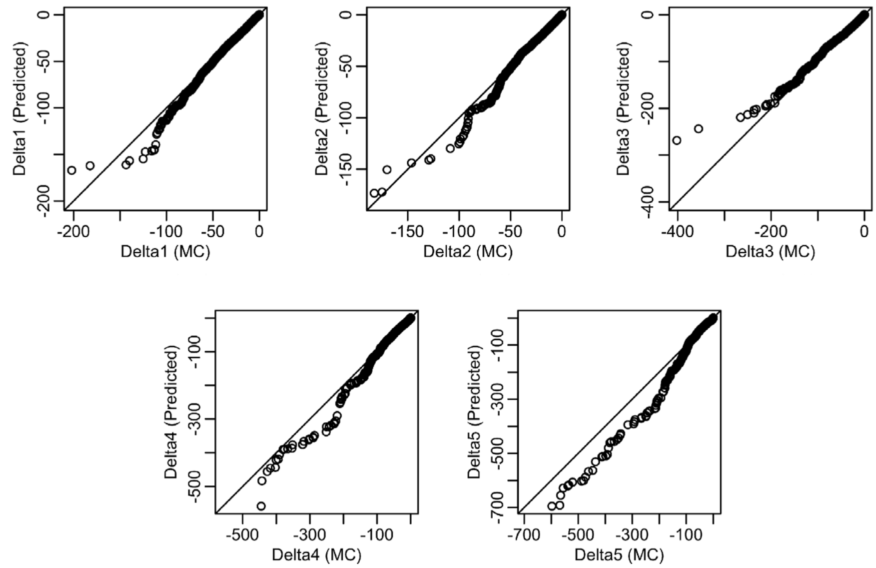

Fig. B.24. The QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the Clayton copula based model for the second portfolio when $s = 320$.

are based on the partial dollar deltas of 10,000 VA policies in the portfolio. The QQ plots show that the metamodels produce similar results.

Table 8 shows the estimated dependence parameters of the various copulas when $s = 320$. Similarly, we observe strong dependence among the partial dollar deltas. The Cramer–von Mises test statistics are shown in Table 9. The test statistics indicate that the t copula provides the best fit among the four copulas.

Table 10 shows the validation measures of the marginal models and the copula models. From the table, we also see that all models produce similar results in terms of the percentage error. The MSE and the CCC measures show that the estimations obtained by marginal models only are more accurate than those obtained by copula models.

If we compare Tables 7 and 10, we see that the accuracy of the metamodels increases when the number of representative VA policies is increased from 160 to 320. When $s = 320$, the percentage errors are reduced to within 2%.

In summary, the numerical results show that incorporating dependence structures in metamodels does not improve the accuracy of those metamodels at the portfolio level. At the individual policy level, copula models produce even worse results due to the small number of simulations used to estimate the partial dollar deltas.

5. Concluding remarks

The financial risks associated with guarantees embedded in variable annuities cannot be adequately addressed by traditional actuarial approaches. In practice, dynamic hedging is usually adopted by insurance companies to mitigate these financial risks. Since the guarantees embedded in VA contracts are complex, Monte Carlo simulation is used to calculate the partial dollar deltas on major market indices that are required by dynamic hedging. However, Monte Carlo simulation is extremely time-consuming when applied to a large portfolio of VA contracts. To address the computational problems associated with dynamic hedging of large portfolios of VA contracts, metamodeling approaches have been proposed in the past few years.

In this paper, we investigated whether modeling the dependence between the partial dollar deltas helps improve the accuracy of the metamodeling approaches. We used several copulas to model the dependence structures of the partial dollar deltas. In particular, we considered the independence copula, the Gaussian copula, the t copula, the Gumbel copula, and the Clayton copula. We performed numerical experiments to compare different metamodels. Our numerical results demonstrate that despite evidence of strong dependence, modeling the dependence structures in the metamodels does not improve the accuracy of the estimations at the portfolio level. The reason is that the dependence between the partial dollar deltas is well captured by the covariates used in the marginal models. In conclusion, the results suggest that we should focus more on modeling the partial dollar deltas on market indices individually than specifying the dependence structure explicitly.

Appendix A. Synthetic VA contracts

Table A.11 shows the parameters used to generate the synthetic VA contracts. In this table, the product types DBRP, DBRU, WB, WBSU, MB refer to the GMDB with return of principle, the GMDB with roll-up, the GMWB with return of principle, the GMWB with step-up, and the GMMB, respectively. More information about the guarantees can be found in The Geneva Association Report (2013). The program used to generate the synthetic VA contracts was written in Java and can be downloaded from https://github.com/ganml/va.

