QUICK GUIDE TO POSITIVE CONES OF NUMERICAL CLASSES

MIHAI FULGER AND BRIAN LEHMANN

Throughout X and Y are projective varieties over an algebraically closed field of arbitrary characteristic, with $\dim X = n$, and $\pi : X \to Y$ is a morphism.

1. Cycles

Let $Z_k(X)$ denote the group of k-cycles on X with $\mathbb{R}$-coefficients. Any subscheme $Z \subset X$ of dimension k has a fundamental cycle denoted $[Z] \in Z_k(X)$ defined as in [4, §1.5]. If $Z = \sum a_i Z_i$ and $Z_i \subset X$ are subvarieties, the *support* of Z is $|Z| := \cup_i Z_i$.

Features of cycle groups:

- Proper pushforwards ([4, §1.4]), that measure degrees of field extensions.
- Flat pullbacks ([4, §1.7]), that may encode scheme structure.
- If Z and V are cycles that meet properly (they have the expected dimension of) intersection) and the support of either one of them is regularly embedded in X, then then one has a well-defined cycle $Z \cdot V$ determined by normal cones in [4, §6.1] and revisited in [4, §7]. If Z and V are *effective* (only nonnegative coefficients), then $Z \cdot V$ is also effective.
- The same happens if neither of Z or V is regularly embedded, but the ambient space is smooth. See [4, §8]. In this case $Z \times V \cap \Delta \subset X \times X$ identifies with $Z \cap V$, and the diagonal is regularly embedded.

2. Chow groups

The Chow groups $CH_k(X)$ are defined as the quotient of $Z_k(X)$ modulo rational equivalence. There are two equivalent definitions for rational equivalence. See [4, §1.3 and §1.6].

Features of Chow groups:

- Proper pushforward (they pass to rational equivalence from pushforwards of cycles). See [4, §1.4].
- Flat pullbacks (again these are induced by cycle flat pullbacks). See [4, §1.7].
- A restriction sequence: If X is closed in Y with complement U, then there exist exact sequences $CH_k(X) \to CH_k(Y) \to CH_k(U) \to 0$. See [4, §1.8].
- Pullbacks by regular embeddings and more generally by l.c.i maps. See [4, §6.1 and §6.6]. These are constructed by deformation to the normal cone, not by moving lemmas.
- Pullbacks from smooth varieties. See [4, §8]. These again use that the diagonal is l.c.i in $X \times X$.
- There exist refined Gysin pullbacks in the previous two settings which also keep track of supports, not just of cycle classes in an ambient space.
- When X is smooth, then by denoting $CH^k(X) = CH_{n-k}(X)$, we get a graded ring structure on $CH^*(X)$. See [4, §8]. This uses that the diagonal map $X \hookrightarrow X \times X$ is a regular embedding.
- There are Chern class actions on Chow groups: If E is a vector bundle, then for each k and m there exists a linear function $CH_m(X) \xrightarrow{c_k(E)\cap \text{-}} CH_{m-k}(X)$. See [4, §3.2]. These satisfy the following properties:
  i) $c_0(E) \cap \alpha = \alpha$.
  ii) **Commutativity:** $c_k(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_k(E) \cap \alpha)$.
  iii) **Projection formula:** $\pi_*(c_k(\pi^* E) \cap \alpha) = c_k(E) \cap \pi_* \alpha$.
  iv) **Naturality for flat maps:** If π is flat, then $\pi^*(c_k(E) \cap \alpha) = c_k(\pi^* E) \cap \pi^* \alpha$.
  v) If E is a line bundle, then $c_1(E) \cap [X]$ is the image in the class group of the Cartier divisor associated to E. See [4, §2].
  vi) If E has rank r and s is a global section that vanishes along a subset $V(s)$ of codimension exactly r, then $[V(s)] = c_r(E) \cap [X]$.

  The Chern class operations are also compatible with the l.c.i pullbacks, intersections on smooth varieties, or Gysin maps.

