Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

I) Suppose that I draw 2 cards from a deck:

(a) What is the probability that the 2^{nd} card is heart?

Solution: $P(2 = \heartsuit) = P((1 = \heartsuit \text{ and } 2 = \heartsuit) \text{ or } (1 \neq \heartsuit \text{ and } 2 = \heartsuit))$

$$= P(1 = \heartsuit) \times P(2 = \heartsuit | 1 = \heartsuit) + P(1 \neq \heartsuit) \times P(2 = \heartsuit | 1 \neq \heartsuit)$$

$$= \left(\frac{13}{52}\right)\left(\frac{12}{51}\right) + \left(\frac{39}{52}\right)\left(\frac{13}{51}\right) = \frac{1}{4}$$

(b) We look at the second card, and find that it is a heart. What is the probability that the **first** card was a heart?

Solution:

We need to find $P(A|B)$ where:

A= the event that the 1st card is a $\heartsuit$

B= the event that the 2nd card is a $\heartsuit$

Then to use Bayes' Theorem, we need to know:

$P(B|A) = P(2 = \heartsuit | 1 = \heartsuit) = \frac{12}{51}$

$P(A) = P(1 = \heartsuit) = \frac{13}{52}$

$P(B) = P(2 = \heartsuit) = \frac{1}{4}$

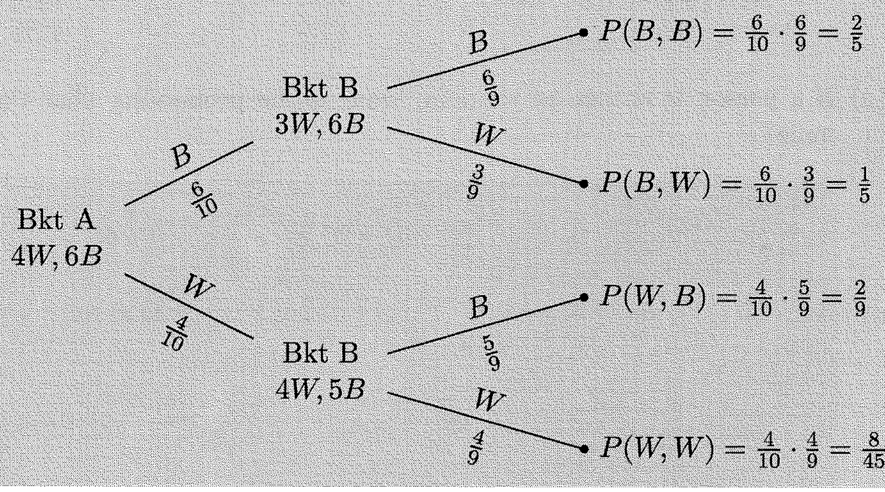
Then $P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{(\frac{12}{51})(\frac{13}{52})}{(\frac{1}{4})} = \frac{12}{51} \approx 23.53$

(c) What is the probability that the **first** card is a heart, given that the second card is a Diamond?

II) Bucket A contains four white balls and six black balls. Bucket B contains three white balls and five black balls. A ball is drawn from bucket A and then transferred to bucket B. A ball is then drawn from bucket B.

(a) Create a tree diagram representing the possible outcomes of this experiment.

Solution:



(b) What is the probability that the transferred ball was **white** given that the second ball was **white** ?

Solution:

$$P(1 = W|2 = W) = \frac{P(2 = W|1 = W)P(1 = W)}{P(2 = W)} = \frac{\left(\frac{4}{9}\right)\left(\frac{4}{10}\right)}{\left(\frac{1}{5} + \frac{8}{45}\right)} = \frac{8}{17}$$

(c) What is the probability that the transferred ball was **black** given that the second ball was **white** ?

Solution:

$$P(1 = B|2 = W) = \frac{P(2 = W|1 = B)P(1 = B)}{P(2 = W)} = \frac{\left(\frac{3}{9}\right)\left(\frac{6}{10}\right)}{\left(\frac{1}{5} + \frac{8}{45}\right)} = \frac{9}{17}$$

(d) What is the probability that the transferred ball was **black** given that the second ball was **black**?

Solution:

$$P(1 = B|2 = B) = \frac{P(2 = B|1 = B)P(1 = B)}{P(2 = B)} = \frac{\left(\frac{6}{9}\right)\left(\frac{6}{10}\right)}{\left(\frac{2}{5} + \frac{2}{9}\right)} = \frac{9}{14}$$

V) A doctor is called to see a sick child. The doctor has prior information that 90% of sick children in that neighborhood have the flu, while the other 10% are sick with measles. Let F stand for an event of a child being sick with flu and M stand for an event of a child being sick with measles. Assume for simplicity that there no other maladies in that neighborhood.

A well-known (and common) symptom of measles is a rash: $P(Rash\ |M) = .95$. However, very occasionally, children with flu also develop rash: $P(Rash\ |F) = 0.08$.

Upon examining the child, the doctor finds a rash. What is the probability that the child has measles?

Solution: In other words, the question is asking for $P(M|\ Rash)$. Bayes' Theorem says . . .

$$P(M|\ Rash) = \frac{P(Rash|\ M)P(M)}{P(Rash)} = \frac{(.95) * (.10)}{(.167)} \approx 56.89\%$$