INTRODUCTION TO ALGEBRAIC GEOMETRY

Contents

1. Affine Geometry 2
1.1. Closed algebraic subsets of affine spaces 2
1.2. Regular functions 4
1.3. Regular maps 5
1.4. Irreducible subsets 8
1.5. Rational functions 10
1.6. Rational maps 11
1.7. Composition of rational maps 12
2. Projective Geometry 15
2.1. Closed subsets of projective space 15
2.2. Example of projective varieties 17
2.3. Regular functions and regular maps on quasiprojective algebraic sets 20
2.4. Rational functions and rational maps for quasiprojective varieties 24
2.5. Projective algebraic sets are universally closed 25
3. Finite maps 29
3.1. Local study of finite maps 31
4. Dimension 34
4.1. Dimension of intersection with a hypersurface 35
4.2. The dimension of the fibers of a regular map 36
4.3. Lines on surfaces 38
5. Nonsingular varieties 41
5.1. Tangent space 41
5.2. Singular points 47
5.3. Codimension one subvarieties 48
5.4. Nonsingular subvarieties of nonsingular varieties 49
6. Blow-ups 51
6.1. The blow-up of $\mathbb{P}^2$ at one point 51
7. Divisors and Class Group 53
8. Bézout's Theorem 55
8.1. Arbitrary smooth projective surfaces 58
9. Appendix 61
9.1. Classical algebraic structures 61
9.2. Commutative algebra 62
9.3. Topology 72
9.4. Categories 73
References 75

1. Affine Geometry

1.1. **Closed algebraic subsets of affine spaces.** Throughout this course, unless otherwise specified, we work over an *algebraically closed field* $k = \bar{k}$. Let's see what space we will work in (for now):

Definition 1.1. The **n-dimensional affine space over** k is $\mathbb{A}_k^n$. As a set this is just

$$k^n := \underbrace{k \times \ldots \times k}_{n \text{ times}},$$

but we will put more structure on it: a *topology* such that the only *continuous* functions $\mathbb{A}_k^n \to \mathbb{A}_k^1$ are *polynomial*. We also ignore the vector space structure on k^n.

A *polynomial function* on $\mathbb{A}_k^n$ is a *polynomial* $P(X_1, \ldots, X_n)$ with coefficients in k. The set of all such is the *polynomial ring* $k[X_1, \ldots, X_n]$. We may also denote by $P(\underline{X})$ when we don't want to write all indices $X_1, \ldots, X_n$.

Definition 1.2. An **affine algebraic variety**[1] or **closed subset of affine space** is a subset $Y \subset \mathbb{A}_k^n$ given by the vanishing of a family of polynomials $P_i(X_1, \ldots, X_n)$ and we denote $Y = V((P_i)_i)$.

Similarly we write $V(T)$ for the common vanishing locus of all polynomials in a set $T \subset \mathrm{k}[\underline{\mathrm{X}}]$.

We may allow non-algebraically closed fields k in this definition.

Example 1.3. The following are examples of closed algebraic subsets:

- A line in $\mathbb{A}^2$ is $V(aX + bY + c)$.
- In general, a d-dimensional linear subspace of $\mathbb{A}^n$ is given by the simultaneous vanishing of d linear equations.
- A *curve* in the *affine plane* is the set of zeros of one nonzero polynomial $P(X, Y)$.
- The union of closed subsets in $\mathbb{A}_k^n$ is also closed: If $Y_1 = V((P_i)_i)$, and $Y_2 = V((Q_j)_j)$, then $Y_1 \cup Y_2 = V((P_i \cdot Q_j)_{i,j})$.
- The intersection of close subsets in $\mathbb{A}_k^n$ is also closed: If Y_1, Y_2 are as above, then $Y_1 \cap Y_2 = V((P_i)_i, (Q_j)_j)$.
- If $Y = V(f)$ in $\mathbb{A}_k^n$ is a **hypersurface** (given by the vanishing of just one polynomial), then

  $$D(f) := \mathbb{A}_k^n \setminus Y$$

  is also an affine variety... but in $\mathbb{A}_k^{n+1}$. In fact $D(f) = V(X_{n+1}f - 1)$. We'll believe this more when we learn about morphisms and isomorphisms.
- The closed subsets of $\mathbb{A}^1$ are:
  - $\emptyset$.
  - Finite subsets of points.
  - $\mathbb{A}^1$.

  This is because a polynomial of degree n in one variable X has at most n zeros. Also, if $Y = \{x_1, \ldots, x_n\}$ are n points in $\mathbb{A}_k^1$, then $P(X) = (X - x_1) \cdot \ldots \cdot (X - x_n)$ is a polynomial with $Y = V(P)$.

Definition/Theorem 1.4. The **Zariski** topology on $\mathbb{A}_k^n$ is the topology whose closed sets are all the closed algebraic subsets in $\mathbb{A}_k^n$.

[1] Soon, *affine algebraic variety* will mean *irreducible* closed algebraic subset

Proof. What needs checking is that the union of two closed subsets is closed, the intersection of arbitrarily (indexed by any family, not necessarily finite or countable) many closed subsets is closed, and that the empty set and $\mathbb{A}^n_k$ are both closed. All are clear. □

Question. How can we change the equations of an affine variety without changing the variety itself?

Example 1.5. Let $Y = V((P_i)_i)$. If we add to the given list of equations of Y one or arbitrarily many equations of the form $\sum Q_j P_j$, where Q_j are finitely many polynomials in $k[\underline{X}]$ and P_j are among $\{P_i\}_{i\in I}$, then we do not change Y. In particular, if we replace $\{P_i\}_{i\in I}$ by the *ideal* $(P_i)_i$ that they generate inside $k[\underline{X}]$, we still get the same common vanishing locus Y.

Related to this we make the following definition.

Definition/Theorem 1.6. If $Y \subset \mathbb{A}^n_k$ is a subset (usually an affine variety), the **ideal of** Y is the ideal $\mathcal{I}(Y) \trianglelefteq k[\underline{X}]$ containing <u>all</u> the polynomials P such that P vanishes on Y (i.e. $Y \subseteq V(P)$).

Proof. All you need to check is that $I(Y)$ is indeed an ideal, and this is a consequence of the previous example. □

Directly from the definition we see $Y \subseteq V(I(Y))$. In fact equality holds, and several other strong results hold as well.

Theorem 1.7.
(i) $V(a) = V(\sqrt{a})$ *for any* $a \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$.
(ii) $I(Y)$ *is a* ***radical*** *ideal, i.e.* $P^r \in I(Y) \Rightarrow P \in I(Y)$.
(iii) If $Y_1 \subset Y_2$, *then* $I(Y_1) \supset I(Y_2)$.
(iv) If $a \subset b$, *then* $V(a) \supset V(b)$.
(v) More generally, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2) \supseteq I(Y_1) \cdot I(Y_2)$.
(vi) If $Y \subset \mathbb{A}^n_k$ *is an affine variety, then* $Y = V(I(Y))$.
(vii) More generally, if $Y \subset \mathbb{A}^n_k$ *is just a subset, then* $V(I(Y)) = \overline{Y}$ *is the closure of* Y *in the Zariski topology on* $\mathbb{A}^n_k$.
(viii) **Hilbert Nullstellensatz:** *If* $\mathfrak{a} \trianglelefteq k[\underline{X}]$ *is an ideal, then* $\sqrt{\mathfrak{a}} = I(V(\mathfrak{a}))$.

Proof. Since (i) through (v) are easy and $(vii) \to (vi)$, it remains to prove the Nullstellensatz and (vii). We prove the latter and leave $(viii)$ for later.

Clearly $Y \subset V(I(Y))$, hence $\overline{Y} \subseteq V(I(Y))$. Conversely, let W be closed with $W \supseteq Y$. The definition of closed sets implies $W = V(a)$ for some $a \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$. Then $V(a) \supseteq Y$ implies $a \subseteq I(V(a)) \subseteq I(Y)$ and $W = V(a) \supset V(I(Y))$. In particular $V(I(Y))$ is the smallest closed subset containing Y, which is by definition $\overline{Y}$.

□

Corollary 1.8. *There is a natural correspondence given by* $V(\cdot)$ *and* $I(\cdot)$ *between*

$$\{\textit{closed subsets of } \mathbb{A}^n_k\} \rightleftharpoons \{\textit{radical ideals of } \mathrm{k}[\underline{\mathrm{X}}]\}.$$

Corollary 1.9. *If* T *and* S *are two sets of equations (polynomials in* $\mathrm{k}[\underline{\mathrm{X}}]$*). Then they describe the same affine variety* $Y \subset \mathbb{A}^n_k$ *(this means* $Y = V(T) = V(S)$*) iff* $\sqrt{(T)} = \sqrt{(S)}$, *where* (T) *and* (S) *are the ideals generated by* T *and* S *in* $\mathrm{k}[\underline{\mathrm{X}}]$.

Question. OK, going the other way: If we have infinitely many equations for an affine variety Y, can we extract finitely many whose vanishing locus is still exactly Y? More precisely: If $T \subset k[\underline{X}]$ is a an arbitrary subset of polynomials, does there exist a finite subset $\{P_1, \ldots, P_r\} \subset T$ such that $V(P_1, \ldots, P_r) = V(T)$?

The answer is yes, and it comes from the:

Theorem 1.10 (Hilbert Basis Theorem)**.** *Any ideal* $I \trianglelefteq k[\underline{X}]$ *is generated by finitely many elements. Equivalently,* $\mathrm{k}[\underline{\mathrm{X}}]$ *is a Noetherian ring.*

Proof. See appendix §9.2.5. □

Assuming this theorem, then $Y = V(T) = V((T)) = V(P_1, \ldots, P_r)$, where $P_1, \ldots, P_r$ are a finite set of generators of (T).

The theorem also shows that $\mathbb{A}^n_k$ is a **Noetherian space** (decreasing sequences of closed subsets are eventually constant). Indeed if $Y_1 \supset Y_2 \supset \ldots$ is a decreasing sequence of closed algebraic subsets, then $I(Y_1) \subset I(Y_2) \subset \ldots$ is an increasing sequence of ideals of $\mathrm{k}[\underline{\mathrm{X}}]$. Since this is a Noetherian ring, this sequence is eventually constant. But $Y_i = V(I(Y_i))$ for all i and the conclusion follows.

1.2. Regular functions.

Definition 1.11. Let $Y \subset \mathbb{A}^n_k$ be a closed algebraic subset. A function $f : Y \to k$ is called **regular** if there exists a polynomial $F \in \mathrm{k}[\underline{\mathrm{X}}]$ such that $F(y) = f(y)$ for all $y \in Y$.

In the above, if we know F then we know f, but if we know f, then usually there are several options for F. More precisely, if $F|_Y = F'|_Y = f$, then $(F - F')|_Y = 0$, which means that the polynomial $F - F'$ vanishes on Y. By definition, this happens precisely when $(F - F') \in I(Y)$. So two polynomials give the same regular function on Y when they are equl modulo $I(Y)$.

Definition/Theorem 1.12. The set of regular functions on the closed $Y \subset \mathbb{A}^n_k$ is

$$k[Y] := \frac{\mathrm{k}[\underline{\mathrm{X}}]}{I(Y)},$$

which is an algebra of finite type over k.

Example 1.13.
- $k[\mathbb{A}^n_k] = \mathrm{k}[\underline{\mathrm{X}}]/(0) = \mathrm{k}[\underline{\mathrm{X}}]$.
- $k[\emptyset] = \mathrm{k}[\underline{\mathrm{X}}]/(1) = 0$.
- If $Y = (1, \ldots, 1)$ is a point in $\mathbb{A}^n_k$, then $k[Y] = \mathrm{k}[\underline{\mathrm{X}}]/I(Y)$. Observe that $I(Y)$ is the *maximal* ideal $m_{\underline{1}} := (X_1 - 1, \ldots, X_n - 1)$, and $k[Y] = \mathrm{k}[\underline{\mathrm{X}}]/m_{\underline{1}} \simeq k$. The same would be true for any other point in $\mathbb{A}^n_k$.
- If Y is the union of the x and y axes in $\mathbb{A}^2_k$, then $k[Y] = k[x, y]/(xy)$.
- If Y is the hyperbola of equation $xy = 1$, then $k[Y] = k[x, y]/(xy - 1)$. This is isomorphic to $k[x, x^{-1}]$.

□

One should look at $k[Y]$ as an invariant of the affine variety Y. For example the hyperbola $V(xy - 1) \subset \mathbb{A}^2_{\mathbb{C}}$ (that last time we saw it should be a lot like $\mathbb{C}^*$) is not "the same" as $\mathbb{A}^1_{\mathbb{C}}$ because their algebras of regular functions $\mathbb{C}[X, X^{-1}]$ and $\mathbb{C}[X]$ are not isomorphic (Because any isomorphism $\mathbb{C}[X, X^{-1}] \to \mathbb{C}[X]$ needs to send X to something invertible, and there are not many options.)

In fact $k[Y]$ is more than an invariant of Y. It actually determines Y. This is a common instance in mathematics: A space in some category (e.g. topological, differential, holomorphic, analytic, algebraic) is actually determines by the admissible functions defined on it (e.g. continuous, differentiable, holomorphic, analytic, regular).

Theorem 1.14. *There is a one-to-one correspondence between points on a closed subset* $Y \subset \mathbb{A}^n_k$ *and maximal ideals of* $k[Y]$.

Proof. Let's look at $\mathbb{A}^n_k$ first. Here the claim is that the only maximal ideals of $k[\underline{X}]$ are the ones of form $m_{\underline{x}}$, where $m_{\underline{x}} = (X_1 - x_1, X_2 - x_2, \dots, X_n - x_n)$ is the maximal ideal corresponding to the point $\underline{x} = (x_1, \dots, x_n) \in \mathbb{A}^n_k$. (this is indeed maximal, because the $k[\underline{X}]/m_{\underline{x}} \simeq k$ is a field.) So let $m \trianglelefteq k[\underline{X}]$ be an arbitrary maximal ideal, and as such radical. Then $V(m)$ is non-empty (because $m \neq (1)$). Pick some point $\underline{x} \in V(m)$. Then by the Nullstellensatz, we have $I(\underline{x}) \supseteq I(V(m)) = \sqrt{m} = m$, so $m_{\underline{x}} := I(\underline{x}) = m$ by maximality.

For general Y, the maximal ideals of $k[Y] = k[\underline{X}]/I(Y)$ are in a one-to-one correspondence with the maximal ideals of $k[\underline{X}]$ that contain $I(Y)$. And $m_{\underline{x}} \supset I(Y)$ iff $\underline{x} \in Y$. □

Replacing polynomials with regular functions we see that we can perform the constructions of today and of last time on Y instead of $\mathbb{A}^n$. So we can change perspective by changing the ambient space:

If $Y \subset \mathbb{A}^n_k$ is a closed subset, and $T \subset k[Y]$ is a subset of regular functions, we define $V(T) = V_Y(T)$ as the common vanishing locus on Y of the functions from T. Then these $V(T)$'s are the closed sets of a topology on Y. Let's test the compatibilities:

Start with a closed subset $Z \subset Y$ (with respect to the topology on Y). Denote by $I_Y(Z)$ the ideal of regular functions from $k[Y]$ that vanish on Z. This is a radical ideal. Let $\varphi : k[\underline{X}] \to k[\underline{X}]/I(Y) \simeq k[Y]$ be the quotient map. Then $\varphi^{-1} I_Y(Z) \trianglelefteq k[\underline{X}]$ is also radical and its vanishing locus in $\mathbb{A}^n_k$ is Z, so Z is also closed in $\mathbb{A}^n_k$.

Conversely, if $Z \subset \mathbb{A}^n_k$ and $Y \subset \mathbb{A}^n_k$ are closed subsets such that it happens that $Z \subset Y$, we show that Z is closed on Y (in Y's topology). We have $I(Y) \subset I(Z)$. Then $I(Z)/I(Y)$ is a radical ideal of regular functions from $k[Y] = k[\underline{X}]/I(Y)$ and it vanishes precisely on Z, so Z is closed in Y. We have proved:

- That the Zariski topology on Y is the one induced from $\mathbb{A}^n_k$.
- That the previous theorem generalizes to a correspondence between closed subsets of Y and radical ideals of $k[\underline{X}]$ containing $I(Y)$.

Even more, for Z closed in Y closed in $\mathbb{A}^n_k$ we have $k[Z] = k[\underline{X}]/I(Z) = k[Y]/I_Y(Z)$ because of the Third Isomorphism Theorem $\left(k[Y]/I_Y(Z) = \frac{k[\underline{X}]/I(Y)}{I(Z)/I(Y)} \simeq k[\underline{X}]/I(Z).\right)$

It is not hard to see that an appropriate Nullstellensatz also holds in this case: If $\mathfrak{a} \trianglelefteq k[Y]$ where $Y \subset \mathbb{A}^n_k$ is closed, then $\sqrt{\mathfrak{a}} = I_Y(V(\mathfrak{a}))$. (Let $\varphi : k[\underline{X}] \to k[Y]$ be the quotient morphism. By the considerations above and the Nullstellensatz from $\mathbb{A}^n_k$, we have $I_Y(V(\mathfrak{a})) = I(V(\varphi^{-1}\mathfrak{a}))/I(Y) = \sqrt{\varphi^{-1}\mathfrak{a}}/I(Y)$. It is an easy exercise to see that this coincides with $\sqrt{\mathfrak{a}}$.)

The analogue of Theorem 1.7 also holds on Y and $k[Y]$.

1.3. **Regular maps.**

We have learned what a regular function on a closed algebraic subset is. Now let's see what functions we allow between two closed algebraic subsets.

Definition 1.15. A function $\varphi : X \to Y$ between closed algebraic subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively is a **regular map** (or **morphism**) if there exist regular functions $f_1, \dots, f_m \in k[X]$ such that

$$\varphi(\underline{x}) = (f_1(\underline{x}), \dots, f_m(\underline{x})),$$

for any $\underline{x} \in X \subset \mathbb{A}^n_k$ (i.e. φ is given by polynomial functions).

Example 1.16. A regular function $f : X \to k$ is the same as a regular map $f : X \to \mathbb{A}^1_k$.

Example 1.17. If $Y \subset \mathbb{A}^n$ is a closed subset, then the inclusion map $\imath : Y \to \mathbb{A}^n$ is regular.