3. Numerical equivalence

3.1. **Smooth case.** We have an intersection pairing $CH_k(X) \times CH^k(X) \to \mathbb{R}$ determined by the ring structure on $CH^*(X)$ and by the natural point counting degree function $\deg : CH_0(X) \to \mathbb{R}$ (For the degree to respect rational equivalence, we need X to be proper). Then one defines the *numerical groups* $N_k(X)$ by quotienting $CH_k(X)$ by the kernel of this pairing and similarly $N^k(X) = N_{n-k}(X)$. This induces a perfect pairing $N_k(X) \times N^k(X) \to \mathbb{R}$ which proves that $N^k(X) \simeq (N_k(X))^\vee$.

3.2. **Singular case.** When X is singular, we do not have an intersection pairing. Instead we can use the Chern class action. Following [4, §19], we say that a k-cycle Z is *numerically trivial*, and denote $Z \equiv 0$, if

$$\deg(P \cap [Z]_{Chow}) = 0$$

for any weight k polynomial P in Chern classes of maybe several vector bundles on X (e.g. $c_1^3(E) - c_1(F)c_2(G)$ is a Chern polynomial of weight 3). A Chern polynomial is naturally seen as an operator on Chow groups using the linearity and commutativity of Chern class actions. The *numerical groups* are the quotients $N_k(X) = CH_k(X)/\equiv$.

The *dual numerical groups* $N^k(X) := (N_k(X))^\vee$ are no longer isomorphic to $N_{n-k}(X)$. They can be defined also as weight k Chern polynomials P modulo those that verify $\deg(P \cap \alpha) = 0$ for any $\alpha \in CH_k(X)$.

Features of the numerical groups $N_k(X)$:

- They are finite dimensional. See [4, Example 19.1.4].
- They admit proper pushforwards (induced from those defined for cycles).
- **Caution:** It is not known whether flat pullbacks descend from CH_k to N_k.
- They admit l.c.i pullbacks. See [4, Example 19.2.3].
- $N_0(X)$ and $N_n(X)$ are both isomorphic to $\mathbb{R}$.

Features of the dual numerical groups $N^k(X) := (N_k(X))^\vee$:

- They admit proper pullbacks (dual to the proper pushforwards for numerical groups).
- Multiplication of polynomials induces a graded ring structure on $N^*(X)$.
- The Chern class actions induce linear maps $N^k(X) \times N_m(X) \to N_{m-k}(X)$ that we continue to denote $P \cap \alpha$, or $P \cdot \alpha$.
- From the above there exist a "cyclification" map $P \mapsto P \cap [X] : N^k(X) \to N_{n-k}(X)$.
- The cyclification $N^1(X) \to N_{n-1}(X)$ is injective (see [4, Example 19.3.3]). Dually $N^{n-1}(X) \to N_1(X)$ is onto.
- More generally $N_*(X)$ is a module over $N^*(X)$.
- **Projection formula**: $\pi_*(\pi^* P \cap \alpha) = P \cap \pi_* \alpha$ for any $P \in N^k(Y)$ and $\alpha \in N_m(X)$.
- The cap pairings are natural whenever there exist pullbacks for numerical groups.
- There exists a map $ch : K^0(X)_{\mathbb{R}} \to N^*(X)$ from the K-theory of vector bundles on X that sends a bundle E to its Chern character [4, Example 3.2.3]. This is a ring morphism (the operations on $K^0(X)_{\mathbb{R}}$ are induced by direct sums and tensor products of bundles).
- When X is smooth, $ch(\_) \cap [X]$ is surjective onto $N_{n-k}(X)$ (see [4, Example 15.2.16.(b)]). This implies that the singular definition agrees with the smooth one in the smooth case.

4. Positive cones in $N_k(X)$

The numerical group is the natural ambient space for any positivity notion that one expects would be preserved by pushforwards.