Table A.11
Parameters used to generate synthetic VA contracts.

Feature	Value
Policyholder birth date	[1/1/1950, 1/1/1980]
Issue date	[1/1/2000, 1/1/2014]
Valuation date	1/1/2014
Maturity	[15, 30] years
Account value	[50000, 500000]
Female percent	40%
Product type	DBRP, DBRU, WB, WBSU, MB (20% of each type)
Fund fee	30, 50, 60, 80, 10, 38, 45, 55, 57, 46 bps for funds 1–10, respectively
Base fee	200 bps
Rider fee	20, 50, 60, 50, 50 bps for DBRP, DBRU, WB, WBSU, MB, respectively
Number of funds invested	[1,10]

Table B.12
Estimated dependence parameters of the fitted copulas for the second portfolio when $s = 320$.

	Normal	t	Gumbel	Clayton
Estimated parameters	$\rho = 0.29$	$\rho = 0.448, df = 8.296$	$\alpha = 1.342$	$\alpha = 0.087$

Table B.13
Cramer–von Mises test statistics of copula models for the second portfolio when $s = 320$.

	Normal	t	Gumbel	Clayton
S_n	0.285	0.347	1.195	0.284

Table B.14
Validation measures of different models for the second portfolio when $s = 320$.

	Delta1	Delta2	Delta3	Delta4	Delta5
(a) Marginal models					
PE	0.002	0.020	−0.032	−0.007	0.024
MSE	34.682	41.240	41.866	61.565	154.335
CCC	0.953	0.865	0.946	0.962	0.934
(b) Independence copula					
PE	0.002	0.019	−0.031	−0.007	0.022
MSE	38.093	42.305	43.948	66.789	170.298
CCC	0.949	0.862	0.944	0.959	0.926
(c) Gaussian copula					
PE	0.001	0.019	−0.031	−0.006	0.027
MSE	38.037	43.814	45.458	66.360	174.121
CCC	0.949	0.858	0.942	0.959	0.927
(d) t copula					
PE	0.002	0.022	−0.032	−0.008	0.023
MSE	37.995	43.464	43.780	67.258	157.383
CCC	0.949	0.860	0.944	0.958	0.933
(e) Gumbel copula					
PE	0.001	0.017	−0.034	−0.009	0.020
MSE	37.801	41.578	44.977	64.953	159.075
CCC	0.949	0.863	0.942	0.960	0.931
(f) Clayton copula					
PE	0.001	0.019	−0.031	−0.007	0.029
MSE	37.504	41.563	43.232	62.852	181.065
CCC	0.950	0.864	0.945	0.961	0.924

Appendix B. Impact of different portfolio compositions

The original synthetic portfolio contains a uniform distribution of the five products. To examine the impact of different portfolio composition on the proposed models, we created a second portfolio, which contains 20% DBRP, 10% MB, 20% WB, and 50% WBSU products. This portfolio also contains 10,000 contracts. Table B.12 shows the estimated dependence parameters of the various copulas for the second portfolio when $s = 320$. Table B.13 shows the Cramer–von Mises test statistics. Table B.14 shows the validation measures of the marginal models and the copula models.

Fig. B.18 shows the QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by the marginal models for the second portfolio when $s = 320$. Fig. B.19 shows the scatter plots of the uniform marginals when $s = 320$. Figs. B.20, B.21, B.22, B.23, and B.24 show the QQ plots of the partial dollar deltas calculated by Monte Carlo simulation and those estimated by various copula based models for the second portfolio when $s = 320$. These tables and figures show similar results as in the original portfolio. We expect that the proposed models are not sensitive to changes in portfolio compositions.