Example 1.18. The first projection $(x, y) \mapsto x \colon V(xy - 1) \to \mathbb{A}^1$ from the hyperbola to the affine line is a regular map. Note that the image $\mathbb{A}^1 \setminus \{0\}$ is not closed.

Example 1.19. The function $t \overset{\varphi}{\mapsto} (t^2, t^3) : \mathbb{A}^1 \to V(x^3 - y^2)$ is a regular map. It is actually bijective. Its inverse as a function is $(x, y) \overset{\varphi^{-1}}{\mapsto} \begin{cases} \frac{y}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

Let's see that φ^{-1} is not a regular map. Put $Y = V(x^3 - y^2)$ and assume there exists $f \in k[Y]$ such that $f(x, y) = \begin{cases} \frac{y}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ for any $(x, y) \in Y$. Pick $F \in k[X, Y]$ such that $F|_Y = f$. Multiplying with x (In the rational function field $k(X, Y)$ if we're looking for an ambient space), we get $(xF(x, y) - y)|_Y = 0$ when $x \neq 0$. It is easy to check that it also holds when $x = 0$ (Either becayse $x = 0$ implies $y = 0$ on Y, or because $xF(x, y) - y$ must vanish on a closed subset of Y, and $\overline{Y \setminus \{(0, 0)\}} = Y$.) Then $xF - y \in I(V(Y)) = (x^3 - y^2)$ (This is because $k[X, Y]$ is an UFD, and $x^3 - y^2$ is irreducible), so $xF - y = P \cdot (x^3 - y^2)$ for some $P \in k[X, Y]$. Making $x = 0$ we get a contradiction.

However, φ^{-1} is continuous: We check that it returns closed sets to closed sets. Since φ^{-1} is invertible, it is equivalent to verify that φ is *closed* (takes closed subsets to closed subsets). Closed subsets of $\mathbb{A}^1$ are finite sets of points. They map to finite sets of points, and these are always closed. □

Definition 1.20. If X, Y are closed algebraic subsets (of maybe different affine spaces), then $\varphi : X \to Y$ is an **isomorphism** if it is bijective, and φ and φ^{-1} are both regular maps (morphisms).

Example 1.21. The map from the previous example is not an isomorphism even though it is bijective, and actually a homeomorphism for the Zariski topologies. This is because φ^{-1} is not regular.

Example 1.22. The map $t \mapsto (t, t^m) : \mathbb{A}^1 \to V(y - x^m)$ from the affine line to the (generalized) parabola is an isomorphism. Its inverse is $(x, y) \to x$. □

Regular maps "act" on regular functions. If $\varphi : X \to Y$ is a regular map given by regular functions $f_1, \ldots, f_m$ in $k[X]$, and $g \in k[Y]$, we define $\varphi^*(g) \in k[X]$ as the function

$$(x_1, \ldots, x_n) = \underline{x} \overset{\varphi^*(g)}{\mapsto} g(f_1(\underline{x}), \ldots, f_m(\underline{x})).$$

Theorem 1.23. *Throughout $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are closed algebraic subsets.*

a) If $\varphi : X \to Y$ is a regular map, then $\varphi^* : k[Y] \to k[X]$ is a morphism of k-algebras.
b) If $\varphi : X \to Y$ is a regular map, then it is an isomorphism iff $\varphi^* : k[Y] \to k[X]$ is an isomorphism of k-algebras.
c) X and Y are isomorphic iff $k[X]$ and $k[Y]$ are isomorphic as k-algebras.
d) There is a contravariant equivalence of categories between

$$\left\{ \begin{matrix} \textit{closed algebraic subsets} \\ \textit{regular maps between them} \end{matrix} \right\} \rightleftharpoons \left\{ \begin{matrix} \textit{reduced algebras of finite type over } k \\ \textit{morphisms of } k\textit{-algebras} \end{matrix} \right\}$$

To go from left to right, send $X \to k[X]$ and $\varphi \to \varphi^*$. Conversely, if $\mathcal{A}$ is a reduced k-algebra of finite type, then there exists a surjective k-algebra morphism $k[X_1, \ldots, X_n] \to \mathcal{A}$ for some $n \geq 0$. Let $\mathcal{I}$ be the kernel. Putting $Y := V(\mathcal{I}) \subset \mathbb{A}^n$, we have $k[Y] = \mathcal{A}$.

Proof. Part a) is a consequence of the fact that if $y \in Y$ is a point, then the evaluation at y map $g \mapsto g(y) : k[Y] \to k$ is a k-algebra morphism. For the remaining parts of the problem, the only interesting part is proving that if $\psi : k[Y] \to k[X]$ is a morphism of k-algebras, then there exists $\varphi : X \to Y$ a regular map such that $\psi = \varphi^*$. And this is homework.

We check the functoriality of φ^*, i.e. if X, Y, Z are closed algebraic subsets, and $\varphi : X \to Y$ and $\phi : Y \to Z$ are regular maps, then $\phi \circ \varphi$ is also a regular map, and $(\phi \circ \varphi)^* = \varphi^* \circ \phi^*$. For the regularity, observe that if φ is given by regular functions $f_1(\underline{x}), \ldots, f_m(\underline{x})$ in $k[X]$, and ϕ by regular functions $g_1(\underline{y}), \ldots, g_p(\underline{y})$ in $k[Y]$, then $\phi \circ \varphi$ is a regular map given by functions $g_1(f_1(\underline{x}), \ldots, f_m(\underline{x})), \ldots, g_p(f_1(\underline{x}), \ldots, f_m(\underline{x}))$ in $k[X]$. For the composition rule, take $h \in k[Z]$ and observe that

$(\phi \circ \varphi)^*(h) = \varphi^*(\phi^*(h)) = h(g_1(f_1(\underline{x}), \ldots, f_m(\underline{x})), \ldots, g_p(f_1(\underline{x}), \ldots, f_m(\underline{x})))$. In particular, if φ is an isomorphism, then so is φ^*, and $(\varphi^*)^{-1} = (\varphi^{-1})^*$. □

Example 1.24. If $Y \subset \mathbb{A}^n$ is a closed subset and $\imath : Y \to \mathbb{A}^n$ is the inclusion map, then $\imath^*$ is the quotient morphism $\mathrm{k}[\underline{\mathrm{X}}] \to \mathrm{k}[\underline{\mathrm{X}}]/_{\mathcal{I}(Y)}$.

In fact whenever we have a surjective morphism $\varphi^* : k[Y] \to k[X]$ is follows that $\varphi : X \to Y$ is the inclusion of X as a closed subset of Y. $\big(X = V_Y(\ker \varphi^*)\big)$.

Example 1.25. The map $t \overset{\varphi}{\to} (t^2, t^3)$ from the affine line to the cusp is also not an isomorphism because $k[X,Y]/_{(X^3 - Y^2)} \overset{\varphi^*}{\to} k[X]$ is not an isomorphism of k-algebras. (Denoting by x, y the classes of X, Y modulo $(X^3 - Y^2)$, we see that φ^* is determined by $\varphi^*(x) = X^2 \in k[X]$ and $\varphi^*(y) = X^3 \in k[X]$. Then φ^* sends the maximal ideal (x, y) to the ideal $(X^2, X^3) = (X^2) \trianglelefteq k[X]$ which is not maximal, so it cannot be an isomorphism.)

In fact we saw in class that the affine line and the cusp are not isomorphic by *any* morphism between them, because $k[X,Y]/_{(X^3 - Y^2)}$ and $k[X]$ are not isomorphic via *any* k-algebra morphism. This is because every maximal ideal $\mathfrak{m} \trianglelefteq k[X]$ has that $\dim_k \mathfrak{m}/_{\mathfrak{m}^2} = 1$, while for $(x, y) \trianglelefteq k[X,Y]/_{(X^3 - Y^2)}$ the analogous quotient is 2-dimensional with a basis given by the classes of x and y modulo $(x, y)^2$.

We will see later that $\mathfrak{m}/_{\mathfrak{m}^2}$ is the *tangent space* at the point corresponding to $\mathfrak{m}$, so this construction is not unnatural. In fact it is quite geometric: the affine line and the cusp are not isomorphic because the line is *smooth*, while the cusp is *singular* at $(0, 0)$. □

Example 1.26. The k-algebra morphism $X \mapsto x : k[X] \to k[X,Y]/_{(X^3 - Y^2)}$ is φ^* for the projection onto the first component morphism $(x, y) \overset{\varphi}{\mapsto} x$ from the cusp to the affine line (seen as the x-axis).

Similarly $X \mapsto y : k[X] \to k[X,Y]/_{(X^3 - Y^2)}$ corresponds to the second projection $(x, y) \to y$ from the cusp to the affine line (y-axis this time).

Except over 0, the first projection is 2-to-1, while the second is 3-to-1 (even though the real picture suggests one-to-one, there exist nonreal third roots of unity). □

Example 1.27. We saw in Example 1.24 that φ^* surjective corresponds to inclusions of closed subsets. What does φ^* injective correspond to?

Let $h \in k[Y]$ and assume φ is given by $f_1, \ldots, f_m \in k[X]$. Let's see what it means for $h \in k[Y]$ to be in the kernel of φ^*. Well, $\varphi^*(h) = 0$ means $h(f_1(\underline{x}), \ldots, f_m(\underline{x})) = 0$ for all $\underline{x} \in X$. This happens precisely when h vanishes along $\varphi(X) \subset Y$. If φ^* is injective, this is supposed to imply that $h = 0$ on Y. But if the only function vanishing along $\varphi(X) \subset Y$ is the zero function, then $\overline{\varphi(X)} = Y$, i.e. the image of φ is dense in Y (This is the Weak Nullstellensatz). We say that X *dominates* Y.

The example of the hyperbola projecting to the affine line, shows that the image of a regular map could be dense without the regular map being surjective.

Example 1.28. More generally, every morphism $\varphi^* : k[Y] \to k[X]$ can be written as the composition of a surjection with an inclusion of reduced k-algebras of finite type

$$k[Y] \twoheadrightarrow \mathrm{Im}(\varphi^*) \hookrightarrow k[X].$$

On the geometric side, this corresponds with writing $\varphi : X \to Y$ as the composition of a dominant map with an inclusion map:

$$X \to \overline{\varphi(X)} \hookrightarrow Y.$$

□

1.4. **Irreducible subsets.**

Definition 1.29. If $Y \subset X$ is a closed subset, we say that Y is **irreducible** if whenever Y_1 and Y_2 are closed subsets of X such that $Y = Y_1 \cup Y_2$, then either $Y = Y_1$ or $Y = Y_2$.

If $U \subset X$ is any subset, we say that it is **irreducible** if its closure $\overline{U} \subset X$ is irreducible.

If U is not irreducible, we say that it is **reducible**.

If $Y \subset \mathbb{A}^n$ is an irreducible closed algebraic subset, we say that Y is an **affine variety**. In §1 defined an affine variety to be the same as closed algebraic subset, but now we also ask for irreducibility.

Remark 1.30. The definition looks similar to that of a connected subset, and indeed an irreducible subset is connected, but note that we allow Y_1 and Y_2 to have nonemty intersection. For example $V(xy) = V(x) \cup V(y) \subset \mathbb{A}^2$ is connected, but not irreducible.

Example 1.31. $\mathbb{A}^n$ is irreducible as a closed algebraic subset of itself. Note that here it is important that we work with algebraically closed fields. If k is a finite field, then $\mathbb{A}^1$ which can be identified with k is a finite union of points.

Example 1.32. If $U \subset X$ is a nonempty open subset of an affine variety (now irreducible by definition), then U is dense in X. (We have $X = (X \setminus U) \cup \overline{U}$ is a union of closed subsets. Now use the definition.)

Example 1.33. If $f \in \mathrm{k}[\underline{X}]$ is a reducible polynomial that is not a power of an irreducible polynomial, then $V(f) \subset \mathbb{A}^n$ is not irreducible. (If f is as above, then we can write $f = gh$ for some nonconstant polynomials g, h without common factors in the UFD (cf. §9.2.8) $\mathrm{k}[\underline{X}]$. Then $V(f) = V(g) \cup V(h)$. We have $V(f) \supsetneq V(g)$ because otherwise $g \in I(V(f))$ implies by the Nullstellensatz $g^m \in (f)$ for some $m \geq 0$, and so there exists $l \in \mathrm{k}[\underline{X}]$ such that $g^m = fl = ghl$ which is a contradiction because g and h have no common factors.)

Example 1.34. If $\varphi : X \to Y$ is a regular map of closed algebraic subsets and X is irreducible, then so is $\varphi(X)$. (If $\overline{\varphi(X)} = V_1 \cup V_2$ is a union of closed sets, then $X = \varphi^{-1}(V_1) \cup \varphi^{-1}(V_2)$ is also a union of closed subsets (potentially empty because φ is not assumed to be surjective). Since X is irreducible, it is equal to one of them. Say $X = \varphi^{-1}(V_1)$. Then $V_1 \supset \varphi(\varphi^{-1}(V_1)) = \varphi(X)$, hence $V_1 \supset \overline{\varphi(X)}$, since V_1 is closed.)

Let's see what is on the algebraic side.

Theorem 1.35. *Let $Y \subset X$ be a closed subset of a closed subset of the affine space. Then Y is irreducible iff $\mathcal{I}_X(Y) \trianglelefteq k[X]$ is prime, iff $k[Y]$ is a domain.*

Proof. These are simple consequences of the dictionary that we have between geometry and algebra.

Say $\mathcal{I}(Y) := \mathcal{I}_X(Y)$ is prime, and write $Y = Y_1 \cup Y_2$ as a union of closed subsets. Then $\mathcal{I}(Y) = \mathcal{I}(Y_1) \cap \mathcal{I}(Y_2)$. But it is impossible to write a prime ideal as an intersection of two radical ideals unless the prime ideal is one of them: If $y_1 \in \mathcal{I}(Y_1) \setminus \mathcal{I}(Y_2)$ and $y_2 \in \mathcal{I}(Y_2) \setminus \mathcal{I}(Y_1)$, then $y_1 y_2 \in \mathcal{I}(Y_1) \cdot \mathcal{I}(Y_2) \subset \mathcal{I}(Y_1) \cap \mathcal{I}(Y_2) = \mathcal{I}(Y)$ gives a contradiction. Therefore either $\mathcal{I}(Y) = \mathcal{I}(Y_1) \subset \mathcal{I}(Y_2)$ hence $Y = Y_1$, or $\mathcal{I}(Y) = \mathcal{I}(Y_2) \subset \mathcal{I}(Y_1)$ hence $Y = Y_2$.

Say Y is irreducible, and let $fg \in \mathcal{I}_X(Y)$. In particular $fg|_Y = 0$. Then $Y = V_Y(f) \cup V_Y(g)$ is a union of closed subsets, and by the definition of irreducibility, $Y = V_Y(f)$ or $Y = V_Y(g)$. Then $f|_Y = 0$ or $g|_Y = 0$, hence f or g are in $\mathcal{I}_X(Y)$.

The equivalence between $\mathcal{I}_X(Y) \trianglelefteq k[X]$ being prime and $k[Y] = k[X]/\mathcal{I}_X(Y)$ being a domain is a classical algebraic result. See §9.2.6. □

Example 1.36. If $X = V(f) \subset \mathbb{A}^n$ for some irreducible $f \in \mathrm{k}[\underline{X}]$, then X is irreducible. (We show in §9.2.8 that $\mathrm{k}[\underline{X}]$ is an UFD. In an UFD, irreducible elements are prime, therefore $(f) \trianglelefteq \mathrm{k}[\underline{X}]$ is prime. In particular $(f) = \sqrt{(f)} = \mathcal{I}(V(f))$, hence $V(f)$ is irreducible.)

In fact $V(f) \subset \mathbb{A}^n$ is irreducible iff f is a power of an irreducible polynomial in $\mathrm{k}[\underline{X}]$. (Use the previous paragraph and Example 1.33.)

Example 1.37. It was important that $\mathrm{k}[\underline{X}]$ was an UFD in the previous example. Consider $R := k[x,y,z]/(x^2 - yz)$. This is a domain, and it is the ring of regular functions for the cone $Y = V(x^2 - yz)$. Furthermore x is irreducible, but not prime (hence R is not an UFD), and $V_Y(x) = V_Y(x^2) = V_Y(yz) = V_Y(y) \cup V_Y(z)$ is the union of the lines through $(0,0,1)$ and $(0,1,0)$, hence not an irreducible closed subset. (To see that x is irreducible, use that R is a graded domain. This is because $k[x,y,z]$ is graded and $(x^2 - yz)$ is a homogeneous prime ideal.)

Example 1.38. If $\mathcal{I} \trianglelefteq k[X]$ is a prime ideal, then $Y := V_X(\mathcal{I}) \subset X$ is irreducible. (This is because if $\mathcal{I}$ is prime, then $\mathcal{I}_X(V_X(\mathcal{I})) = \sqrt{\mathcal{I}} = \mathcal{I}$, hence $\mathcal{I}_X(Y)$ is prime.)

Definition 1.39. A closed subset $Z \subset Y$ of a closed algebraic subset of an affine space is an **irreducible component** of Y if Z is irreducible and $Z \not\subset \overline{Y \setminus Z}$.

Proposition 1.40. *Any closed algebraic subset Y of an affine space has only finitely many irreducible components $Y_1, \dots, Y_r$ and $Y = \cup_{i=1}^r Y_i$. This decompositions is unique (among decompositions of Y as a finite union of irreducible closed subsets W_j such that $W_j \not\subset W_{j'}$ for $j \neq j'$).*

Proof. One can give a proof by contradiction using that Y is a Noetherian topological space as in the book, or we can apply the primary ideal decompositions from §9.2.7 to the radical ideal $\mathcal{I}(Y)$ to write it uniquely as an intersection of minimal prime ideals. □

1.4.1. *Products.* If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are closed algebraic subsets given by equations $f_i \in k[X_1, \dots, X_n]$ and $g_j \in k[Y_1, \dots, Y_m]$ respectively, then $X \times Y$ is a closed algebraic subset of $\mathbb{A}^{n+m}$. It is given by the equations f_i and g_j seen as polynomials (or regular functions) in the bigger ring $k[X_1, \dots, X_n, Y_1, \dots Y_m]$.