4.1. **The pseudoeffective cone.** The closure of the cone generated by numerical classes of effective k-cycles is the *pseudoeffective cone* $\overline{\mathrm{Eff}}_k(X)$. Its interior is the *big cone.*

Features of the pseudoeffective cone:

- $\overline{\mathrm{Eff}}_k(X)$ generates $N_k(X)$.
- $\overline{\mathrm{Eff}}_k(X)$ is a pointed cone (it does not contain lines, only half-lines).
- $\pi_* \overline{\mathrm{Eff}}_k(X) \subseteq \overline{\mathrm{Eff}}_k(Y)$. If π is surjective, then equality holds for psef cones, and also for big cones.
- If π is flat of relative dimension d and π^* exists (e.g. π is smooth or Y is smooth), then $\pi^* \overline{\mathrm{Eff}}_k(Y) \subset \overline{\mathrm{Eff}}_{k+d}(X)$.
- If $h \in N^1(X)$ is the class of an ample divisor, then $h^{n-k} \cap [X]$ is big, i.e. in the interior of $\overline{\mathrm{Eff}}_k(X)$. The same is true of any complete intersection of possibly different ample divisor classes.

- For any $\alpha \in \overline{\mathrm{Eff}}_k(X)$ we can define its degree $\deg_h(\alpha) := h^k \cap \alpha$. We have that $\alpha = 0$ if and only if $\deg_h(\alpha) = 0$.
- There exists a norm $|\cdot|$ on $N_k(X)$ such that $|\alpha| = \deg_h(\alpha)$ for any $\alpha \in \overline{\mathrm{Eff}}_k(X)$.
- The effective cone of numerical Cartier classes $\overline{\mathrm{Eff}}(X)$ is a subcone of $\overline{\mathrm{Eff}}_{n-1}(X)$. If X is normal, then $\overline{\mathrm{Eff}}(X) = N^1(X) \cap \overline{\mathrm{Eff}}_{n-1}(X) \subset N_{n-1}(X)$.
- If $P \in \mathrm{PL}^k(X)$ is pliant or $P \in \mathrm{BPF}^k$ (see below for both), then $P \cap \overline{\mathrm{Eff}}_m\, k(X) \subset \overline{\mathrm{Eff}}_{m-k}(X)$. In particular the intersection between a psef class and a nef divisor is a psef class.
- [5] introduced a continuous function $\mathrm{mob} : N_k(X) \to \mathbb{R}$ called *mobility* such that
  - i) $\mathrm{mob}(\alpha) > 0$ if and only if α is a big class (i.e. interior to $\overline{\mathrm{Eff}}_k(X)$).
  - ii) mob is homogeneous of degree $\frac{n}{n-k}$.
  - iii) If X is smooth and $\delta \in \overline{\mathrm{Eff}}(X) \subset N^1(X)$, then $\mathrm{mob}(\alpha) = \mathrm{vol}(\alpha)$, with the latter being the classical volume of a divisor.
  - iv) The mobility of a class α measures the asymptotic growth of the *mobility count* (defined below) of multiples of α.

Many of the statements above are nontrivial for pseudoeffective classes that are not effective.

Definition 4.1 (Families of cycles)**.** A family of k cycles on X is a proper generically equidimensional morphism $p : U \to W$ of relative dimension k, where W is an irreducible quasiprojective variety, and U is a closed subset of $W \times X$, such that p is the restriction of the first projection. There is a well-defined numerical class for the fundamental cycle of the general fiber of p. We denote this class by $[p]$. We can easily modify the definition to include the case when U is a subscheme, or a cycle in $W \times X$.

Definition 4.2 (Mobility count)**.** The *mobility count* of p is the maximal integer m such that for every m (very) general points of X there exists a cycle Z in the family p that contains them. The mobility count of a class α is the maximal mobility count among all families p with $[p] = \alpha$. We denote it by $\mathrm{mc}(\alpha)$.

Then $\mathrm{mob}(\alpha) := \limsup_{m\to\infty} \frac{\mathrm{mc}(m\alpha)}{m^{\frac{n}{n-k}}/n!}$

Motivation: Working with families is a substitute for working with linear series which no longer exist in any relevant way.

4.2. **The movable cone.** Intuitively speaking, the movable classes are represented by cycles that cover X without always having a component in a fixed proper subset of X.