References

Barton, R., 1994. Metamodeling: a state of the art review. In: Winter Simulation Conference Proceedings, pp. 237–244.
Cherubini, U., Luciano, E., Vecchiato, W., 2004. Copula Methods in Finance. Wiley, Hoboken, NJ.
Dardis, T., 2016. Model efficiency in the US life insurance industry. Model. Platform (3), 9–16.
Forsyth, P., Vetzal, K., 2014. An optimal stochastic control framework for determining the cost of hedging of variable annuities. J. Econom. Dynam. Control 44, 29–53.
Frees, E.W., 2009. Regression Modeling with Actuarial and Financial Applications. Cambridge University Press.
Frees, E.W., Valdez, E.A., 1998. Understanding relationships using copulas. N. Am. Actuar. J. 2 (1), 1–25.
Friedman, L.W., 1996. The Simulation Metamodel. Kluwer Academic Publishers, Norwell, MA, USA.
Gan, G., 2011. Data Clustering in C++: An Object-Oriented Approach. In: Data Mining and Knowledge Discovery Series, Chapman & Hall/CRC Press, Boca Raton, FL, USA.
Gan, G., 2013. Application of data clustering and machine learning in variable annuity valuation. Insurance Math. Econom. 53 (3), 795–801.
Gan, G., 2015a. Application of metamodeling to the valuation of large variable annuity portfolios. In: Proceedings of the Winter Simulation Conference, pp. 1103–1114.
Gan, G., 2015b. A multi-asset monte carlo simulation model for the valuation of variable annuities. In: Proceedings of the Winter Simulation Conference, pp. 3162–3163.
Gan, G., Lan, Q., Ma, C., 2016. Scalable clustering by truncated fuzzy c-means. Big Data Inf. Anal. (in preparation).
Gan, G., Lin, X.S., 2015. Valuation of large variable annuity portfolios under nested simulation: A functional data approach. Insurance Math. Econom. 62, 138–150.
Gan, G., Lin, X.S., 2016. Efficient greek calculation of variable annuity portfolios for dynamic hedging: A two-level metamodeling approach. N. Am. Actuar. J. (in press).
Gan, G., Valdez, E.A., 2016a. An empirical comparison of some experimental designs for the valuation of large variable annuity portfolios. Depend. Model. 4 (1), 382–400 (in preparation).
Gan, G., Valdez, E.A., 2016b. Regression modeling for the valuation of large variable annuity portfolios. North Amer. Actuar. J. (in preparation).
Garcia, R., Tsafack, G., 2011. Dependence structure and extreme comovements in international equity and bond markets. J. Banking Finance 35 (8), 1954–1970.
Genest, C., Mackay, J., 1986. The joy of copulas: Bivariate distributions with uniform marginals. Amer. Statist. 40 (4), 280–283.
Genest, C., Rémillard, B., Beaudoin, D., 2009. Goodness-of-fit tests for copulas: A review and a power study. Insurance Math. Econom. 44 (2), 199–213.
Hardy, M., 2003. Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance. John Wiley & Sons, Inc., Hoboken, New Jersey.
Hejazi, S.A., Jackson, K.R., 2016. A neural network approach to efficient valuation of large portfolios of variable annuities. Insurance Math. Econom. 70, 169–181.
Hu, L., 2006. Dependence patterns across financial markets: a mixed copula approach. Appl. Financ. Econ. 16 (10), 717–729.
Kleinow, T., Willder, M., 2007. The effect of management discretion on hedging and fair valuation of participating policies with maturity guarantees. Insurance Math. Econom. 40, 445–458.
Kolkiewicz, A., Liu, Y., 2012. Semi-static hedging for gmwb in variable annuities. N. Am. Actuar. J. 16 (1), 112–140.
Ledlie, M.C., Corry, D.P., Finkelstein, G.S., Ritchie, A.J., Su, K., Wilson, D.C.E., 2008. Variable annuities. Brit. Actuar. J. 14 (2), 327–389.
Li, J., Szimayer, A., 2011. The uncertain mortality intensity framework: Pricing and hedging unit-linked life insurance contracts. Insurance Math. Econom. 49 (3), 471–486.
Lin, L.I.-K., 1989. A concordance correlation coefficient to evaluate reproducibility. Biometrics 45 (1), 255–268.
Loeppky, J.L., Sacks, J., Welch, W.J., 2009. Choosing the sample size of a computer experiment: A practical guide. Technometrics 51 (4), 366–376.
Melnikov, A., Tong, S., 2014. Quantile hedging on equity-linked life insurance contracts with transaction costs. Insurance Math. Econom. 58 (0), 77–88.
Moller, T., 1998. Risk-minimizing hedging strategies for unit-linked life insurance contracts. Astin Bull. 28 (1), 17–47.
Nelsen, R.B., 2006. An Introduction to Copulas. Springer, New York, NY.
Ng, A.C.-Y., Li, J.S.-H., 2013. Pricing and hedging variable annuity guarantees with multiasset stochastic investment models. N. Am. Actuar. J. 17 (1), 41–62.
The Geneva Association Report 2013. Variable annuities - an analysis of financial stability. Available online at: https://www.genevaassociation.org/media/618236/ga2013-variable_annuities.pdf.
Windcliff, H., Le Roux, M., Forsyth, P., Vetzal, K., 2002. Understanding the behavior and hedging of segregated funds offering the reset feature. N. Am. Actuar. J. 6 (2), 107–124.