Remark 1.41. The Zariski topology on $X \times Y$ is the one induced from $\mathbb{A}^{n+m}$, and this is different from the product topology as we have seen in homework. For example the diagonal $V(x - y)$ is closed in $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, but it is not closed in the product topology (Its complement is not a (finite) union of open subsets of form $U_i \times U_i'$, where $U_i, U_i' \subset \mathbb{A}^1$ are open.) □

The ring of regular functions on $X \times Y \subset \mathbb{A}^{n+m}$ is

$$k[X \times Y] \simeq k[X] \otimes_k k[Y] = \frac{k[\underline{X}, \underline{Y}]}{\mathcal{I}_{\mathbb{A}^n}(X)k[\underline{X}, \underline{Y}] + \mathcal{I}_{\mathbb{A}^m}(Y)k[\underline{X}, \underline{Y}]},$$

where $k[\underline{X}, \underline{Y}] = k[X_1, \dots, X_n, Y_1, \dots, Y_m]$.

The projections $(x, y) \overset{p_1}{\mapsto} x : X \times Y \to Y$ and the corresponding $p_2 : X \times Y \to Y$ are regular maps, and p_1^* is the identification of $k[X]$ with $k[X] \otimes_k 1$ in $k[X \times Y]$, while p_2^* is the identification of $k[Y]$ with $1 \otimes_k k[Y]$.

Any regular map $\varphi : X \to Y$ admits a factorization

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_\varphi} & X \times Y \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle p_2} \\ & & Y \end{array}$$

where Γ_φ is the *graph morphism* $x \mapsto (x, \varphi(x)) : X \to X \times Y$ and p_2 is the second projection $X \times Y \to Y$ as above.

Proposition 1.42. *If X and Y are irreducible, then so is $X \times Y$.*

Proof. Assume $X \times Y = V_1 \cup V_2$ is a union of closed subsets. For $y \in Y$, denote $X_y = p_2^{-1}\{y\}$. This a copy of X_y sitting over $y \in Y$ inside $X \times Y$. Since X is irreducible, and $X_y = (X_y \cap V_1) \cup (X_y \cap V_2)$, we have $X_y \subset V_1$ or $X_y \subset V_2$ for each y (although being in V_1 or V_2 may change as we change y).

Denote $Y_1 = \{y \in Y \mid X_y \subset V_1\}$ and define Y_2 analogously. By the remark above, $Y = Y_1 \cup Y_2$. By the irreducibility of Y, we have $Y = \overline{Y}_1$ or $Y = \overline{Y}_2$. Say $Y = \overline{Y}_1$. Then V_1 contains the open subset $X \times (Y \setminus \overline{Y \setminus Y_1})$.

Let $f \in k[X \times Y]$ be a polynomial that vanishes along V_1. Then it vanishes along $X \times (Y \setminus \overline{Y \setminus Y_1})$. In particular for each $x \in X$, it vanishes along $\{x\} \times (Y \setminus \overline{Y \setminus Y_1})$, hence also along its closure in $Y \times X$ which is Y_x. Since this happens for all $x \in X$, we obtain that $f = 0$. This implies that V_1 is dense in $X \times Y$, and since it was closed to begin with we get $X \times Y = V_1$. This concludes the proof. □

1.5. **Rational functions.** When X is an affine variety, which since last time also means irreducible, then $k[X]$ is a domain and we can talk about its fraction field that we denote $k(X)$.

Definition 1.43. A **rational function** on the affine variety X is an element of $k(X)$.

Every rational function is then a ratio $\frac{f}{g}$ of a regular function f by a nonzero (meaning not zero everywhere, but maybe somewhere) regular function g.

Remark 1.44. Because they are elements of the field of fractions, $\frac{f}{g} = \frac{f'}{g'}$ in $k(X)$ if and only if $fg' = f'g$ in $k[X]$.

Example 1.45. $\frac{1}{x}$ is a rational function on $\mathbb{A}^1$.

Observe that rational functions are not necessarily defined over the entire X. In the preceding example, $\frac{1}{x}$ is not defined at $x = 0$. But where is a rational function defined? If we write $h \in k(X)$ as $\frac{f}{g}$, then a temporary answer is that h is defined where g does not vanish, and indeed it is defined *at least* at those points.

Definition 1.46. Let $h \in k(X)$. We say that h is **defined** (or **regular**) at a point $x \in X$ if there exist $f, g \in k[X]$ such that $h = \frac{f}{g}$ *and* $g(x) \neq 0$.

There is however an issue that makes finding where rational functions are defined a subtle question: The representation $h = \frac{f}{g}$ is not unique. A simple example is $\frac{x-1}{x^2-x} = \frac{1}{x}$, and if we look at the first formula we are tempted to say that the function is not defined at 0 and 1. But if we use the second formula, then it is defined at 1. In this particular case we would "simplify" as much as possible and then look at where the denominator does not vanish. In general though, more specifically when working with rings that are not UFD, then an "optimal" representation does not exist.

Example 1.47. If $X = V(x^3 - x^2 + y^2 - y)$, then X is irreducible, and $\frac{y-1}{x} = \frac{x-x^2}{y}$ because of the formula $x^3 - x^2 + y^2 - y = 0$. The first formula says that the function $h := \frac{y-1}{x}$ is defined when $x \neq 0$, which means $(x, y) \notin \{(0, 0), (0, 1)\}$, while the second formula says that the function h (with a different formula) is defined when $y \neq 0$, which means when $(x, y) \notin \{(0, 0), (1, 0)\}$. Neither of these is optimal. We actually put them together instead of picking the larger one to decide that h is defined outside $(0, 0)$. □

Definition 1.48. Let $h \in k(X)$ be a rational function on X. Then the **domain of definition** of h is the union of all open subsets $D(g) = X \setminus V(g)$, where g varies over *all* denominators of representations $h = \frac{f}{g}$ with $f, g \in k[X]$ and $g \neq 0$.

A rephrasing is that $x \in X$ is in the domain of definition of h if h is defined at x

Remark 1.49. The domain of definition of a rational function is open (because an arbitrary union of open sets is open in any topological space). Since we are working in a *Noetherian* topological space, any cover by open sets admits a finite subcover. This means that the domain of h is actually a union of finitely many $D(g_i)$ in the previous definition.

Proposition 1.50. *Two rational functions $h, h' \in k(X)$ are equal if and only if they are both defined and agree on a nonempty open subset $U \subset X$.*

Proof. If $r := h - h'$ it comes down to showing that if $r|_U = 0$, then $r = 0$. Write $r = \frac{f}{g}$. Then $r|_U = 0$ implies $f|_U = 0$, which in turn means $U \subset V(f)$. This implies $V(0) = X = \overline{U} \subseteq V(f)$. (We saw last time that a nonempty open subset of an irreducible space is dense). But $V(0) = V(f)$ implies $f = 0$ by the Nullstellensatz. □

The domain of definition gives us a way of testing when a rational function is regular.

Proposition 1.51. *Let X be an affine variety. If the domain of definition of $h \in k(X)$ is the entire X, i.e. h is defined everywhere, then h is actually a regular function, i.e. $h \in k[X]$. The converse is immediate.*

Proof. The domain of h is the union of all $D(g)$ with $h = \frac{f}{g}$ for some $f, g \in k[X]$, and $g \neq 0$. The complement of this union is the intersections of the complements, which means the intersection of all $V(g)$, which we know is $V(\mathfrak{a})$, where $\mathfrak{a}$ is the ideal generated by all such g. If the domain of h is X, this complement is empty. But $V(\mathfrak{a}) = \emptyset$ only when $\mathfrak{a} = (1)$. By the definition of $\mathfrak{a}$, there exist finitely many nonzero $g_i \in k[X]$ and correspondingly finitely many $f_i \in k[X]$ with $h = \frac{f_i}{g_i}$ for all i, but also finitely many $r_i \in k[X]$ such that $\sum_i r_i g_i = 1$. Multiplying this by h we get $h = \sum_i r_i f_i$. There are no longer any fractions in this expression, hence h is regular. □

Remark 1.52. Given finitely many rational functions $h_i \in k(X)$, there exists a nonempty open subset $U \subset X$ where all h_i are defined. (The domain of definition of each h_i is open and nonempty. It is enough to show that the intersection finitely many nonempty open subsets is always open nonempty. Indeed if U_i are nonempty open sets and $V_i := X \setminus U_i$ are their complements, then if $\cap_i U_i = \emptyset$, then $\cup_i V_i = X$. The irreducibility of X then says that X is one of the V_i's, which was excluded by the nonemptyness of each U_i)

We were able to understand regular functions on X as polynomial functions restricted to X. Let's see about rational functions: Say $h = \frac{f}{g}$ with $f, g \in k[X]$ and $g \neq 0$. Choose $F, G \in \mathrm{k}[\underline{\mathrm{X}}]$ polynomial functions that restrict to f, g. The condition $g \neq 0$ is equivalent to $G \notin \mathcal{I}(X)$. So $h(x) = \frac{F(x)}{G(x)}$ whenever $x \in X$, but $x \notin V(G)$. Moreover $h = 0$ if and only if $F|_X = 0$, i.e. $F \in \mathcal{I}(X)$. Then we have the following presentation of rational functions:

Proposition 1.53. *Let $\mathcal{O}_X$ be the subring of $k(\underline{X})$ generated by elements of form $\frac{F}{G}$ with $G \notin \mathcal{I}(X)$, and let $\mathfrak{m}$ be the set of such functions such that $F \in \mathcal{I}(X)$. Then $\mathfrak{m} \trianglelefteq \mathcal{O}_X$ and $k(X) \simeq \mathcal{O}_X/_{\mathfrak{m}}$. In particular $\mathfrak{m}$ is a maximal ideal.*

Proof. It is easy to check that $\mathcal{O}_X$ is a ring and that the function $\frac{F}{G} \mapsto \frac{F}{G}|_X : \mathcal{O}_X \to k(X)$ is a ring morphism. Its kernel is all fractions $\frac{F}{G}$ such that $F|_X = 0$, which is by definition $\mathfrak{m}$. Conclude by the first isomorphism theorem. □

Localization can help phrase this as $k(X) \simeq \mathrm{k}[\underline{\mathrm{X}}]_{(\mathcal{I}(X))}/\mathcal{I}(X)\, \mathrm{k}[\underline{\mathrm{X}}]_{(\mathcal{I}(X))}$, where $\mathrm{k}[\underline{\mathrm{X}}]_{(\mathcal{I}(X))}$ is the *localization of* $\mathrm{k}[\underline{\mathrm{X}}]$ at the prime ideal $\mathcal{I}(X)$ (prime because X is irreducible).

1.6. **Rational maps.** Recall that a regular map was given by finitely many regular functions. We do the same to define rational maps.

Definition 1.54. Let X be an affine variety, and consider $f_1, \ldots, f_m \in K(X)$ rational functions. They define a **rational map** $\varphi : X \dashrightarrow \mathbb{A}^m$ by the formula $\varphi(x) = (f_1(x), \ldots, f_m(x))$ valid when all f_i are *defined* at x. We say that φ is **regular** at x.

If $Y \subset \mathbb{A}^m$ is a closed subset, a rational map $\varphi : X \dashrightarrow Y$ is rational map $X \dashrightarrow \mathbb{A}^m$ such that $\varphi(x) \in Y$ for all $x \in X$ where φ is defined.

Definition 1.55. A rational map $f : X \dashrightarrow \mathbb{A}^1$ is the same as a rational function $f \in k(X)$.

Remark 1.52 tells us that a rational map is defined on a nonempty open subset $U \subset X$. For example one can take the intersection of the domains of definition of all f_i's. Recall that nonempty open subsets of irreducible spaces are dense.

Definition 1.56. The **domain of definition** of a rational map $\varphi = (f_1, \ldots, f_m)$ is the intersection of the domains of definition of the f_i's.

The **image** $\varphi(X)$ of φ is the set of all $\varphi(x)$, where x is in the domain of definition of φ.

Example 1.57. Consider the *stereographic projection* in $\mathbb{A}^2$ from the origin $(0,0)$ to the line $x=1$. This is the rational function $p : \mathbb{A}^2 \dashrightarrow \mathbb{A}^1$ that computes slopes of points, i.e. $p(x,y) = \frac{y}{x}$. Then p is defined at every point except on the line $x = 0$. □

Remark 1.58 (When are rational maps equal?)**.** Say φ and ψ are rational maps from the affine variety X given by rational functions f_i and g_i in $k(X)$. Then $\varphi = \psi$ if $f_i = g_i$ for all i as elements of $k(X)$. At the level of functions, using Lemma 1.50, this is the same as asking that there exists $U \subset X$ open contained in the domain of definition (e.g. the intersection of the domain of definition of all f_i and g_i) of both φ and ψ such that $\varphi|_U = \psi|_U$ as functions, meaning $\varphi(x) = \psi(x)$ for all $x \in U$.

Example 1.59. Let $X \subset \mathbb{A}^n_k$ be an affine variety and let $\varphi : X \dashrightarrow Y$ be a rational map. Then $\varphi = 1_X$ as rational maps if and only if φ is a regular map and $\varphi = 1_X$ as regular maps.

Proof. The other implication being clear, let's assume that $\varphi = 1_X$ as ratinonal maps. By definition φ is given by n rational functions $f_1, \dots, f_n \in k(X)$. By the previous remark, if $\varphi = 1_X$ as rational maps, then $f_i = x_i$ for all i, where x_i is the restriction to X of the i-th coordinate function X_i on $\mathbb{A}^n_k$. Then f_i is regular and equal to x_i for all i not just as rational functions, but as regular functions in $k[X]$. Then φ is a regular map and $\varphi = 1_X$. □

1.7. **Composition of rational maps.** Funny things can happen when we have rational maps $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$. (Note that by the definition of a rational map, this forces both X and Y to be varieties, i.e. irreducible.) For example the image of φ could land outside the domain of ψ, and then $\psi \circ \varphi$ doesn't make sense anywhere.

Example 1.60. We cannot compose the stereographic projection from $(0,0)$ to $V(x-1)$ with the stereographic projection from $(1,0)$ to $V(x)$, both seen as rational maps $\mathbb{A}^2 \dashrightarrow \mathbb{A}^2$. Neither of them is defined on the image of the other. □

To solve this issue we ask that the domain of ψ meets the image of φ. Note that the image of φ is irreducible, in particular the intersection with the domain of ψ is dense under our assumption that it is nonempty.

Definition 1.61. Let $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ be rational maps with X and Y affine varieties, and Z a closed algebraic subset of some affine space. if $\varphi(X)$ meets the domain of ψ, then we define the **composition** $\psi \circ \varphi : X \dashrightarrow Z$. This is defined on the inverse image through φ of the domain of ψ wich is nonempty by assumption.

Recall that if $\varphi : X \to Y$ was a regular map, we defined a *pullback* morphism $\varphi^* : k[Y] \to k[X]$ by $\varphi^*(f)(x) = f(\varphi(x))$ for any regular function $f : Y \to k$. This is the same as $\varphi^*(f) = f \circ \varphi$, with f seen now as a regular *map* (as opposed to function) $f : Y \to \mathbb{A}^1$.

We try to do the same for a rational function $\varphi : X \dashrightarrow Y$. If we want to talk about rational functions on Y, then Y must also be irreducible. We would like the composition $f \circ \varphi$ to be defined for *all* rational functions $f \in k(Y)$, which means that we want $\varphi(X)$ to meet the domain of definition of *every* rational function on Y. As the following lemma shows, this is only possible when $\varphi(X)$ is dense in Y.

Lemma 1.62. *Let Y be an affine variety. If a subset $T \subset Y$ (think $\varphi(X)$ which doesn't have to be neither open, nor closed, nor open in its closure) intersects the domain of definition of* every *rational function $f \in k(Y)$, then T is dense in Y.*

Proof. Assume that T is not dense. Then $U := Y \setminus \overline{T}$ is open and nonempty. Let $g \in \mathcal{I}_X(\overline{T}) \trianglelefteq k[X]$ be a nonzero function. A nonzero function exists because $\mathcal{I}_X(\overline{T}) = 0$ if and only if $\overline{T} = X$, and we assumed that this is not the case. Then $\overline{T} \subset V(g)$, hence by passing to complements we have $D(g) \subset U$. Let's show that $D(g)$ is the domain of definition of some rational function f. This will show that T does not meet the domain of f, which is a contradiction.

The first guess would be $f = \frac{1}{g}$, but we know we have to be careful to check that f is not defined (by any other representation as a fraction) at any point of $V(g)$. Assume that f is defined at $x \in V(g)$. Then there exist $u, v \in k(X)$ with $v(x) \neq 0$ such that $\frac{1}{g} = f = \frac{u}{v}$. This means $v = ug$, and if we evaluate at x we get $v(x) = u(x)g(x) = u(x) \cdot 0 = 0$, because $x \in V(g)$. This is a contradiction. □

Definition 1.63. Let $\varphi : X \dashrightarrow Y$ be a rational map between affine varieties. We say that φ is **dominant** if $\varphi(X)$ is dense in Y.

Definition 1.64. Let $\varphi : X \dashrightarrow Y$ be a dominant rational map between affine varieties. The **pullback** $\varphi^* : k(Y) \to k(X)$ is the field morphism defined by $\varphi^*(f) = f \circ \varphi$ for all $f \in k(Y)$.

Since morphisms between fields are injective (actually any morphism of rings from a field is injective, unless the target is the 0 ring), this means that φ^* is injective. Now that we know what a composition of rational maps is, we can talk about what "isomorphism" should mean for this kind of maps.

Definition 1.65. The affine varieties X and Y are **birational** if there exist dominant rational maps $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ such that $\varphi \circ \psi = 1_Y$ and $\psi \circ \varphi = 1_X$.

Then φ and ψ are called **birational isomorphisms**, or just **birational**, and they are **inverses** (birationally) of each other.