If $p : U \to W$ is a family of k-cycles on X and U is irreducible and the second projection $s : U \to X$ is dominant, we say that p is *strongly movable*, and that $[p]$ is a *strongly movable class* represented by the family p. If U is reducible, but every irreducible component still dominates X, we say that p is *strictly movable.*

The closure of the cone in $N_k(X)$ generated by strongly (or strictly) movable classes is called the *movable cone* $\overline{\mathrm{Mov}}_k(X)$.

Features of the movable cone:

- It generates $N_k(X)$.
- It is pointed.
- Complete intersections are in the interior of $\overline{\mathrm{Mov}}_k(X)$.
- If π is surjective, then $\pi_* \overline{\mathrm{Mov}}_k(X) = \overline{\mathrm{Mov}}_k(Y)$.
- If π is flat of relative dimension d and p is a strongly or strictly movable family, then the fiber product $\pi^* p$ is strictly movable. Additionally if π^* exists for numerical groups (e.g. π smooth, or Y smooth), then $\pi^* \overline{\mathrm{Mov}}_k(Y) \subset \overline{\mathrm{Mov}}_{k+d}(X)$.
- If P is pliant or bpf (see below), then $P \cap \overline{\mathrm{Mov}}_m(X) \subset \overline{\mathrm{Mov}}_{m-k}(X)$. This holds for example when P is an intersection of nef divisor classes.
- If $\delta \in \overline{\mathrm{Eff}}(X) \subset N^1(X)$ is a pseudoeffective Cartier divisor class, then $\delta \cap \overline{\mathrm{Mov}}_k(X) \subset \overline{\mathrm{Eff}}_{k-1}(X)$.
- If δ is a big Cartier divisor class, and $\alpha \in \overline{\mathrm{Mov}}_k(X)$, then $\alpha = 0$ if and only if $\delta \cap \alpha = 0$.
- Movable Cartier classes (limits of basepoint free classes in codimension one) are in $\overline{\mathrm{Mov}}_{n-1}(X)$.
- If X is smooth, then $N^1(X) \cap \overline{\mathrm{Mov}}_{n-1}(X) = \overline{\mathrm{Mov}}(X)$.
- By [1] Any $\alpha \in \overline{\mathrm{Eff}}_k(X)$ can be written as $\alpha = P + N$, where $P \in \overline{\mathrm{Mov}}_k(X)$, and $N \in \overline{\mathrm{Eff}}_k(X)$, such that $\mathrm{mob}(\alpha) = \mathrm{mob}(P)$. This coincides with the Nakayama σ-decomposition of Cartier divisors on smooth varieties, which is a suitable higher-dimensional generalization of the Zariski decomposition of divisors on surfaces.

- Except for divisor classes on smooth varieties, it is unknown whether the decomposition is unique.
- It is conjectured that if $\alpha \in \overline{\mathrm{Eff}}_k(X)$ is extremal and not movable, so that $\alpha = 0 + N$ is the unique Zariski decomposition, then there exists a proper subvariety $Z \subsetneq X$ with inclusion map $\imath$ and $\beta \in \overline{\mathrm{Eff}}_k(Z)$ such that $\imath_*\beta = N = \alpha$. When α is a Cartier divisor class on a smooth variety, then this would be the equivalent of N being effective.

Motivation: Statements about psef classes tend to be easier for movable classes, because morally speaking we can move these away from "special" subvarieties of X. Via the conjecture that extremal negative parts are pushed from proper closed subvarieties, one would then hope to reduce to the movable case by induction using Zariski decompositions. This strategy is exploited in [2].

There is hope that movable classes are birational objects. Unfortunately even if π is a birational morphism, and p is a strongly movable family on Y, then it is not π^*p that gives a strongly (or even strictly) movable family on X (the numerical class $\pi^*[p]$ could potentially even fail to be psef), but the strict transform of p, and taking strict transforms is not a numerical operation.

N.B.: In some cases it is useful to work with alternate definitions of movability. One could avoid families altogether and say that a class is (weakly) movable if it is a limit of classes of cycles Z_i such that for each (reducible) divisor $D \subset X$, infinitely many of the Z_i's meet D properly (i.e. they have no components in D). With this definition, at least for extremal classes α one has that either α is (weakly) movable, or α is pushed from a psef class on a subvariety of X. This is good for some questions requiring only positivity of intersections, but it allows only few geometric arguments for lack of any obvious finiteness/boundedness that was provided by working with families.