Example 1.66.
i) If X and Y are isomorphic, then they are also birational. The converse may fail. See examples below.
ii) Let $\phi : \mathbb{A}^2 \to \mathbb{A}^1$ be the stereographic projection from $(0,0)$ onto the line $V(x-1) \subset \mathbb{A}^2$. Let $X = V(x^3 - y^2)$ be the cusp and let $\varphi : X \dashrightarrow \mathbb{A}^1$ be the restriction $\varphi := \psi|_X$. Then φ is birational. An inverse is the morphism (regular map) $\psi : \mathbb{A}^1 \to X$ given by $\psi(t) = (t^2, t^3)$. Observe though that φ itself is not regular as it is not defined at the origin. Also, if φ was regular, then X and $\mathbb{A}^1$ would be isomorphic, which we know they are not.
iii) Similarly the node $V(x^3 - x^2 + y^2)$ is birational to $\mathbb{A}^1$.

Just like with regular maps, birational isomorphism can be checked with algebra.

Theorem 1.67. *Let X and Y be affine varieties and let $\varphi : X \dashrightarrow Y$ be a dominant rational map. Then φ is a birational isomorphism if and only if $\varphi^* : k(Y) \to k(X)$ is an isomorphism of k-algebras (and fields).*

Moreover, X and Y are birational if and only if there exists $\phi : k(Y) \to k(X)$ an isomorphism of k-algebras (and fields).

Proof. Just like for regular maps, the first part boils down to checking that the pullback is *functorial*, meaning that it respects composition: if $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ are rational maps that can be composed, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. And also $(1_X)^* = 1_{k(X)}$. These are proved just like for regular maps.

If $Y \subset \mathbb{A}^m$, denote the coordinate functions $y_1, \ldots, y_m \in k[Y] \subset k(Y)$. Define $\varphi : X \dashrightarrow Y$ by $\varphi = (\phi(y_1), \ldots, \phi(y_m))$. This is only a rational map because we don't know that $\phi : k(Y) \to k(X)$ sends $k[Y]$ to $k[X]$. One constructs analogously $\psi : Y \dashrightarrow X$ from $\phi^{-1} : k(X) \to k(Y)$. To check that φ and ψ are birational isomorphisms, inverses to each other, by the composition tricks in the previous paragraph, it is enough to check that if $X = Y$, then $\varphi = 1_X$ as rational maps if and only if $\varphi^* = 1_{k(X)}$. This is easy. □

Example 1.68. $\mathbb{A}^2$ and $\mathbb{A}^1$ are not birational. This is because their function fields $k(X)$ and $k(X,Y)$ are not isomorphic as k-algebras. (They have different *transcendence degrees* over k. The first is 1, the second is 2.)

Example 1.69.
i) The map $(x,y) \mapsto (x, xy) : \mathbb{A}^2 \to \mathbb{A}^2$ is a birational isomorphism. (This is homework.)
ii) $t \mapsto t^2 : \mathbb{A}^1 \to \mathbb{A}^1$ is not birational. (X is not in the image of $X \mapsto X^2 : k(X) \to k(X)$.)
iii) The *elliptic* curve $V(x^3 - x + y^2)$ and the affine line $\mathbb{A}^1$ are not birational. (Also homework.).

Definition 1.70. If X is an affine variety birational to some $\mathbb{A}^n$, we say that X is **rational**.

We will see at some point that n in the above definition is uniquely determined if it exists (it is the *dimension* of X). The motivation for studying rational varieties X is that a birational isomorphism $\mathbb{A}^n \dashrightarrow X$ gives a *parameterization* of X. In fact we are usually happy when this map is just dominant and then we call X **unirational**. There exist nonrational unirational varieties, but the examples are not that easy.

Example 1.71. i) If $f \in k[\underline{X}]$ is irreducible of total degree 2, then $V(f)$ is rational. (The stereographic projection from a point $x \in V(f)$ to some linear hyperplane in $\mathbb{A}^n$ that does not pass through x restricts to a birational isomorphism $V(f) \dashrightarrow \mathbb{A}^{n-1}$. This is because lines through x in $\mathbb{A}^n$ intersect $V(f)$ in at most one more point other than x.)

ii) $t \to (t^2, t^3)$ parameterizes the cusp $V(x^3 - y^2)$ birationally.

iii) $t \to (t^2+1, t(t^2+1))$ parameterizes the node $V(x^3 - x^2 + y^2)$ birationally.

iv) $t \to t^2$ parameterizes $\mathbb{A}^1$, but not birationally.

Example 1.72. $X := V(x^3 + y^3 + z^3 - 1) \subset \mathbb{A}^3$ is rational if the characteristic of k is not 3. (X contains two skew (not in a same plane) lines $\ell_1 := V(x+y,\ z)$ and $\ell_2 := V(x+\epsilon y,\ z-\epsilon)$, where $\epsilon^3 = 1$, but $\epsilon \neq 1$ (such ϵ exists because $x^3 - 1 = 0$ has nonzero discriminant if the characteristic is not 3).

For any pair of points $x_i \in \ell_i$, the line through x_1 and x_2 intersects X at at most one point (if this line is parameterized by $t \to (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3)$, then to compute the intersection with X, we plug these into the equation of X, and the result is a polynomial of degree 3 in t which has 3 roots, but sometimes they have multiplicities). This determines a rational map $\varphi : \ell_1 \times \ell_2 \dashrightarrow X$.

To construct the inverse, recall that if we have two skew lines ℓ_1 and ℓ_2 in space, then for x in space, not on either of the lines, there exists a unique line through x that intersects the lines ℓ_1 and ℓ_2. This way we get a rational map $\psi : X \dashrightarrow \ell_1 \times \ell_2$ that sends x to the intersection of this line through x with the lines ℓ_1 and ℓ_2 respectively.

The uniqueness of the line in the previous paragraph shows that φ and ψ are inverses to each other. We are then done because $\ell_1 \times \ell_2$ is isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{A}^2$.)

Theorem 1.73. *Every affine variety is birational to a hypersurface of some* $\mathbb{A}^{d+1}$.

Proof. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $K = k(X)$ be its function field. Then K is finitely generated over k (for example by $x_1, \ldots, x_n$, the restrictions of the coordinate functions on $\mathbb{A}^n$ to X). Let d be the maximal number of elements in a subset of $\{x_1, \ldots, x_n\}$ that are algebraically independent over k. The Theorem of the Primitive Element then tells us that there exist $t_1, \ldots, t_d, t_{d+1}$ such that $t_1, \ldots, t_d$ are algebraically independent over k, and $K = k(t_1, \ldots, t_d)(t_{d+1})$ is the finite extension over $L := k(t_1, \ldots, t_d)$ generated by the algebraic (over L) element t_{d+1}. In particular t_{d+1} is the root of some irreducible polynomial $f \in L[T]$. Up to clearing denominators, we can assume that f has coefficients in $k[t_1, \ldots, t_d]$, and it is still irreducible. If we include T, then f is an irreducible element of $k[t_1, \ldots, t_d, T]$ which is a polynomial ring in $d+1$ variables. Then $Y := V(f)$ is a hypersurface of some affine space and $k(Y) \simeq K = k(X)$. □

Remark 1.74. In fact, the t_i's in the previous proof can be chosen as some linear combinations (with coefficients in k) of the x_j's. Moreover, the Theorem of the Primitive Element actually guarantees that t_{d+1} can be chosen to be separable over L, so that the inclusion $L \subset K$ is finite and separable.

2. Projective Geometry

2.1. Closed subsets of projective space.

Definition 2.1. The projective space $\mathbb{P}^n_k$ is the set of equivalence classes $\mathbb{A}^{n+1} \setminus \{(0,\dots,0)\}/\sim$, where $(a_0,\dots,a_n) \sim (b_0,\dots,b_n)$ iff there exists $\lambda \in k$ (automatically nonzero) such that $b_i = \lambda a_i$ for all i.

Remark 2.2. $\mathbb{P}^n$ can be identified with

- $\mathbb{A}^{n+1} \setminus \{(0,\dots,0)\}/k^*$.
- The set of lines in $\mathbb{A}^{n+1}$ through the origin (two points are equivalent iff they are on the same line through the origin).
- The set of linear hyperplanes in $\mathbb{A}^{n+1}$ through the origin (a hyperplane through the origin is given by an equation $a_0x_0 + \ldots + a_nx_n$, but the coefficients a_i are determined only up to multiplication by a nonzero scalar).
- The set of one-dimensional quotients $k^{n+1} \twoheadrightarrow k$ up to multiplication by nonzero scalars (the kernel of such is a linear hyperplane).

Definition 2.3. The class of $(a_0,\dots,a_n)$ from $\mathbb{A}^{n+1}\setminus\{(0,\dots,0\}$ in $\mathbb{P}^n$ is denoted $[a_0 : \ldots : a_n]$. We call these **homogeneous coordinates**.

We want to treat $\mathbb{P}^n$ like we did $\mathbb{A}^n$. We want to put a Zariski topology on it and talk about ideals, algebra of regular functions, and fields of rational functions. First we should understand what $V(f)$ should be. A first problem is that, except for constant polynomials, we cannot evaluate a polynomial $f \in k[X_0,\dots,X_n]$ at all projective points $[a_0 : \ldots : a_n]$. This is because $[\lambda a_0 : \ldots : \lambda a_n] = [a_0 : \ldots : a_n]$ for any $\lambda \in k^*$, but for some choice of a_i's and λ, we have $f(a_0,\dots,a_n) \neq f(\lambda a_0,\dots,\lambda a_n)$, unless of course f was constant. But for our specified goal we don't care as much about *evaluating* polynomials at points as we care about deciding whether they are zero or not. With this we have more success if we ask that f be homogeneous.

Definition 2.4. Say that $f \in k[X_0,\dots,X_n]$ is **homogeneous of degree d** if it is a linear combination of monomials of the same degree d. This is equivalent to $f(\lambda \cdot \underline{a}) = \lambda^d \cdot f(\underline{a})$ for all $\underline{a} \in \mathbb{A}^{n+1}$ and $\lambda \in k$.

Remark 2.5. If f is *homogeneous*, then we have a clear idea of what it means for f to *vanish* at a point $[a_0 : \ldots : a_n] \in \mathbb{P}^n$. This is because $f(\underline{a}) = 0 \Leftrightarrow f(\lambda \cdot \underline{a}) = 0$ for some $\lambda \in k^*$.

If f is not homogeneous, then we can break it into finitely many homogeneous pieces: $f = \sum_i f_i$, where f_i is homogenous of degree i. Then we can also say that f vanishes at $[a_0 : \ldots : a_n]$ if all f_i vanish there.

Definition 2.6. Let $\mathfrak{a} \trianglelefteq \mathrm{k}[\underline{X}]$ be an ideal generated by (finitely many) homogeneous polynomials (of maybe different degrees). We say that $\mathfrak{a}$ is a *homogeneous ideal.* Define $V(\mathfrak{a}) \subset \mathbb{P}^n$ as the common vanishing locus of every (homogeneous) polynomial $f \in \mathfrak{a}$.

If a subset $X \subset \mathbb{P}^n$ is equal to some $V(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a} \trianglelefteq \mathrm{k}[\underline{X}]$, we say that X is a **closed algebraic subset** in $\mathbb{P}^n$.

Conversely, if $X \subset \mathbb{P}^n$ is any subset, we define $\mathcal{I}(X) \trianglelefteq \mathrm{k}[\underline{X}]$ as the ideal generated by all homogeneous polynomials $f \in \mathrm{k}[\underline{X}]$ that vanish on X.

Example 2.7.

- $V(a_0x_0 + \ldots + a_nx_n) \subset \mathbb{P}^n$ is a **linear hyperplane** of $\mathbb{P}^n$. In general $V(f)$, where $f \in \mathrm{k}[\underline{X}]$ is homogeneous, is a **projective hypersurface**. Its **degree** is the degree of f.
- $V(f(x,y,z)) \subset \mathbb{P}^2$ is called a **projective curve**.

Remark 2.8. Using that $\mathrm{k}[\underline{X}]$ is a graded ring, one can show the following properties of homogeneous ideals:

(i) An ideal $\mathfrak{a}$ of a graded ring R is homogeneous iff $r \in \mathfrak{a} \Leftrightarrow r_i \in \mathfrak{a}\ \forall i$, where r_i is the degree i component of r.

(ii) The sum, product, or intersection of homogeneous ideals is homogeneous.
(iii) If $\mathfrak{a}$ is homogeneous, then $\sqrt{\mathfrak{a}}$ is also homogeneous.

Remark 2.9. If $V(\mathfrak{a}) \subset \mathbb{P}^n$ is closed given by a homogeneous ideal, then $V(\mathfrak{a}) \subset \mathbb{A}^{n+1}$ is the *affine cone* over it, and also a closed subset.

Just like in $\mathbb{A}^n$, the sets $V(\mathfrak{a})$ are the closed subsets of a topology on $\mathbb{P}^n$ that we continue to call the Zariski topology. This relies on showing that the union of finitely many closed algebraic subsets and that the intersection of finitely many closed algebraic subsets is still closed. One can do this with cones.

Since irreducibility was topological, we can continue to talk about that, and define a **projective variety** as an irreducible closed algebraic subset of $\mathbb{P}^n$.

We still have analogous results to Theorem 1.7:

(i) $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$ for homogeneous $\mathfrak{a} \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$.
(ii) $\mathcal{I}(X)$ is a radical ideal for any subset $X \subset \mathbb{P}^n$.
(iii) $Y_1 \subset Y_2 \Rightarrow \mathcal{I}(Y_1) \supset \mathcal{I}(Y_2)$.
(iv) $\mathfrak{a} \subset \mathfrak{b} \Rightarrow V(\mathfrak{a}) \supset V(\mathfrak{b})$.
(v) $\mathcal{I}(Y_1 \cup Y_2) = \mathcal{I}(Y_1) \cap \mathcal{I}(Y_2) \supseteq \mathcal{I}(Y_1) \cdot \mathcal{I}(Y_2)$.
(vi) $\cap_{t\in T} V(\mathfrak{a}_t) = V(\sum_{t\in T} \mathfrak{a}_t)$.
(vii) $V(\mathcal{I}(Y)) = \overline{Y}$ for any subset $Y \subset \mathbb{P}^n$.
(viii) **Homogeneous Strong Nullstellensatz:** $\mathcal{I}(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for any homogeneous $\mathfrak{a} \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$.

Something different is the

Homogeneous Weak Nullstellensatz: If $\mathfrak{a} \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$ is homogeneous, and $\mathfrak{a} \neq (1)$, then $V(\mathfrak{a}) = \emptyset \subset \mathbb{P}^n$ iff $\sqrt{\mathfrak{a}} = (X_0, \ldots, X_n) \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$.

(This is because if $\mathfrak{a} \neq (1)$, then one always has $\sqrt{\mathfrak{a}} \subset (X_0, \ldots, X_n)$. Also $V(X_0, \ldots, X_n) = \emptyset$, because $\mathbb{P}^n$ does not contain the point $[0 : \ldots : 0]$ by construction. The confusion comes from the fact the the affine cone over $\emptyset \subset \mathbb{P}^n$ is $(0, \ldots, 0)$, not $\emptyset \subset \mathbb{A}^{n+1}$)

Let $I := (X_0, \ldots, X_n)$ be the *irrelevant* ideal of $\mathrm{k}[\underline{\mathrm{X}}]$. This is the only maximal homogeneous ideal. For example if $x = [1 : 0 : \ldots : 0]$, then its ideal in $\mathrm{k}[\underline{\mathrm{X}}]$ is $(X_1, \ldots, X_n) \subset (X_0, \ldots, X_n) \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$ (so the first X_0 is missing from the ideal). This ideal is not maximal. In general points on X correspond to homogeneous ideals that contain $\mathcal{I}(X)$, maximal not among all ideals, but among those contained in the irrelevant ideal I and not equal to it. □

The geometry of the projective space is a bit different and actually better:

Example 2.10 (Homework)**.** Every two lines in $\mathbb{P}^2$ intersect. Looking at the cones over the lines is useful.

The underlying reason is that $\mathbb{P}^n$ is a "compactification" of $\mathbb{A}^n$, and "points at infinity" of $\mathbb{A}^n$ are contained in $\mathbb{P}^n$. Let's see some of these. First observe that since $\mathbb{P}^n$ does not contain $[0 : \ldots : 0]$, it is covered by the $n+1$ open subsets $D(x_i) = \mathbb{P}^n \setminus V(x_i)$ for all $i \in \{0, \ldots, n\}$.

If $x_0 \neq 0$, so if we are on $D(x_0)$, then

$$[x_0 : \ldots : x_n] = [1 : \frac{x_1}{x_0} : \ldots : \frac{x_n}{x_0}],$$

so actually $[\underline{x}]$ is unambiguously identified with $(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in \mathbb{A}^n$. Actually more is true:

Proposition 2.11. *The identification* $[x_0 : \ldots : x_n] \mapsto (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) : D(x_0) \overset{\varphi_0}{\to} \mathbb{A}^n$ *is a homeomorphism*[2] *for the Zariski topology. Of course we can replace* x_0 *by any other* x_i *to obtain a different* φ_i*.*

[2] it will be an isomorphism once we define regular maps between quasiprojective varieties

Proof. The map is clearly bijective. To check continuity it is enough to show that $\varphi^{-1}(V(f(T_1,\dots,T_n)))$ is closed for any $f \in k[T_1,\dots,T_n] = k[\mathbb{A}^n]$. This is made by "homogenizing" f: Write $T_i = \frac{x_i}{x_0}$. Then choose $N \gg 0$ such that $g(x_0,\dots,x_n) = x_0^N \cdot f(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0})$ clears the denominators. Observe that g is homogeneous and in $\mathbb{P}^n$ we have $[a_0 : \dots : a_n] \in V(g) \cap D(x_0) \Leftrightarrow f(\frac{a_1}{a_0},\dots,\frac{a_n}{a_0}) = 0 \Leftrightarrow \varphi_0([a_0 : \dots : a_n]) = (\frac{a_1}{a_0},\dots,\frac{a_n}{a_0}) \in V(f) \subset \mathbb{A}^n$.

We also want to check that φ_0^{-1} is continuous. It is enough to check that φ_0 is closed, and for this it is enough to check that $\varphi_0(V(f) \cap D(x_0)) \subset \mathbb{A}^n$ is closed for any homogeneous $f \in k[\underline{X}]$. This set coincides with $V(f(1,x_1,\dots,x_n))$, so it is closed. □

This is more in line with what we will eventually call an algebraic variety: $\mathbb{P}^n$ is covered by open subsets that are "isomorphic" to affine varieties.

Remark 2.12. $V(x_0) \subset \mathbb{P}^n$ is identified with $\mathbb{P}^{n-1}$ with homogeneous coordinates $[x_1 : \dots : x_n]$. By iterating we obtain a stratification

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \underbrace{A^{n-1} \sqcup \overbrace{\mathbb{A}^{n-2} \dots \sqcup \mathbb{A}^0}^{\mathbb{P}^{n-2}}}_{\mathbb{P}^{n-1}}.$$

2.2. **Example of projective varieties.** More details and examples than I am giving in these notes can be found here.

2.2.1. *Veronese subvarieties.* Are a nonlinear way of realizing $\mathbb{P}^n$ as a subvariety of a larger projective space. Choose a positive integer $d > 0$. Then the homogeneous polynomials of degree d in $k[X_0,\dots,X_n]$ form a k-vector space generated by the monomials of degree d. An elementary counting argument shows that this space has dimension $N_{n,d} := \binom{n+d}{d}$.

Definition 2.13. The **Veronese subvariety** $V_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is the (closure of the) image of the **Veronese embedding** function (no regularity assumptions for now, but eventually it will be the)

$$[x_0 : \dots : x_n] \overset{\varphi_{n,d}}{\mapsto} [x_0^d : x_0^{d-1}x_1 : \dots : x_n^d].$$

All monomials in $x_0,\dots,x_n$ of degree d are supposed to appear in the above expression.

Theorem 2.14. *$\mathcal{I}(V_{n,d}) \trianglelefteq k[T_{\underline{i}}]$, where $\underline{i} = (i_0,\dots,i_n)$ ranges over multi-indices with nonnegative entries and $i_0 + \dots + i_n = d$, so that we have one for $T_{\underline{i}}$ each monomial of degree d in the x_i's, is generated by*

$$T_{\underline{i}}T_{\underline{j}} - T_{\underline{i}'}T_{\underline{j}'}$$

for all $\underline{i} + \underline{j} = \underline{i}' + \underline{j}'$.

Sketch of proof: Show that the ideal defined by such expressions is the kernel of the algebra morphism $k[T_{\underline{i}}] \to k[\underline{X}]$ which sends $T_{\underline{i}} \to X_0^{i_0} \cdot \dots \cdot X_n^{i_n}$. □

Remark 2.15. The map $\varphi_{n,d}$ gives a homeomorphism between $\mathbb{P}^n$ and $V_{n,d}$. We will later see that it is an isomorphism.

Example 2.16. $\varphi_{1,2}([x : y]) = [x^2 : xy : y^2]$ is the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Its image $V_{1,2}$ is given by the equation $xz - y^2 = 0$.

Corollary 2.17. *If f is a homogeneous polynomial of degree d, then $D(f) \subset \mathbb{P}^n$ is homeomorphic*[3] *to an affine variety.*

Proof. We can look at f as a linear combination of $T_{\underline{i}}$'s. Call this $g \in k[T_{\underline{i}}]$. Then $\varphi_{n,d}$ identifies $D(f)$ with $V_{n,d} \cap D(g)$ homeomorphically. But since g is linear, $D(g)$ is homeomorphic to some $\mathbb{A}^{N_{n,d}-1}$ and then $V_{n,d} \cap D(g)$ is homeomorphic to a (irreducible) closed subset of $\mathbb{A}^{N_{n,d}-1}$, which by definition is an affine variety. □

[3] actually isomorphic once we learn what that is

2.2.2. *Segre varieties.* tell us that products of projective spaces are projective varieties as well.

Definition 2.18. Let $\mathbb{P}^n$ and $\mathbb{P}^m$ be projective spaces with homogeneous coordinates $[x_0 : \ldots : x_n]$ and $[y_0 : \ldots : y_m]$ respectively. The **Segre variety** $S_{n,m}$ is (the closure of) the image of the function $([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) \mapsto [x_0y_0 : x_0y_1 : \ldots : x_ny_m] : \mathbb{P}^n \times \mathbb{P}^m \stackrel{s_{n,m}}{\to} V(\mathcal{I})$

Theorem 2.19. *Let $k[T_{ij}]$ be a polynomial ring in $(n+1)(m+1)$ variables with indices $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m+1\}$. Let $\mathcal{I} \trianglelefteq k[T_{ij}]$ be the ideal generate by elements of form*

$$T_{ij}T_{kl} - T_{il}T_{kj}.$$

The function $s_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to V(\mathcal{I})$ is a bijection[4].

Remark 2.20. This will also show that the product of any two projective varieties is a projective variety.

2.2.3. *Grassmann varieties.* The **Grassmannian** $G(d, n)$ is the set of d-dimensional linear subspaces of k^n. We will give it the structure of a projective variety.

Example 2.21.
- $G(0, n)$ is just the origin in $\mathbb{A}^n$.
- $G(1, n+1)$ is the set of lines through the origin in $\mathbb{A}^{n+1}$, so we know that we can identify this with $\mathbb{P}^n$.
- $G(n, n+1)$ is the set of hyperplanes through the origin in $\mathbb{A}^{n+1}$. This is also identified with $\mathbb{P}^n$.
- $G(n, n)$ is also just a point: $\mathbb{A}^n$ is the only n-dimensional subspace of $\mathbb{A}^n$.

Things look more interesting for $G(2, 4)$. A plane $P \subset k^4$ is the span of two nonzero and non-collinear vectors $\vec{x}$ and $\vec{y}$ from $V := k^4$. Out of this information we want to obtain a line in some (other) vector space. This line is

$$\wedge^2 P \to \wedge^2 V.$$

As such it is well determined (up to multiplication by scalars) by the "simple wedge" element $\vec{x} \wedge \vec{y}$ of $\wedge^2 V$:

One can look at $\wedge^2 V$ as a $\binom{4}{2} = 6$-dimensional k-vector space (and P is a $\binom{2}{2} = 1$-dimensional k-vector space). If $e_1, \ldots, e_4$ is a basis for V, then $\{e_i \wedge e_j \mid i < j\}$ is a basis for $\wedge^2 V$. We have a function $f : V \times V \to \wedge^2 V$ that one usually denotes $f(v, w) = v \wedge w$. If we want to write this with respect to coordinates e_i and $e_i \wedge e_j$, then

(1) $(a_1e_1 + \ldots + a_4e_4) \wedge (b_1e_1 + \ldots + b_4e_4) = (a_1b_2 - a_2b_1)e_1 \wedge e_2 + \ldots + (a_3b_4 - a_4b_3)e_3 \wedge e_4,$

where the coefficient of $e_i \wedge e_j$ is $\det \begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix}$. These all come from asking that f is bilinear, that $f(e_i, e_j) = e_i \wedge e_j$ if $i < j$, and $v \wedge v = 0$ for all $v \in V$. This also works when the characteristic is 2. In this case $e_i \wedge e_j + e_j \wedge e_j = 0$ for all i, j because $(e_i + e_j) \wedge (e_i + e_j) = 0$.

Observe that $\vec{u} \wedge \vec{v} = 0$ iff $\vec{u}$ and $\vec{v}$ are linearly dependent: From (1) we obtain that $\vec{u} \wedge \vec{v} = 0$ iff $\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$ has rank less than 2 (because the 2×2 minors are all 0). This precisely means that $\vec{u}$ and $\vec{v}$ are linearly dependent.

[4] also isomorphism

Linearly independent vectors $\vec{x}$ and $\vec{y}$ span the same plane as the vectors $\vec{x}'$ and $\vec{y}'$, if and only if there exist $a, b, c, d \in k$ with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible and

$$\vec{x}' = a\vec{x} + b\vec{y}$$
$$\vec{y}' = c\vec{x} + d\vec{y}$$

When these formulas hold, then using the bilinearity of $\wedge$ we have

$$\vec{x}' \wedge \vec{y}' = (a\vec{x} + b\vec{y}) \wedge (c\vec{x} + d\vec{y}) = (ad - bc) \cdot \vec{x} \wedge \vec{y}.$$

Conversely, if $\vec{0} \neq \vec{x}' \wedge \vec{y}' = \lambda \cdot \vec{x} \wedge \vec{y}$ for some $\lambda \in k^*$, then $\{\vec{x}, \vec{y}\}$ and $\{\vec{x}', \vec{y}'\}$ are bases for the same plane in V: One defines $\wedge^3 V$ similarly to $\wedge^2 V$, and observes that $\vec{u} \wedge \vec{v} \wedge w = 0$ iff $\{\vec{u}, \vec{v}, \vec{w}\}$ are linearly dependent. In particular $\vec{x} \wedge \vec{y} \wedge \vec{y}' = \frac{1}{\lambda} \vec{x}' \wedge \vec{y}' \wedge \vec{y}' = 0$ because $\{\vec{x}', \vec{y}', \vec{y}'\}$ are linearly dependent. Therefore $\{\vec{x}, \vec{y}, \vec{y}'\}$ are linearly dependent, so $\vec{y}'$ is in the plane spanned by $\vec{x}$ and $\vec{y}$. Similarly $\vec{x}'$ is also in the plane.

In conclusion the set $G(2, 4)$ of planes in $V = k^4$ is identified with the set of all nonzero $\vec{u} \wedge \vec{v} \in \wedge^2 V$ up to multiplication by nonzero scalars. This means that $G(2, 4)$ identifies with a subset of $\mathbb{P}(\wedge^2 V)$. The thing to pay attention to is that not every element of $\wedge^2 V$ is a "simple wedge", i.e. of form $\vec{u} \wedge \vec{v}$. For example $e_1 \wedge e_2 + e_3 \wedge e_4$ is not of this form as we shall see.

If $\sum_{i<j} c_{ij} e_i \wedge e_j$ is in the image of f, then there exist a_i and b_i such that

$$c_{ij} = \det \begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix}$$

for all $i < j$. The claim that f is not onto suggests that there exist "relations" between these 2×2 determinants. Such a relation is

$$c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23} = 0, \tag{2}$$

which for example $e_1 \wedge e_2 + e_3 \wedge e_4$ does not verify. The relation comes from the Laplace expansion using 2×2 cofactors for the noninvertible matrix (regardless of the choice of a_i and b_i)

$$\det \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} = 0. \tag{3}$$

Conversely, if (2) is satisfied, then we show that we can find a_i and b_i such that $c_{ij} = \det \begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix}$. Because we are "working projectively", it is enough to solve equations up to multiplication by scalars. Then WLOG assume $c_{12} = 1$, so that

$$c_{34} = c_{13}c_{24} - c_{14}c_{23} \tag{4}$$

Put $a_1 = b_2 = 1$ and $a_2 = b_1 = 0$. Then c_{ij} are the 2×2 minors of

$$\begin{bmatrix} 1 & 0 & a_3 & a_4 \\ 0 & 1 & b_3 & b_4 \end{bmatrix}.$$

Then

$$a_3 = -c_{23} \qquad b_3 = c_{13}$$
$$a_4 = -c_{24} \qquad b_4 = c_{14}$$

This is a solution because it also verifies the equation $a_3b_4 - a_4b_3 = c_{34}$ by (4). A similar argument can use to prove the following:

Theorem 2.22. *Assume $d \leq n/2$. Then $G(d,n)$ is the projective subvariety of $\mathbb{P}^2(\wedge^d k^n)$ given by the* **Plücker** *quadratic equations arising as Laplace expansions (using $d\times d$ cofactors) for all the $2d \times 2d$ minors of the $2d \times n$ matrix analogous to the one in (3).*

$G(d,n)$ is the image of the "wedging" map $\mathbb{P}^{dn-1} \to \mathbb{P}(\wedge^d k^n)$ that sends d vectors in k^n (up to multiplying all of them by the same scalar) to their wedge product. In coordinates, the map takes a $d \times n$ matrix to all its $d \times d$ minors.

Remark 2.23. As opposed to the Segre or Veronese embeddings, the map above is not a bijection onto its image. This is because a plane has many bases, so the fibers are huge.

When $d > n/2$, the issue is that (3) does not have any $2d \times 2d$ minors. In this case one could us an identification $G(d,n) \simeq G(n-d,n)$ to reduce to the previous case. For example one could use the same equations as for $G(n-d,n)$, but the coefficients c_I, where $I \subset \{1,\ldots,n\}$ is a subset with d elements, refer to the coefficients $c_{I'}$ for $G(n-d,n)$, where I' is the complement of I, consequently having $n-d$ elements.

2.3. **Regular functions and regular maps on quasiprojective algebraic sets.** In today's talk we will define regular functions and maps and see examples. The idea to take away is that as opposed to the affine case, it is hard and not very productive to work with these as global objects. The most natural way to define things is locally, where you hope to use affine results, and if the definition was good, local data glues globally.

2.3.1. *Functions.*

Definition 2.24. A **quasiprojective** variety is an open subset of a projective variety. We define similarly **quasiprojective algebraic subsets** by removing the irreducibility assumption.

Example 2.25. Affine varieties and open subsets of affine varieties are quasiprojective by identifying $\mathbb{A}^n$ with $D(x_0)$ in $\mathbb{P}^n$.

On $\mathbb{P}^n$, we have seen that we cannot evaluate a nonconstant homogeneous polynomial $f \in \mathrm{k}[\underline{\mathrm{X}}]$ at any projective point, unless it is zero at that point. The reason was the degree of homogeneity $d = \deg f$. One can think that things are so great for constants because they have degree 0. We cannot find other homogeneous polynomials of degree 0, but we can find rational functions of degree 0. These are ratios $f = \frac{g}{h}$, where g, h are homogeneous polynomials of the same degree d. Then we have $f(\lambda \cdot \underline{x}) = \frac{\lambda^d g(\underline{x})}{\lambda^d h(\underline{x})} = \frac{g}{h}(\underline{x})$ for all $\lambda \neq 0$. This means that f is well-defined at the projective point $[x_0 : \ldots : x_n]$.

We have a better shot as defining regular functions as rational functions defined everywhere, meaning as functions on X, instead of just elements of a ring defined by algebra. This is supported by the constructions of §1.5, more specifically Proposition 1.51.

Definition 2.26. A **form of degree d** is a homogeneous polynomial $f \in \mathrm{k}[\underline{\mathrm{X}}]$ of degree d.

Let $X \subset \mathbb{P}^n$ be quasiprojective. A **regular function** at $x \in X$ is a ratio $f = \frac{g}{h}$ of forms of the same degree with $h(x) \neq 0$. We also say that f is *defined in a neighborhood of x*. (The neighborhood could for example be $D(h)$ or any open subset in it.). The set of all functions regular at x is $\mathcal{O}_{X,x}$.

If f is regular at every point $x \in X$, we say that f is **regular on** X. The set of all such forms a ring that we denote $k[X]$.

Remark 2.27. Two ratios $\frac{g}{h}$ and $\frac{g'}{h'}$ give the same regular function in a neighborhood of x, if apart from $h(x) \neq 0$ and $h'(x) \neq 0$, we have $gh' = g'h$ on every irreducible component C_x of X that contains x, (i.e. $gh' - g'h \in \mathcal{I}(C_x)$). If X is irreducible, we can of course just look at $\mathcal{I}(X)$.

Therefore we identify functions at x if they agree in a neighborhood of x.

(To see this, note that if $\frac{g}{h}(y) = \frac{g'}{h'}(y)$ for every y in a neighborhood U of x, then $\frac{gh'-g'h}{hh'}(y) = 0 \Rightarrow (gh' - g'h)(y) = 0$ for all $y \in U$, hence $gh' = g'h$ on the closure $\overline{U}$ which is precisely the claimed union (after possibly shrinking U).

The first two examples show that if we look at affine varieties in $\mathbb{A}^n$ as quasiprojective in $D(x_0)$, then this sections's definition of a regular function agrees with the one back in §1.5.

Example 2.28. i) If $X = D(x_0) \subset \mathbb{P}^n$ is the familiar copy of $\mathbb{A}^n$, then by dehomogenizing a regular function $f = \frac{g}{h}$, i.e. making $x_0 = 1$, we get a rational function on $\mathbb{A}^n$ in the sense of §1.5 that is defined everywhere, hence it is regular by Proposition 1.51. Conversely, a polynomail $f \in k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$ of degree d is a ratio $f = \frac{g}{x_0^d}$ which is defined at all $x \in D(x_0)$.

ii) The same works if X is closed and irreducible in $D(x_0)$. A slight change in the argument of Proposition 1.51 also shows that a regular function on any closed subset of the affine space (not necessarily irreducible) in the sense of this section is regular in the sense of §1.3, i.e. an element of $k[X]$. (The argument is in the book. The idea is to do a "partition of unity" on the irreducible components of X.)

iii) If $X = \mathbb{P}^n$, then $k[X] = k$. (If $f = \frac{g}{h}$ is a ration of forms of the same degree, then f is not defined on $D(h)$.) We will see that the same is true of any projective variety.

iv) $k[\mathbb{A}^2 \setminus \{(0,0)\}] = k[\mathbb{A}^2]$ as you will show.

Caution. The ring of regular functions $k[X]$ for quasiprojective algebraic sets X does not determine X as was the case for closed affine algebraic sets. For example

- If X is a projective variety, then $k[X] = k$ (as we will see later, but we at least saw it for $\mathbb{P}^n$), so only the constants are regular, but there are many projective varieties. If X is reducible, then $k[X] = k^{\pi_0(X)}$, where $\pi_0(X)$ is the (finite) set of connected (not the irreducible ones) components of X.
- If X is closed in $D(x_0)$, then $k[X]$ is a finitely generated k-algebra, because it is a closed affine set.
- If X is arbitrary quasiprojective, it may happen that $k[X]$ is not finitely generated. Rees and Nagata construct examples of such.

For these reasons, particularly the first, we don't focus on regular functions on the entire X, but rather on functions regular on (affine) open subsets which give us a better grasp on X itself.

Remark 2.29. If $X \subset \mathbb{P}^n$ is closed, then we can talk about the **homogeneous coordinate ring** $S(X) := \mathrm{k}[\underline{\mathrm{X}}]/_{\mathcal{I}(X)}$, where $\mathcal{I}(X) \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$ is the ideal generated by the forms vanishing on X.

If X is closed on $\mathbb{P}^n$, then $S(X)$ is the ring of regular functions not on X, but on the affine variety that is the cone in $\mathbb{A}^{n+1}$ over X.