4.3. **The basepoint free cone.** If W is a quasiprojective variety, and $p : U \to W$ is proper, with general equidimensional fiber of dimension k, and $s : U \to X$ is flat (not necessarily proper), we say that p is a *basepoint free family.* A general cycle theoretic fiber U_w of p has proper support and determines a well defined numerical class on X by pushforward (from its support, not from U, because s may fail to be proper). We may denote this class by $[s(p)]$ and call it basepoint free.

The **basepointfree cone** $\mathrm{BPF}_k(X) \subset N_k(X)$ is the closure of the cone generated by classes $[s(p)]$ for basepoint free families p.

An important distinction to note here is that basepoint free families are not necessarily families of cycles on X, in that we do not require that U lives on $W \times X$.

Features of the bpf cone:

- If X is smooth, then $\mathrm{BPF}_k(X)$ generates $N_k(X)$.
- If $Z \subset X$ is a closed subset, and $p : U \to W$ is a basepoint free family, then for general $w \in W$, the intersection $Z \cap s(U_w)$ has the expected dimension or less.
- If X is smooth in the previous point, then Z and $s(U_w)$ meet properly.
- If X is smooth, then $\mathrm{BPF}^k(X) \subset \mathrm{Upsef}^k(X)$. (The universally pseudoeffective cone is defined below).
- **Bpf is stable under flat pushforward.** If $\pi : X \to Y$ is flat, and $p : U \to W$ is a bpf family on X, with evaluation map $s : U \to X$, then it is also bpf on Y with evaluation map $s' := \pi \circ s$, and $\pi_*[s(p)] = [s'(p)]$.
- **Bpf is preserved by pullback from a smooth base.** If Y is smooth and $\alpha \in \mathrm{BPF}^k(Y)$, then $\pi^*\alpha \cap [X] \in \mathrm{BPF}_{n-k}(X)$.
- If $p : U \to W$ is a strongly movable family of cycles on Y, and $s : U \to Y$ denotes the second projection, and $\pi : X \to Y$ is a proper birational morphism that flattens s to $s' : U' \to X$, then the induced $p' : U' \to W$ is a basepoint free family on X.
- If π is dominant, then $\pi_* \mathrm{BPF}_k(X) \subset \overline{\mathrm{Mov}}_k(Y)$.
- $\mathrm{PL}^k(X) \cap [X] \subset \mathrm{BPF}_{n-k}(X)$. (The pliant cone is defined below.)

Motivation: The preservation under flat pushforward, and the flattening trick to transform strictly movable families into basepoint free ones advertise the BPF cone as a versatile tool.

N.B.: Basepoint freeness seems to be an awkward condition in $N_k(X)$. It is not preserved by proper pushforward, and despite its definition as a cone in $N_k(X)$, it is used mostly when X is smooth for intersection theoretic purposes, so it is best behaved as a cone in $N^k(X)$ (when X is smooth).

One could find purely numerical definitions by asking only that α is bpf if it is a limit of classes of cycles Z_i such that for any subvariety $T \subset X$, infinitely many of the Z_i have at most the expected dimension of intersection with T. Again this retains the intersection theoretic positivity, but loses geometry.

June Huh conjectures that if X is smooth, and $\alpha \in \overline{\mathrm{Eff}}_{n-k}(X) \cap \mathrm{Nef}^k(X)$ (intersection of subsets of $N^k(X) = N_{n-k}(X)$, not a cap pairing), then α is basepoint free in this weaker sense.

5. Cones in $N^k(X)$

The dual numerical group is the natural setting for positivity notions that are preserved by proper pullbacks.

5.1. **The nef cone.** Via the duality pairing $N^k(X) \times N_k(X) \to \mathbb{R}$, the *nef cone* $\mathrm{Nef}^k(X) \subset N^k(X)$ is the dual of $\overline{\mathrm{Eff}}_k(X) \subset N_k(X)$.