This is a slightly better invariant of X than $k[X]$. For example [Har, Theorem 3.4] verifies that if $x \in X$ with corresponding homogeneous maximal ideal (among the homogeneous ideals inside the irrelevant ideal) $\mathfrak{m}$, then $\mathcal{O}_{X,x} = S(X)_{(\mathfrak{m})}$, where the latter denotes the set of ratios of (classes of) homogeneous forms in the localization $S(X)_{\mathfrak{m}}$.

Similarly $S(X)$ determines $k[X]$ and the not yet defined ring of rational functions on X.

However $S(X)$ is not intrinsic to X, but depends on the choice of embedding $X \subset \mathbb{P}^n$. For example $\mathbb{P}^1$ identifies with $V(x_0x_2 - x_1^1) \subset \mathbb{P}^2$ via the second Veronese embedding. However the homogeneous coordinate ring for $\mathbb{P}^1$ is $k[x,y]$, but for its second Veronese it is $k[x,y,z]/_{(xz - y^2)}$. These are not isomorphic. The "tangent space" $\mathfrak{m}/_{\mathfrak{m}^2}$ is 2-dimensional for every point in $\mathbb{A}^2$, but it is 3-dimensional for $(0,0,0)$ on the cone $V(xz - y^2) \subset \mathbb{A}^3$.

Another downside of $S(X)$ is that its elements, except for the constants, are not regular functions on X.

2.3.2. *Maps.*

Definition 2.30. A function $\varphi : X \to \mathbb{P}^m$ from a quasiprojective algebraic subset of $\mathbb{P}^n$ is **regular** if for every $x \in X$ there exists $0 \leq i \leq m$ and a neighborhood $x \in U \subset \varphi^{-1}(D(y_i))$ such that $\varphi|_U : U \to \mathbb{A}^m = D(y_i)$ is regular, i.e. given by m regular functions from $k[U]$.

If the image of φ is contained in Y, we talk about a regular map $X \to Y$.

When X is irreducible, by taking a homogenization we have an alternate definition that is nicer to work with. The book proves that these are defined well and equivalent in the irreducible case.

Definition 2.31. A **regular map** $\varphi : X \to \mathbb{P}^m$ from a quasiprojective variety is an equivalence class of $m+1$-tuples of homogeneous forms of the same degree $f_0, \ldots, f_m$, where $(f_0, \ldots, f_m) \sim (g_0, \ldots, g_m)$ if $f_i g_j = f_j g_i$ for all i, j, meaning that $f_i g_j - f_j g_i \in \mathcal{I}(X)$ (This is basically saying that $[f_i]$ and $[g_i]$ should differ by multiplying all terms by one function.), with an extra regularity condition: For every $x \in X$ there exists such an expression where not all f_i's vanish at x. We denote by $\varphi = [f_0 : \ldots : f_m]$ the equivalence class of the $m+1$-tuple $(f_0, \ldots, f_m)$.

The point of $[f_0 : \ldots : f_m]$ is that $[f_0(\lambda x) : \ldots : f_m(\lambda x)] = [\lambda^d f_0(x) : \ldots : \lambda^d f_m(x)] = [f_0(x) : \ldots : f_m(x)]$, so when we take homogeneous coordinates we no longer have the problem that f_i themselves are not functions on X.

Example 2.32.
- In projective coordinates, the parametrization of the cusp looks like $[s : t] \mapsto [s^2 t : s^3 : t^3]\ :\ \mathbb{P}^1 \to V(x^3 - y^2 z) \subset \mathbb{P}^2$.
- The second Veronese embedding of $\mathbb{P}^1$ is $[s : t] \mapsto [s^2 : st : t^2]$.
- $[x : y : z] \mapsto \begin{cases} [x^2 : yz] & \text{, away from } [0:1:0]\ ,\ [0:0:1] \\ [y : x - z] & \text{, away from } [1:0:1] \end{cases}$ going from the projective node $V(y^2 z - x^3 + x^2 z)$ to $\mathbb{P}^1$ is a regular map. (Note that we cannot find f, g forms of the same degree such that $[x^2 : yz] = [f : g]$ and f and g have no common zero on the node. The justification uses Bézout's Theorem. First note by Bézout that if f, g have common factors h, then $V(h) \subset V(f) \cap V(g)$ is a curve that must intersect the node, which contradicts the assumption.

  By Definition 2.31, $x^2 g - yzf \in (y^2 z - x^3 + x^2 z)$ in $k[x, y, z]$, so

  $$x^2 g - yzf = p \cdot (y^2 z - x^3 + x^2 z)$$

  and $V(f) \cap V(g)$ lies outside the node. Say $d = \deg f = \deg g$. Then $\deg p = d - 1$. If $d = 1$, then $V(f) \cap V(g)$ is a nonempty subset of the node.

  The intersection $V(f) \cap V(g)$ is d^2 points (with multiplicities). Then $V(f) \cap V(p)$ contains these d^2 points (with multiplicities), but itself is $d(d-1)$ points with multiplicities if $d > 1$, unless $V(f)$ and $V(p)$ have common components. If $V(h)$ is a common component, with h irreducible, then $h | x^2 g$, so $h = x$ or $h | g$. Since f, g don't have common factos, $h = x$. Then $x | f$ and by repeating the argumet for $V(g)$ and $V(p)$ we get that $y | g$ or $z | g$. We have $x\bar{g} - y\bar{f} = \bar{p} \cdot (y^2 z - x^3 + x^2 z)$ or $x\bar{g} - z\bar{f} = \bar{p} \cdot (y^2 z - x^3 + x^2 z)$ after substituting $f = x\bar{f}$ and simplifying, etc. Now $\deg \bar{f} = \deg \bar{g} = d - 1$ and $\deg \bar{p} = d - 3$. Repeat the full argument to get $\hat{g} - \hat{f} = \hat{p} \cdot (y^2 z - x^3 + x^2 z)$ with $\deg \hat{f} = \deg \hat{g} = d - 2$, and $\deg \hat{p} = d - 5$. Now the argument shows that $\hat{f}$ and $\hat{g}$ have common factors, hence so do f and g, unless $d \leq 2$, i.e. $d = 2$. When $d = 2$, we get that up to scalars, $f = x^2$ and $g = yz$, which both vanish at the points $[0 : 1 : 0]$ and $[0 : 0 : 1]$ of the node.)

The usual paraphernalia of regular maps carries through in the projective setting:

Definition 2.33. A regular map between closed projective sets is an **isomorphism** if its inverse exists and is regular.

A regular map $\varphi : X \to Y$ induces a morphism of algebras $\varphi^* : k[Y] \to k[X]$ by $\varphi(f) = f \circ \varphi$. By the same formula, working not globally but around a fixed $x \in X$, it induces a morphism of algebras $\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ between rings of regular functions at $f(x)$ and x respectively.

Example 2.34. The Veronese embedding $\mathbb{P}^n \xrightarrow{\varphi_{n,d}} \mathbb{P}^{\binom{n+d}{n} - 1}$ is an isomorphism onto its image.

Something with more nuance is the following

Definition 2.35. A quasiprojective variety that is isomorphic to an irreducible closed algebraic subset of $\mathbb{A}^n$ is called an affine variety.

This yet again improves our definition of affine variety, because now we no longer look at them as closed in $\mathbb{A}^n$, but *locally closed* in some $\mathbb{P}^m$, or even locally closed, but not closed in another $\mathbb{A}^m$.

Example 2.36. If X is a closed subvariety of $\mathbb{A}^n$ and $f \in k[X]$ is a regular function, then $D(f)$ which is quasiprojective as an open subset of $X \subset \mathbb{A}^n = D(x_0) \subset \mathbb{P}^n$ is also an affine variety: It is isomorphic to $V(fT-1)$ sitting as a closed subset in $X \times \mathbb{A}^1 \subset \mathbb{A}^{n+1} = D(x_0) \subset \mathbb{P}^{n+1}$. (The function $x \mapsto (x, \frac{1}{f(x)}) : D(f) \to V(fT-1)$ is regular on $D(f)$, and its inverse is the regular map $(x,t) \mapsto x : V(fT-1) \to D(f)$.) $D(f) \subset X$ is called a **principal affine subset** of the affine variety X.

The ring of regular functions on $D(f)$ is the ring of regular functions on $V(fT-1)$. Then $k[D(f)] = k[V(fT-1)] = k[X \times \mathbb{A}^1]/(fT-1) = k[X][T]/(fT-1) = k[X][\frac{1}{f}] = k[X]_f$, where the latter is the localization of $k[X]$ at the multiplicative system $1, f, f^2, \ldots$.

Caution. It is not true that every quasiprojective variety is projective or affine. For example $\mathbb{A}^2 \setminus \{(0,0)\}$ is not projective (because it is not "compact"), and it is not affine because it is not isomorphic to $\mathbb{A}^2$ and they have isomorphic rings of regular functions.

We will see later that if a quasiprojective variety $X \subset \mathbb{P}^n$ is isomorphic to a closed subvariety of some $\mathbb{P}^m$, then it was already closed in $\mathbb{P}^n$.

We will show that regular maps are continuous. The method of proof is to work locally, which shouldn't be surprising because this is how we work with continuous functions in almost every other branch of mathematics. But the actual details are more algebraic.

Lemma 2.37. *Let X be a quasiprojective subset of $\mathbb{P}^n$. Then the Zariski topology on X has a basis of open subsets that are affine varieties (isomorphic to closed subvarieties in some affine space). Equivalently every point $x \in X$ has an affine neighborhood.*

Proof. Since an open subset of a quasiprojective variety is again quasiprojective, it is enough to check that X itself is covered by open affine varieties. Then because the sets $D(x_i) \cap X$ are open in X and cover it, it is enough to check say that $D(x_0) \cap X$ is covered by affine varieties.

The new problem is: If X is open in the closure $\bar{X} \subset \mathbb{A}^n$, then X is covered by affine varieties. The set $\bar{X} \setminus X$ is closed in $\bar{X}$, hence closed in $\mathbb{A}^n$. There exists a set $f_i \in k[\bar{X}]$ of regular functions on $\bar{X}$ such that $\bar{X} \setminus X = \cap_i V(f_i)$. Then $X = \cup_i D(f_i)$. □

Remark 2.38. To work with regular maps $\varphi : X \to Y$, one can cover Y by affines U_i. Then cover each $\varphi^{-1}U_i$ by affines V_{ij}. Then $\varphi|_{V_{ij}}^{U_i}$ is a regular map between affine varieties in the sense of §1.3, so just given by polynomials.

Corollary 2.39. *If $\varphi : X \to Y$ is a regular map of quasiprojective subsets of projective spaces with $Y \subset \mathbb{P}^m$. Then φ is continuous for the Zariski topology.*

Proof. Let $Z \subset Y$ be closed. For each $x \in X$ there exists an open subset $x \in U \subset X$ such that $\varphi(U) \subset D(y_i) = A^m$ is regular. By shrinking U around x, we can assume that it is affine. Then φ is a regular map between affine varieties, hence it is continuous. Then $U \cap \varphi^{-1}Z$ is closed in U. Since the U's cover X, this implies that $\varphi^{-1}Z$ is closed in X. □

Yet another definition for regular maps, which is the one that [Har, §1.3] adopts is

Definition 2.40. A **regular map** $\varphi : X \to Y$ is a continuous function such that the induced pullback $\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is a (well-defined) morphism of rings for all $x \in X$.

We practically checked that our initial definition implies this one. Conversely, tf it verifies this one, let $x \in X$ and choose $D(y_i) = \mathbb{A}^m$ such that $U := \varphi^{-1}(D(y_i))$ contains x. Note that U is open by the assumed continuity. Then on a smaller neighborhood $V \subset U$ of x where all $g_j := \varphi_x^*(\frac{y_j}{y_i})$ are regular functions, $\varphi : V \to \mathbb{A}^m$ is given by the formula $\varphi(x) = (g_0(x), \ldots, g_m(x))$. The i-th g_i is missing. It is one, and it shows up if you write homogeneous coordinates $[g_0 : \ldots : g_m]$.

2.4. **Rational functions and rational maps for quasiprojective varieties.** The thing to take away from this section is that a rational function or map on a quasiprojective variety X (recall that for us variety means irreducible) is the same as a regular function or map defined on a nonempty open (and dense) subset.

We do not have enough regular functions on projective varieties to just say that a rational function is a ratio of regular functions, but things are better if we work with ratios of forms of the same degree.

Definition 2.41. Let $X \subset \mathbb{P}^n$ be a quasiprojective variety. A **rational function** on X is a ratio $\frac{f}{g}$ of forms on $\mathbb{P}^n$ of the same degree with $g|_X \neq 0$, (i.e. $g \notin \mathcal{I}(X)$). We have $\frac{f}{g} = \frac{f'}{g'}$ iff $fg' = f'g$ on X, (i.e. $fg' - f'g \in \mathcal{I}(X)$).

The set of rational functions forms a field denoted $k(X)$.

A rational function $f = \frac{g}{h}$ is **defined** at $x \in X$ if $h(x) \neq 0$.

The **domain of definition** of f is the dense open subset $\cup_{f=\frac{g}{h}} D(h)$. A rational function is regular on its domain of definition.

Remark 2.42. Two rational functions $f_1, f_2 \in k(X)$ are equal iff they are equal as functions on an open subset contained in the intersection of their domains. (Put $f_1 = \frac{g_1}{h_1}$ and $f_2 = \frac{g_2}{h_2}$. They are both regular on $U = D(h_1) \cap D(h_2)$. The definitions of when two rational functions that are regular on U agree and when two regular functions on U agree are the same. See Remark 2.27.)

Remark 2.43. Let U be open in X. Then a regular function on U is the same as a rational function on X that is defined (regular) on U.

Remark 2.44. If $U \subset X$ is open, then $k(U) = k(X)$. (If a form f vanishes on U, then it vanished on X and conversely).

Moreover, $k(X)$ is the fraction field of $\mathcal{O}_{X,x}$ for every $x \in X$. (It is enough to observe that the obvious map $\mathcal{O}_{X,x} \to k(X)$ is injective, and every rational function is a ratio of functions regular at x: Let $f = \frac{g}{h}$ be a rational function. If $h(x) \neq 0$, then f is regular at x. If not, then choose some form p of degree $\deg g = \deg h$ that does not vanish at x. Then $f = \frac{g/p}{h/p}$ is a ratio of regular functions at x.)

Example 2.45. i) If $X \subset D(x_0) \subset \mathbb{P}^n$ is an affine variety, then the definition given in this section for a rational function agrees with the one in §1.5. (This can be seen by (de)homogenizing).
ii) If $U \subset X$ is an open affine subset of a quasiprojective variety, then $k(X) = k(U)$ can be computed as in §1.5. In particular $k(\mathbb{P}^2) = k(x, y)$ and $k(\mathbb{P}^n) = k(\mathbb{A}^n)$.

Definition 2.46. A **rational map** $\varphi : X \dashrightarrow \mathbb{P}^m$ from a quasiprojective variety is an equivalence class of $m+1$-tuples of homogeneous forms of the same degree $f_0, \ldots, f_m$, where $(f_0, \ldots, f_m) \sim (g_0, \ldots, g_m)$ if $f_i g_j = f_j g_i$ for all i, j, meaning that $f_i g_j - f_j g_i \in \mathcal{I}(X)$ (This is basically saying that $[f_i]$ and $[g_i]$ should differ by multiplying all terms by one function.), with the extra condition that not all f_i vanish on all of X. We denote by $\varphi = [f_0 : \ldots : f_m]$ the equivalence class of the $m+1$-tuple $(f_0, \ldots, f_m)$.

The map φ is **defined** (or **regular**) at x if for some $m+1$-tuple, not all f_i vanish at x. The **domain** of φ is the open subset where it is regular. The **image** of φ is the image of the domain of φ.

If $\mathrm{Im}(\varphi)$ meets $Y \subset \mathbb{P}^m$, we may write $\varphi : X \dashrightarrow Y$. We say that φ is **dominant** if its image is dense in Y.

If $\varphi : X \dashrightarrow Y$ is dominant, then we have an induced $\varphi^* : k(Y) \hookrightarrow k(X)$. This construction is functorial (respects compositions of dominant rational maps).

Remark 2.47. We can also see $[f_0 : \dots : f_m] = [\frac{f_0}{f} : \dots : \frac{f_m}{f}]$, where f is a form of the same degree as the f_i's, and f does not vanish on all of X. In this way we can see rational functions as given by $m+1$ rational functions, not all identically zero on X.

If $f = f_i$ for some i such that f_i does not vanish on all of X, then we can see the rational function $[f_0 : \dots : f_m] = (\frac{f_0}{f_i}, \dots, \frac{f_m}{f_i})$ as a rational function to $\mathbb{A}^m = D(y_i)$.

Remark 2.48. To give a rational map $X \dashrightarrow Y$ is the same as giving a regular map $U \to Y$, where $U \subset X$ is nonempty open (and dense).

In particular, two rational maps agree iff they agree as regular functions on an open subset.

Definition 2.49. $\varphi : X \dashrightarrow Y$ is a **birational isomorphism** if it admits an inverse which is also a rational map. In this case we may also say that X and Y are **birational**.

Proposition 2.50. *Let X and Y be quasiprojective varieties. The following are equivalent*

i) *X and Y are birational.*
ii) *There exists a birational isomorphism $\varphi : X \dashrightarrow Y$.*
iii) *There exist open subsets $U \subset X$ and $V \subset Y$ that are isomorphic.*
iv) *There exists a rational map $\varphi : X \dashrightarrow Y$ and open subsets $U \subset X$ such that φ is defined on U and an isomorphism onto its image in Y.*
v) *There exists a dominant rational map $\varphi : X \dashrightarrow Y$ such that $\varphi^* : k(Y) \hookrightarrow k(X)$ is an isomorphism.*
vi) *$k(Y)$ and $k(X)$ are isomorphic as k-algebras.*
vii) *There exists a rational map $\varphi : X \dashrightarrow Y$ such that $\varphi^*_x : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is an isomorphism for every x in the domain of φ.*
viii) *There exists a rational map $\varphi : X \dashrightarrow Y$ such that $\varphi^*_x : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is an isomorphism for* one *x in the domain of φ.*

Proof. Only some of the implications are non-trivial.

ii) $\to$ *iii)*. Let $\psi = \varphi^{-1}$. Let U be the domain of φ and let V be the domain of ψ. Then $U' := \varphi^{-1}V \cap U$ is open because φ is continuous as a function $U \to Y$. Similarly $V' := \psi^{-1}U \cap V$ is open. Then $\varphi : U' \to V'$ is an isomorphism with inverse $\psi|_{V'}^{U'}$.

vi) $\to$ *ii)*. From a nonzero morphism $f : k(Y) \to k(X)$ which is automatically injective (because the source is a field), we construct a dominant rational map $\varphi : X \dashrightarrow Y$ such that $f = \varphi^*$. The result would then follow by functoriality. Since $k(X) = k(U)$ for any nonempty open subset $U \subset X$, we may assume that $X \subset \mathbb{A}^p$ and $Y \subset \mathbb{A}^q$ are affine varieties. Then $k(Y)$ is generated by the coordinate functions $y_1, \dots, y_q$. Define $\varphi : X \dashrightarrow Y$ by $\varphi = (f(y_1), \dots, f(y_q))$. By shrinking X, we can assume that φ is regular, while X is still affine. This is automatically dominant because $\varphi^* : k[Y] \to k[X]$ is injective on *regular* functions. The claim follows after observing that a (dominant) rational map $\eta : X \dashrightarrow X$ for which $\eta^* = 1_{k(X)}$ is birational. This is also handled by restricting to affine subsets (although the one on the left may be different from the one on the right). □

2.5. **Projective algebraic sets are universally closed.**

The goal of this section is to prove the following theorem.

Theorem 2.51. *Let $X \subset \mathbb{P}^n$ be a (closed) projective algebraic subset and let $f : X \to Y$ be a regular map to a quasiprojective variety. Then $f(X)$ is closed in Y.*

Remark 2.52. The projectivity assumption is necessary. For example the image of the regular map $(x, y) \mapsto x : V(xy-1) \to \mathbb{A}^1$ is $\mathbb{A}^1 \setminus \{(0,0)\}$ which is not closed in $\mathbb{A}^1$.

Corollary 2.53. *If X is a connected projective algebraic set, then $k[X] = k$. Consequently, if Y is (quasi)affine and $f : X \to Y$ is regular, then f is constant.*

Proof. Indeed a regular function $f : X \to k$ can be seen as a regular map $f : X \to \mathbb{A}^1$, or as a regular map $f : X \to \mathbb{P}^1$ whose image does not contain the "point at infinity" $[1:0]$. But $f(X) \subset \mathbb{P}^1$ is closed by the theorem. If it is not everything (because $[1:0]$ is not there), then it is just a finite subset of $\mathbb{P}^1$. Since X is connected, so is this subset. Therefore $f(X)$ is just a point, meaning that f is constant. □

Corollary 2.54. *If X is simultaneously a closed affine and a closed projective set, then X is finite.*

Proof. Let X' be a connected component of X. It is still both affine and projective. In particular there exists a regular map $X' \to \mathbb{A}^n$ for some n, and this map is an isomorphism onto its image. The previous corollary tells us that the image is just a point, therefore X' is just a point, and X is finite. □

Corollary 2.55. *If $X \subset \mathbb{P}^n$ is an infinite closed subset and f is a nonzero form, then X intersects the hypersurface $V(f)$.*

Proof. By Corollary 2.17, we know that $D(f)$ is affine. By the previous corollary, since X is infinite, it cannot be contained fully in $D(f)$, or else it would also be closed in the affine set $D(f)$. Therefore X and $V(f)$ must intersect. □

The proof of the theorem uses an understanding of graphs of regular maps between quasiprojective algebraic sets and of products of such sets.

2.5.1. *Products and graphs.* If $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are quasiprojective algebraic subsets, then $X \times Y$ has a structure of qausiprojective set. Here is how it works:

- If X and Y are affine, then $X \times Y$ is the affine variety whose algebra of regular functions is $k[X \times Y] = k[X] \otimes_k k[Y]$. (If $X \subset \mathbb{A}^n$ is given by equations f_i and $Y \subset \mathbb{A}^m$ is given by equations g_j, then $X \times Y \subset \mathbb{A}^{n+m}$ is given by equations f_i and g_j.)
- If X and Y are just open subsets of affine varieties, then $X \times Y = \overline{X} \times \overline{Y} \setminus \{(\overline{X} \setminus X) \times \overline{Y} \cup \overline{X} \times (\overline{Y} \setminus Y)\}$ shows that the product is also open in an affine variety.
- If $X = \mathbb{P}^n$ and $Y = \mathbb{P}^m$, then $X \times Y$ can be seen as a subvariety of $\mathbb{P}^{(n+1)(m+1)-1}$ via the Segre embedding:
$$([x_i], [y_j]) \mapsto w_{ij} = [x_i y_j] : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}.$$
This map is a bijection onto its image, and its image is closed, given by equations $w_{ij} w_{kl} = w_{il} w_{kj}$ for all $i, k \in \{0, \dots, n\}$ and $j, l \in \{0, \dots, m\}$. Therefore $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety. The topology on $\mathbb{P}^n \times \mathbb{P}^m$ is the one induced from $\mathbb{P}^{(n+1)(m+1)-1}$.
  We should make a little effort to understand the closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$. They are given by vanishing loci of homogeneous forms in w_{ij} restricted to $\mathbb{P}^n \times \mathbb{P}^m$. But after restriction, $w_{ij} = x_i y_j$. Then a monomial of degree d in w_{ij}'s is the same as the product between a monomial of degree d in x_i and one of degree d in y_j. Therefore the closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ are given by the vanishing of polynomials $h(\underline{x}, \underline{y})$ which are bihomogeneous, meaning homogeneous in the x_i and homogeneous in the y_j, for now of the same degree. But if $h(\underline{x}, \underline{y})$ is bihomogeneous of maybe different degrees in the x_i's than y_j's, then we can still manage. For example $x_1^2 y_1 - 2x_2^2 y_0$ vanishes on $\mathbb{P}^2 \times \mathbb{P}^1$ on the same points as the two polynomials $x_1^2 y_0 y_1 - 2x_2^2 y_0^2$ and $x_1^2 y_1^2 - 2x_2^2 y_0 y_1$ together.
- If $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are closed subsets given by equations $f_i(\underline{x})$ and $g_j(\underline{y})$, then $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$ is given inside $\mathbb{P}^n \times \mathbb{P}^m$ by the equations f_i and g_j which can be seen as bihomogeneous, so $X \times Y$ is a closed projective set.
- If X and Y are quasiprojective, then we define $X \times Y$ as we did for quasiaffine sets.

Remark 2.56. Here are some properties of products:

i) The projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are regular.
ii) To give morphisms $f : Z \to X$ and $g : Z \to Y$ is the same as giving one morphism $h : Z \to X \times Y$. Then f and g are recovered by composing with p_X and p_Y. In fact $X \times Y$ is determined up to isomorphism by this universality property.

iii) If $f : X \to Y$ is a morphism, then the *graph of* f is $\Gamma_f \subset X \times Y$ which is the set of points $\{(x,y) \in X \times Y \mid y = f(x)\}$. Note that $\Gamma_f \subset \mathbb{X} \times Y$ is closed (With a bit of work, one can understand $y = f(x)$ as a set of bihomogeneous equations.)

2.5.2. *Proof of Theorem 2.51.* Recall that we want to show that if $f : X \to Y$ is a regular map from a projective algebraic subset of $\mathbb{P}^n$ to a quasiprojective set, then $f(X)$ is closed in Y.

Reduction to projection maps. Since the graph $\Gamma_f \subset X \times Y$ is closed, and $p_Y(\Gamma_f) = f(X)$, it is enough to show that $p_Y : X \times Y \to Y$ maps closed subsets to closed subsets.

Reduction to $X = \mathbb{P}^n$. We have $X \subset \mathbb{P}^n$ closed. Then our work on products shows that $X \times Y \subset \mathbb{P}^n \times Y$ is also closed and closed subsets of $X \times Y$ are closed in $\mathbb{P}^n \times Y$. Also $p_Y : X \times Y \to Y$ is the restriction of $p_Y : \mathbb{P}^n \times Y \to Y$. If we show that $p_Y : \mathbb{P}^n \times Y \to Y$ maps closed subsets to closed subsets, then so did $p_Y : X \times Y \to Y$.

Reduction to Y affine. Let $Z \subset \mathbb{P}^n \times Y$ be closed. The question of $p_Y(Z)$ being closed in Y is local. Then we can cover Y by affine subsets and treat the problem over each of them, i.e. show that $p_U(Z \cap \mathbb{P}^n \times U) \subset U$ is closed if $U \subset Y$ is open affine.

Reduction to $Y = \mathbb{A}^m$. If Y is affine, then it is isomorphic to a closed subset of $\mathbb{A}^m$ for some m. If $Z \subset \mathbb{P}^n \times Y$ is closed, then it is closed in $\mathbb{P}^n \times \mathbb{A}^m$ as well, and $p_Y(Z) \subset Y$ is closed iff it is closed in $\mathbb{A}^m$. But $p_Y(Z) = p_{\mathbb{A}^m}(Z)$.

We want to show that $p := p_{\mathbb{A}^m} : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ sends closed subsets to closed subsets. A closed subset of $\mathbb{P}^n \times \mathbb{A}^m$ is the vanishing locus of polynomials $g_i(\underline{x}, \underline{y})$ that are homogeneous in the $n+1$ coordinates $\underline{x}$ and with no condition on $\underline{y}$ (by dehomogenizing from $\mathbb{P}^m$ to $D(y_0) = \mathbb{A}^m$.) We can assume that g_i all have the same degree d by the same trick from §2.5.1 (when we upgraded from bihomogeneous of the same degree, to bihomogeneous of possibly different degrees in the two sets of variables $\underline{x}$ and $\underline{y}$).

A point $y_0 \in \mathbb{A}^m$ is not in the image of p iff $g_i(\underline{x}, y_0)$ vanish nowhere on $\mathbb{P}^m \times \{y_0\}$. The Weak Projective Nullstellensatz says that this happens precisely when $g_i(\underline{x}, y_0)$ generate an ideal $\mathcal{I}_{y_0} \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$ that contains a power of the irrelevant $(x_0, \dots, x_n) \trianglelefteq \mathrm{k}[\underline{\mathrm{X}}]$.

For every positive integer s, put

$$T_s := \{y_0 \in \mathbb{A}^m \mid \mathcal{I}_{y_0} \not\supset (x_0, \dots, x_n)^s\}.$$

Then $\mathrm{Im}(p) = \cap_s T_s$. It is enough to show that each T_s is closed. Let M be a monomial of degree s in the variables $\underline{x}$. Then $M \in \mathcal{I}_{y_0}$ means $M = \sum_i g_i(\underline{x}, y_0) N_{i,y_0}(\underline{x})$, where $N_{i,y_0}(\underline{x}) \in \mathrm{k}[\underline{\mathrm{X}}]$. By looking at the degree s pieces, we can assume that $N_{i,y_0}(\underline{x})$ are all homogeneous polynomials of degree $s - d$.

Denote by S_d the set of forms of degree d in $\mathrm{k}[\underline{\mathrm{X}}]$. This is a k-vectors space of finite dimension $\binom{n+d}{n}$. Put $G_{y_0} \subset S_d$ the k-subspace generated by the $g_i(\underline{x}, y_0)$'s. The condition $\mathcal{I}_{y_0} \not\supset (x_0, \dots, x_n)^s$ is then

$$G_{y_0} \cdot S_{s-d} \supsetneq S_s,$$

where $G_{y_0} \cdot S_{s_d}$ is the k-vector subspace of S_s generated by products $g_i \cdot N$, where N ranges through monomials of degree $s - d$. (If you understand it, this is saying that $G_{y_0} \otimes_k S_{s-d} \not\twoheadrightarrow S_s$.)

This is an algebraic condition in $\underline{y}$, i.e. given by polynomials: Look at g_i as polynomials in $\underline{x}$ with coefficients in $k[\underline{y}]$. For fixed $\underline{y}$ and each monomial N of degree $s-d$, we have that $g_i(\underline{x}, \underline{y}) \cdot N$ is a vector in S_s. That for fixed $\underline{y}$ that these $t \cdot \dim S_{s-d}$ vectors, where t is the number of g_i's, do not generate S_s means that the matrix that they generate as columns has rank less then $\dim S_s$, so all $\dim S_s \times \dim S_s$ minors vanish. These can be seen as polynomials in $k[\underline{y}]$. Therefore T_s is closed. $\square$

Note that when s gets very large, then $\dim S_s > t \cdot S_{s-d}$, so $G_{y_0} \cdot S_{s-d} \neq S_s$ for dimension reasons, and $T_s = \mathbb{A}^m$. The proof is interesting for small s and it can actually be implemented in a computer.

2.5.3. *Locally projective maps.*

Remark 2.57. The same proof can be used to show that if we have a factorization

$$\begin{array}{ccc} X & \xhookrightarrow{\imath} & \mathbb{P}^n \times Y \\ & \overset{\circlearrowleft}{\underset{f}{\searrow}} & \downarrow p_Y \\ & & Y \end{array},$$

for some n, where $\imath$ is an isomorphism onto a closed subset of $\mathbb{P}^n \times Y$ (also called a **closed embedding**), then $\mathrm{Im}(f) \subset Y$ is closed, without the assumption that X is closed projective.

Morphisms admitting such a factorization are called **projective**. They are an important particular case of a larger class of morphisms called **proper** that also have the property that $\mathrm{Im}(f) \subset Y$ is closed.

If the above factorization exists only locally over Y (meaning that there exists a covering U_i of Y by open subsets such that $f : f^{-1}U_i \to U_i$ is a projective morphism for all i), we say that f is **locally projective**.

Corollary 2.58. *If $f : X \to Y$ is locally projective, then f is closed (sends closed subsets of X to closed subsets of Y).*

Proof. If $Z \subset X$ is closed, we want to show that $f(Z) \subset Y$ is closed. This is a local question. Let U_i be a cover of Y by open subsets such if we denote $V_i := f^{-1}U_i$, then $f_i := f|_{V_i}^{U_i}$ factors through a closed embedding $V_i \subset \mathbb{P}^{n_i} \times U_i$. The closed embedding $Z_i := Z \cap V_i \subset V_i$ extends to a closed embedding $Z_i \subset \mathbb{P}^{n_i} \times U_i$. Then $f(Z) \cap U_i = p_{U_i}(Z_i) \subset U_i$ is closed. $\square$

Remark 2.59. If X is projective, then any regular map $f : X \to Y$ to a quasiprojective set is projective. (If $X \subset \mathbb{P}^n$ is closed, then we have

$$\begin{array}{ccccc} X & \xhookrightarrow{\Gamma_f} & X \times Y & \xhookrightarrow{\imath} & \mathbb{P}^n \times Y \\ & \overset{\circlearrowleft}{\underset{f}{\searrow}} & \downarrow p_Y & \overset{\circlearrowleft}{\underset{p_Y}{\swarrow}} & \\ & & Y & & \end{array}$$

The composition on the top row is a closed embedding $X \subset \mathbb{P}^n \times Y$.)

3. Finite maps

If $f : X \to Y$ is a regular map of quasiprojective varieties such that $f^{-1}(y)$ is a finite set for all $y \in Y$, then we may think that X and Y are somewhat similar. Algebraic Geometry teaches us to expect an algebraic counterpart for rings which will help with proofs. This is the concept of a finite extension of rings: If $f : A \to B$ is a morphism of rings such that B is a finite module over A, then we say that f is *finite*.

Caution. [Sha, §5.3] wants f to be injective, but we don't. The reason as we will see is that this allows to consider inclusions of closed subsets as finite maps, which makes sense because they are definitely finite-to-one.

Definition 3.1.

- Affine version: A regular map $f : X \to Y$ of closed affine sets is **finite** if $f^* : k[Y] \to k[X]$ is a finite.
- Quasiprojective version: A regular map $f : X \to Y$ of quasiprojective sets is **finite** if Y admits a covering by open affine sets U_i (isomorphic to a closed subset of an affine space, not necessarily irreducible) such that $V_i := f^{-1}U_i$ is affine and $f_i := f|_{V_i}^{U_i}$ is a finite map of closed affine sets.

Remark 3.2. Algebra says that a morphism of rings $f : A \to B$ such B is of finite type over A is finite iff B is *integral* over A, i.e. for every $b \in B$ there exists $n > 0$ and $a_1, \dots, a_n \in A$ such that $b^n + \sum_i a_i b^{n-i} = 0$ in B.

Example 3.3. i) If $X \subset \mathbb{P}^n$ is quasiprojective and $Y \subset X$ is the closed subset $V_X(I) := V(I) \cap X$ for some ideal $I \trianglelefteq k[x_0, \dots, x_n]$, then the inclusion $\imath : Y \to X$ is a finite map. (After covering X by affines, we can restrict to the case where X and Y are affine. Then $\imath^* : k[X] \to k[Y]$ is the quotient map $k[X] \to k[Y]/\mathcal{I}_X(Y)$.)

ii) Any isomorphism of quasiprojective sets is finite. (In this case f_i^* is an isomorphism for all i.)

iii) The inclusion $\mathbb{A}^1 \setminus \{(0,0)\} \to \mathbb{A}^1$ is not finite, even though it is one-to-one. (In this case f^* is the inclusion $k[X] \subset k[X,T]/(XT-1)$ and T is not integral over $k[X]$, mainly because $XT - 1$ is not a monic polynomial.) For the same reason the projection $(x, y) \mapsto x \; : \; V(xy-1) \to \mathbb{A}^1$ is not finite.

iv) The map $[x_0 : \ldots : x_n] \mapsto [x_0^m : \ldots : x_n^m] \; : \; \mathbb{P}^n \to \mathbb{P}^n$ is finite for any $m > 0$. More generally we will see that if f_i are $m+1$ forms of the same degree on a (*closed*) projective set X without common zeros on X, then $x \mapsto [f_0(x) : \ldots : f_m(x)] \; : \; X \to \mathbb{P}^m$ is a finite map.

v) A composition of finite maps is finite.* (The algebra statement is that if $A \to B$ and $B \to C$ are finite, then so is the induced $A \to C$. This is clear.)

vi) Let $f : X \to Y$ and $g : Y \to Z$ be regular maps of *affine sets*. If $gf : X \to Z$ is finite, then so is f. (The algebra statement is that if $A \to B \to C$ is such that C is finite over A, then it is also finite over B.)

vii) In particular if $f : X \to Y$ is a finite map of *affine sets* and $Z \subset X$ is closed, then $f|_Z$ and $f|_Z^{f(Z)}$ are also finite.