Features of the nef cone:

- $\mathrm{Nef}^k(X)$ generates $N^k(X)$.
- $\mathrm{Nef}^k(X)$ is pointed.
- $\mathrm{Nef}^k(X)$ contains complete intersections h^k in its interior. Here $h \in N^1(X)$ is ample.
- $\pi^* \mathrm{Nef}^k(Y) \subseteq \mathrm{Nef}^k(X)$.
- If π is dominant, then $\pi^*\alpha$ is nef if and only if α is nef.
- If $h_1, \ldots, h_k$ are nef divisor classes, then $h_1 \cdot \ldots \cdot h_k \in \mathrm{Nef}^k(X)$.
- **Caution:** Debarre, Ein, Lazarsfeld, and Voisin construct examples of nef classes of codimension two whose intersection product is not nef. They also construct nef classes that are not pseudoeffective. This shows that the nice properties of nef divisors do not carry over to higher codimension classes in general.
- The nef cone contains all the other cones in this section. It also contains the bpf cone when X is smooth.
- If X is smooth, then $\mathrm{Nef}^{n-1}(X) = \overline{\mathrm{Mov}}_1(X)$. This is BDPP.

5.2. **The pliant cone.** If E is a globally generated vector bundle, then there exists an induced Gauss map $E \to G$ to some Grassmann variety such that E is the pullback of the universal quotient bundle Q (still a globally generated bundle) on G. Then the Chern and more generally the Schur classes ([4, §14.5] gives a determinental formula for these in terms of the Chern classes) of E are obtained by pulling back the classes of Q, which are represented by (effective and even irreducible) Schubert cycles on G.

The *pliant cone* $\mathrm{PL}^k(X) \subset N^k(X)$ is the closure of the cone generated by monomials in Schur classes of maybe different globally generated bundles on X.

Features of the pliant cone:

- $\mathrm{PL}^k(X)$ generates $N^k(X)$. This is one of the few results that uses the projectivity of X, as opposed to just properness.
- $\mathrm{PL}^k(X)$ is pointed.
- Complete intersections of ample divisor classes are in the interior of $\mathrm{PL}^k(X)$. This is actually the result that proves the similar ones listed before.
- $\mathrm{PL}^1(X) = \mathrm{Nef}^1(X)$.
- $\mathrm{PL}^k(X) \cap \overline{\mathrm{Eff}}_m(X) \subset \overline{\mathrm{Eff}}_{m-k}(X)$ by [4, Example 12.1.7].
- The pliant cone is contained in any positive cone in this section, and its pairing with $[X]$ is contained in any positive cone in the previous section.
- The pliant cone is preserved by pullbacks.
- If G is a Grassmann variety of dimension g, then $\overline{\mathrm{Eff}}_k(G) = \overline{\mathrm{Mov}}_k(G) = \mathrm{BPF}_k(G) = \mathrm{Nef}^{g-k}(X) = \mathrm{PL}^{g-k}(X) = \mathrm{Upsef}^{g-k}(X)$.

Motivation: The nef cone of Cartier divisor classes is the closure of the cone generated by globally generated line bundles. With this perspective, one can at least approximate nef divisor classes by geometric objects (divisor classes obtained by pulling back hyperplanes from projective spaces).

In higher codimension, one could try to do something similar by using complete intersection of globally generated divisors, but these are not sufficient to generate $N^k(X)$. For example $N^2(G(2,4))$ has dimension 2, but $N^1(G(2,4))$ is of dimension 1, and the same is true of the span of complete intersections. Instead we pullback and intersect Schubert cycles from Grassmannians. These satisfy almost every positivity property one could think of, since Grassmannians are homogeneous spaces, and then Kleiman's Lemma applies.

As mentioned before, working with the pliant cone has allowed us to prove that complete intersections are in the interior of the nef cone, which is a key result for many of the features of the pseudoeffective cone.

N.B.: The many positivity properties of the pliant cone make it very rigid. The only property that we do not know for the pliant cone is whether $\pi^*\alpha \in \mathrm{PL}^k(X)$ implies that $\alpha \in \mathrm{PL}^k(Y)$ when π is dominant, a property that is verified by nef divisors, and by nef classes in general. Usually the pliant cone is also very hard to compute, except when we can show that it is equal to one of the other cones, and then we only need to construct sufficiently many pliant classes instead of describing all of them directly.

5.3. **The universally pseudoeffective cone.** We say that $\alpha \in N^k(X)$ is *universally pseudoeffective* if $f^*\alpha \cap [Z] \in \overline{\mathrm{Eff}}_*(Z)$ for any morphism of projective varieties $f : Z \to X$. The cone generated by all such α is denoted $\mathrm{Upsef}^k(X)$.

Features of the universally pseudoeffective cone:

- We have $\mathrm{PL}^k(X) \subseteq \mathrm{Upsef}^k(X) \subseteq \mathrm{Nef}^k(X)$. When X is smooth, we also have $\mathrm{PL}^k(X) \subset \mathrm{BPF}^k(X) \subset \mathrm{Upsef}^k(X) \subset \mathrm{Nef}^k(X)$.
- In particular $\mathrm{Upsef}^k(X)$ generates $N^k(X)$ and is pointed.
- It is preserved by pullbacks.
- $\mathrm{Upsef}^1(X) = \mathrm{Nef}^1(X)$.
- In characteristic zero, to test if $\alpha \in \mathrm{Upsef}^k(X)$, it is enough to check that its pullbacks via maps f that are birational onto their image (which may be a proper subset of X) are pseudoeffective. In arbitrary characteristic, one should replace birational by generically finite, or prove the existence of resolutions of singularities.
- If X is smooth spherical, e.g. smooth toric, then $\mathrm{Upsef}^k(X) = \mathrm{Nef}^k(X) \subset \overline{\mathrm{Eff}}_{n-k}(X)$.
- At least over $\mathbb{C}$, if X is smooth, then $\mathrm{Upsef}^{n-1}(X) = \mathrm{Nef}^{n-1}(X) = \overline{\mathrm{Mov}}_1(X)$.

Motivation: The definition of the universally pseudoeffective cone mimics Kleiman's criterion for nefness. The hope is to show that it is equal to the bpf or pliant cone.

N.B.: At the moment we know very little about this cone. The issue is that the definition is intersection theoretic, and leaves little room for geometric arguments.

6. Questions

Question 6.1. If E is a nef (or ample, meaning that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef or ample) vector bundle of rank $r > 1$ on (smooth) projective X, is it true that $c_k(E) \cap [X]$ is pseudoeffective?

The answer is known to be yes in the following cases:

- $k \in \{0, 1, n-1, n\}$, because the Chern classes are nef (see below).
- When E is globally generated (because then the Chern classes are pliant).
- When E is a split extension of nef line bundles.
- When X is such that $\mathrm{Nef}^k(X) \cap [X] \subset \overline{\mathrm{Eff}}_{n-k}(X)$.

Quite generally it is known that $c_k(E) \in \mathrm{Nef}^k(X)$. This is due to a deep result of Bloch–Gieseker. However the DELV examples of nef non-psef classes suggest that the answer to this question might be no.

Even for ample bundles, the issue is that $c_1^k(E)$ weighs heavier in $c_k(S^mE)$ than $c_k(E)$ as m grows, and so the global generation of S^mE is not helpful. The "splitting principle" is also not an appropriate approach, because in Algebraic Geometry it exhibits a pullback of E as a usually nonsplit extension of line bundles, and then only one of them (the last one on the right) needs to be positive.

Question 6.2. Is it true that any variety of dimension at least 4 is birational to one that supports nef non-psef classes?

One expects that smooth varieties where $\mathrm{Nef}^k(X) \not\subset \overline{\mathrm{Eff}}_{n-k}(X)$ abound, but only the DELV examples on abelian varieties are known at the moment.

Question 6.3. Is it true that if X is smooth then $\overline{\mathrm{Eff}}_{n-k}(X) \cap \mathrm{Nef}^k(X)$ is one of the cones $\mathrm{BPF}^k(X)$, or $\mathrm{Upsef}^k(X)$, or the weaker BPF^k where we only ask that a class α is approximated by classes of cycles that meet any subvariety properly (one different sequence for each subvariety)?

These seem optimistic.

Question 6.4. Is it true that if X is smooth, then $\mathrm{Upsef}^k(X) \subset \overline{\mathrm{Mov}}_{n-k}(X)$?

Question 6.5. If X is smooth, is it true that $\alpha \in \mathrm{Upsef}^k(X)$ if and only if $\alpha \cdot \beta \in \overline{\mathrm{Eff}}_{m-k}(X)$ for any $\beta \in \overline{\mathrm{Eff}}_m(X)$?

How about if $\alpha|_Z \cap [Z] \in \overline{\mathrm{Eff}}_*(Z)$ for any subvariety $Z \subset X$? (Asking that α restricts to a psef class on Z is a stronger condition than asking that the pushforward to X of the restriction to Z is psef).

Question 6.6. Is it true that if $\pi : X \to Y$ is surjective, and $\pi^*\alpha$ is pliant, then α is also pliant?

Question 6.7. Is it true that if X is a G/P, then all the positive cones defined here coincide?

This is true for Grassmannians, and probably at least for the flag manifolds of $GL_n(\mathbb{C})$.

Question 6.8. If $a : X \to A$ is the Albanese map, then is $a^* \overline{\mathrm{Eff}}^k(A) \subset \mathrm{PL}^k(X)$?

How about if we have a map $f : X \to G/P$?

The intention behind the construction of the pliant cone was to collect all the classes that have very good positivity properties and that can be constructed geometrically.

Question 6.9. Is it true that $\overline{\mathrm{Mov}}_k(X)$ is the birationally pliant cone?

We know that pushforwards of bpf classes via birational morphisms span a dense subcone of $\overline{\mathrm{Mov}}_k(X)$.

Question 6.10. Is it true that if $\alpha \in \overline{\mathrm{Eff}}_k(X)$ is extremal and not movable then there exists a subvariety $i : Z \hookrightarrow X$ and $\beta \in \overline{\mathrm{Eff}}_k(Z)$ such that $i_*\beta = \alpha$?

This could help many arguments by induction on dimension via Zariski decompositions.

Question 6.11. If ℓ is the class of a line in $\mathbb{P}^3$, is it true that $\mathrm{mob}(\ell) = 1$? If not, compute it.

Is it true that more generally $\mathrm{mob}(h^k) = (h^n)$ for any ample class on a (smooth) (complex) projective X?

Is there a different and more computable definition of "volume" for numerical classes that is continuous on $N_k(X)$ and positive precisely on the big cone?

[5, Theorem 7.3] shows that $\mathrm{mob}(\ell) \in [1, 3.54)$.

Supporting evidence for the claim on $\mathrm{mob}(h^k)$ is that if $h^k = a^k$ for some ample divisor classes, then $(h^n) = (a^n)$ follows from the Hodge inequalities for intersections of nef classes.

Lazarsfeld suggests that allowing multiplicities at the points in the definition of the mobility count could give a more computable definition for mobility, or for a function with similar properties.

Question 6.12. Do flat, not necessarily l.c.i, pullbacks exist for numerical groups of singular varieties?

Question 6.13. How does Ottem's positivity for subvarieties from here fit into this picture?

The issue is that positivity notions defined on subvarieties rarely descend to numerical equivalence, and it is unclear that they generate $N^k(X)$ even when X is smooth. What is known is that the numerical classes of l.c.i subvarieties with nef normal bundle are themselves nef.

References

[1] Mihai Fulger and Brian Lehmann, *Zariski decompositions of numerical cycle classes*, arXiv:1310.0538 [math.AG]
[2] Mihai Fulger and Brian Lehmann, *Kernels of numerical pushforwards*, in preparation.
[3] Mihai Fulger and Brian Lehmann, *Positive cones of numerical classes*, in preparation.
[4] William Fulton, *Intersection Theory*, (1984)
[5] Brian Lehmann, *Geometric characterizations of big cycles*, arXiv:1309.0880 [math.AG]