* For now this is clear only for the affine version of Definition 3.1, but we will reconcile the two versions.

Lemma 3.4. *Let $f : X \to Y$ be a finite map of quasiprojective sets. Then f is locally projective over Y (cf. Remark 2.57).*

Proof. Since the statement is local, we may work over a cover as in Definition 3.1. Then we may assume that $f : X \to Y$ is a finite map of closed affine sets. We show that f is projective in this case. Let $t_1, \dots, t_n$ be a finite set of generators of $k[X]$ as a $k[Y]$ algebra. For example they can be the coordinates from an affine space containing X. Then we have a surjection $k[Y][t_1, \dots, t_n] \twoheadrightarrow k[X]$ showing that X is isomorphic to a closed subset of $\mathbb{A}^n \times Y$. We aim to show that $X \subset \mathbb{P}^n \times Y$ is also closed, where the homogeneous coordinates z_i on $\mathbb{P}^n$ are such that $t_i = \frac{z_i}{z_0}$. It is enough to prove that $\overline{X} \cap (V(z_0) \times Y) = \emptyset$, i.e. $\overline{X}$ does not meet the hyperplane at infinity over any point of Y.

By the finite condition, each t_i verifies equations $g_i(\underline{t}, y) = 0$, where $g_i(\underline{t}, y) := t_i^{n_i} + \sum_j a_{ij}(y) t_i^{n_i - j}$. If $([0 : z_1 : \ldots : z_n], y) \in \overline{X}$, then it verifies the homogenized (with respect to $\underline{z}$) equations $G_i(\underline{z}, y) = 0$, where

$$G_i(\underline{z}, y) = z_i^{n_i} + \sum_j a_{ij}(y) z_0^j z_i^{n_i - j}.$$

But $z_0 = 0$ and $G_i(\underline{z}, y) = 0$ for all i clearly implies $z_0 = \ldots = z_n = 0$ which is not possible in $\mathbb{P}^n$. □

Corollary 3.5. *If $f : X \to Y$ is finite, then f is closed (sends closed subsets of X to closed subsets of Y). In particular if f is finite and dominant ($f(X)$ is dense in Y), then f is surjective.*

Proof. Immediate from Corollary 2.58. □

Corollary 3.6. *Finite maps are finite-to-one.*

Proof. We may assume that $f : X \to Y$ is a finite map of closed affine subsets. As in the proof of the Lemma, we have that X is identified with a closed subset of $\mathbb{A}^n \times Y$ that is also is also closed in $\mathbb{P}^n \times Y$. For every $y \in Y$ we have that $f^{-1}y$ is closed in $\mathbb{A}^n$ and $\mathbb{P}^n$. But the only closed affine and closed projective sets are the finite sets of points. □

Remark 3.7. Lemma 3.4 and its proof tell us that if $f : X \to Y$ is finite, then the fibers of f don't "disappear to infinity". For example this gives another reason why the projection of the hyperbola $V(xy - 1)$ on the x-axis is not finite.

The next corollary tells us that if X is irreducible, then nonempty open subsets of X contain most fibers of f, so these fibers tend to band together.

Corollary 3.8. *Let $f : X \to Y$ be a finite map of quasiprojective* varieties. *Then for any open subset $U \subset X$ there exists an affine open subset $V \subset Y$ such that $\emptyset \neq f^{-1}V \subset U$.*

There is also a converse to Lemma 3.4.

Lemma 3.9. *If $f : X \to Y$ is a locally projective morphism with finite fibers, then f is finite.*

Proof. Without loss of generality we may assume that Y is affine, and that f is projective, i.e. f factors through a closed embedding $X \subset \mathbb{P}^n \times Y$ for some n. For each $y \in Y$, let $H_y \in \mathbb{P}^n$ be a linear hyperplane such that $H_y \cap X_y = \emptyset$, where $X_y := f^{-1}y$ (H_y exists since X_y is finite). Then $H_y \times Y \subset \mathbb{P}^n \times Y$ is closed and so is the intersection with X. This intersection avoids X_y, so its image through the second projection $\pi := p_Y$ is a closed (because π is projective) subset $V_y \subset Y$ that does not contain y. Let $U_y := Y \setminus V_y$. This is an open subset of Y containing y.

Over U_y, we have that $X \cap (\mathbb{P}^n \times U_y) \subset (\mathbb{P}^n \setminus H_y) \times U_y$ is closed. But $\mathbb{P}^n \setminus H_y$ is isomorphic to $\mathbb{A}^{n-1}$. Therefore $X \cap (\mathbb{P}^n \times U_y)$ is isomorphic to a closed subset of $\mathbb{A}^n \times U_y$, also closed in $\mathbb{P}^n \times U_y$. Let U'_y be an affine neighborhood of y contained in U_y. Put $Y' := U'_y$ and $X' := X \cap (\mathbb{P}^n \times Y')$. We also have that X' is closed in $\mathbb{A}^n \times Y'$, in particular it is affine. Put $f' : X' \to Y'$ the map induced from f. The goal is to show that $k[X']$ is finite over $k[Y']$, generated by the restrictions $t_1, \ldots, t_n$ of the coordinate functions on $\mathbb{A}^n$. In any case the t_i's generate $k[X']$ as a $k[Y']$-algebra, because X' is closed in $\mathbb{A}^n \times Y'$.

Let z_i be the homogeneous coordinates on $\mathbb{P}^n$ so that $t_i = \frac{z_i}{z_0}$. For ease of notation, assume $n = 1$. Then X' is given in $\mathbb{P}^1 \times Y'$ by equations $h_i(z_0, z_1, y) = 0$ homogeneous in the variables $\underline{z}$. We can assume that they have the same degree of homogeneity d. We have that $X' \cap V(z_0) = V(z_0, (h_i)_i) = \emptyset$. Working in the affine chart $z_1 = 1$, the Weak Nullstellensatz implies that $(z_0, (h_i(z_0, 1, y))_i) = (1) \trianglelefteq k[Y'][z_0]$. In particular there exists a relation on $\mathbb{A}^1_{z_0} \times Y'$:

$$1 = z_0 \cdot g(z_0, y) + \sum_i h_i(z_0, 1, y) \cdot g_i(y).$$

(We have collected everything with z_0 that possibly appeared in g_i in g.) Setting $z_0 = 0$ we get

$$1 = \sum_i h_i(0, 1, y) \cdot g_i(y).$$

On X' we then have $\sum_i h_i(1, t_1, y) \cdot g_i(y) = 0$, because $h_i(z_0, z_1, y) = 0$ and $t_1 = \frac{z_1}{z_0}$. Because of the previous formula, this is monic in t_1. Hence t_1 is integral over $k[Y']$. And so $k[X']$ is finite over $k[Y']$. □

Corollary 3.10. *Let $X \subset \mathbb{P}^n$ be closed projective, and let $y \in \mathbb{P}^n \setminus X$. Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ be the stereographic projection from y. Then $\varphi|_X$ is finite (and regular).*

By iterating, we can replace the point y by a linear subspace $L \subset \mathbb{P}^n$ that does not meet X. Then if $\dim L = d$, then the stereographic projection with center L is a map $\mathbb{P}^n \setminus L \to \mathbb{P}^{n-d-1}$.

Proof. Note that φ is regular outside y, so in particular it is regular on X. Observe that the graph of $\varphi|_X$ is closed in $\mathbb{P}^n \times \mathbb{P}^{n-1}$. This implies that $\varphi|_X : X \to \mathbb{P}^n$ is a projective morphism. If we show that it has finite fibers, then Lemma 3.9 finishes the proof. If the fibers are not infinite, then there exists a line ℓ in $\mathbb{P}^n$ through y that meets X at infinitely many points. Since $\ell \cap X$ is closed in ℓ, this is only possible if $\ell \subset X$. But $y \in \ell \setminus X$ is a contradiction. □

Corollary 3.11 (Noether normalization)**.** *Let $X \subset \mathbb{P}^n$ be a projective variety. Then there exists a* **finite dominant** *map $X \to \mathbb{P}^m$ for some m.*

Similarly if $X \subset \mathbb{A}^n$ is an affine variety, then there exists a finite map $X \to \mathbb{A}^m$ for some m.

Proof. Repeat the previous stereographic projection from points until it cannot continue, which is when $\varphi(X) = \mathbb{P}^m$. □

Remark 3.12. This seemingly harmless result will give us geometric understanding of the *dimension* of projective varieties: $\dim X = m$, if the Noether normalization dominates $\mathbb{P}^m$.

Corollary 3.13. *If $X \subset \mathbb{P}^n$ is a projective variety and $X \to \mathbb{P}^m$ is a Noether normalization, then there exists a linear subspace $L \subset \mathbb{P}^n$ of dimension $n - m - 1$ that does not meet X, and every subspace S of dimension at least $n - m$ meets X.*

Proof. L is constructed as the linear span of the points y appearing in the iteration in the proof of Noether normalization. The projection from L maps S to a linear subspace of $\mathbb{P}^m$ of dimension at least $\dim s + m - n$. □

Corollary 3.14. *If $X \subset \mathbb{P}^n$ is a closed projective set and $f_0, \dots, f_s$ are forms of the same degree d that do not vanish on X, then $f : X \to \mathbb{P}^s$ defined by $f(x) = [f_0(x) : \dots : f_s(x)]$ is finite.*

Proof. The Veronese embedding $\nu : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{n}-1}$ sends $[\underline{x}]$ to all degree d monomials in $\underline{x}$. It is a closed embedding. Put $N = \binom{n+d}{n} - 1$. On $\mathbb{P}^N$, we can see f_i as linear forms F_i. Then the rational map $\mathbb{P}^N \dashrightarrow \mathbb{P}^s$ given by these F_i is for our purposes a linear projection, and $\nu(X)$ is contained in its regular locus. The result is a consequence of Corollary 3.10. □

3.1. **Local study of finite maps.** We are also interested in seeing how finiteness behaves with respect to open subsets. If $f : X \to Y$ is a finite morphism of quasiprojective sets with X and Y affine, then can we reconcile the approaches in Definition 3.1 and show that it was already a finite map of affine sets?

Remark 3.15. If $\varphi : X \to Y$ is a morphism of *affine* varieties and $f \in k[Y]$, then $\varphi^{-1}D(f) = D(\varphi^*(f)) \subset X$, i.e. the inverse image of a principal open subset of Y is a principal open subset of X. It could be empty if f vanishes on $\varphi(X)$.

Proposition 3.16. *If $f : X \to Y$ is a finite map of affine quasiprojective sets (in the sense of the second definition), then $f^* : k[Y] \to k[X]$ is finite, i.e. the second definition of 3.1 implies the first for morphisms of affine quasiprojective sets.*

Proof. Let $Y = \cup_i U_i$ be an affine open cover of Y such that $f^{-1}U_i$ is an affine open subset of X and f is finite over U_i in the affine sense (first version of Definition 3.1).

If $V \subset U_i$ is a principal open subset corresponding to $g \in k[U_i]$, then $f^{-1}V$ is a principal open affine subset of $f^{-1}U_i$ by the previous remark. Moreover, if $k[U_i] \to k[f^{-1}U_i]$ is finite, then so is $k[V] \to k[f^{-1}V]$. This is because the latter map is just the localization at g of the previous one, i.e. $k[V] = k[D(g)] = k[U_i]_g$, which is just rational functions on U_i where the denominators are only allowed to range through powers of g.

Cover U_i by principal open subsets of Y. Then they are also principal on U_i by the remark, and f is finite over them in the affine sense. Without loss of generality we can assume $U_i = D(g_i)$ for $g_i \in k[Y]$ and that this is a finite cover.

Let h_{ij} be generators for $k[X]_{f^*(g_i)} = k[f^{-1}D(g_i)]$ as a finite algebra over $k[Y]_{g_i} = k[D(g_i)]$. By clearing denominators, since $\frac{g_i}{1}$ is invertible in $k[Y]_{g_i}$, we can assume that $h_{ij} = \frac{u_{ij}}{1}$ for some $u_{ij} \in k[X]$. (We could write $u_{ij} = h_{ij}$ if $k[X]$ was a domain, but we do not assume this).

We claim that u_{ij} generate $k[X]$ as a $k[Y]$-module: Let $x \in k[X]$. For each i, in $k[X]_{f^*(g_i)}$ write $\frac{x}{1} = \sum_j \frac{a_{ij}}{g_i^{n_{ij}}} \frac{u_{ij}}{1}$ for some $a_{ij} \in k[Y]$ and some $n_{ij} \geq 0$. By clearing denominators, $g_i^{n_i} x = \sum a'_{ij} u_{ij}$ for some $n_i \geq 0$ and $a'_{ij} \in k[Y]$. Since $D(g_i)$ cover Y, it follows that $((g_i)_i) = 1 \trianglelefteq k[Y]$, hence the same is true of $((g_i^{n_i})_i)$. We can find then $b_i \in k[Y]$ such that $\sum_i b_i g_i^{n_i} = 1$. Then $x = \sum_i b_i g_i^{n_i} x$ is in the $k[Y]$-span of u_{ij}. □

Lemma 3.17. *Let X be a quasiprojective set and let U and V be open affine subsets. Then $U \cap V$ is an open affine. More generally, if $f : U \to X$ is a regular map from a closed affine set U to quasiprojective X and $V \subset X$ is affine open, then $f^{-1}V$ is affine open in U.*

Proof. $U \times V \subset X \times X$ is an open subset and affine, because it is the product of two affines. The diagonal $\Delta = \{(x, y) \in X \times X \mid x = y\}$ is a closed subset of $X \times X$ (the graph of the identity morphism). Therefore its intersection with $U \times V$ is closed in $U \times V$ in the induced topology. But $(U \times V) \cap \Delta = U \cap V$, therefore $U \cap V$ is closed in the affine set $U \times V$, meaning that itself is affine. The case when the inclusion $U \subset X$ is replaced by a regular map is similar. □

Proposition 3.18. *If $f : X \to Y$ is a finite map of quasiprojective sets, then f is* **affine**. *This means that if $U \subset Y$ is an open affine, then the open $f^{-1}U \subset X$ is also affine.*

Proof. Cover Y by affines U_i such that $f^{-1}U_i$ is affine and f is finite over U_i in the affine sense. Cover U by open affines W_j that are each principal in one of U_i. By working over the corresponding U_i, we see as in the previous proposition that $f^{-1}W_j$ is affine open and f is finite over W_j. Now we are free to assume $U = Y$.

Up to a further refinement, we can assume that U_i (from $Y = \cup_i U_i$) are principal open subsets of Y with $f^{-1}U_i \subset X$ affine open and f finite over U_i. We want to show that X is itself affine.

To make writing easier, assume that $Y = D(g_1) \cup D(g_2)$ for regular functions $g_1, g_2 \in k[Y]$. Observe that $D(g_1) \cap D(g_2) = D(g_1 g_2)$ is principal in Y, but also in $D(g_1)$ and $D(g_2)$ (in the first it is the nonvanishing of g_2 and in the second of g_1. And g_i is regular on $D(g_j)$ because it is regular on Y). Then $X = D(f^*g_1) \cup D(f^*g_2)$ and $D(f^*g_1) \cap D(f^*g_2)$ is principal in each of them. Let x be a regular function on $D(f^*g_1)$. Then it is regular on the intersection, which is principal in $D(f^*g_2)$, given by the nonvanishing of g_1. Since we have arranged for $D(f^*g_2)$ to be affine, $k[D(f^*(g_1 g_2))] = k[D(f^*g_2)]_{f^*g_1}$. In particular there exists $x' \in k[D(f^*g_2)]$ such that $x = \frac{x'}{g_1^m}$ in $k[D(f^*g_2)]_{f^*g_1}$ for some $m \geq 0$. Then in $k[D(f^*g_2)]$, we have

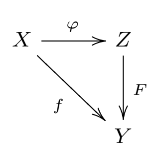
$$g_1^n x' = g_1^{m+n} x \in k[D(f^*g_1)]$$

for some $n \geq 0$ (if X is irreducible, we can choose $n = 0$). This means that $g_1^{n+m} x$ is regular on the entire X. Similarly, for any $x \in k[D(f^*g_2)]$ there exits $m \geq 0$ such that $g_2^m x$ is regular on X. By working through the algebra, this means that the natural maps

$$k[X]_{f^*g_i} \to k[D(f^*g_i)] \tag{5}$$

are surjective (they would be isomorphisms if we knew that X was affine). Quite generally these maps are injective: It is enough to check that if $x \in k[X]$, and $x|_{D(f^*g_1)} = 0$, then $g_1^m x = 0$ for some $m \geq 0$. But in fact $g_1 x$ vanishes everywhere, therefore it is the zero regular function. (Somewhere in here we are using that $k[X]$ is reduced, which is true because the definitions of §2.3.1 do not create nilpotents.)

As in the previous proposition, it can be shown that $k[X]$ is finite over $k[Y]$. In particular it is of finite type over k. Since $k[X]$ is also reduced, we can consider Z the closed affine set having $k[Z] = k[X]$. (We would have $Z = X$ if X was affine.) Using that Y and Z are affine, the morphisms $k[Y] \to k[Z] = k[X]$ induce a factorization



(If t_i generate $k[X] = k[Z]$ as a k-algebra, then $x \mapsto (t_1(x), \dots, t_d(x)) : X \to \mathbb{A}^d$ always lands in Z because it verifies its equations. This gives φ. Note that $\varphi^* = 1_{k[Z]}$. The map F is the only regular map of affine sets with $F^* = f^*$. If y_j are generators for $k[Y]$, then $F(z) = (F^*(y_1)(z), \dots, F^*(y_r)(z))$. In particular

$$F(\varphi(x)) = (F^*(y_1)(\varphi(x)), \dots, F^*(y_r)(\varphi(x))) = (\varphi^* F^*(y_1)(x), \dots, \varphi^* F^*(y_r)(x)) = (f^*(y_1)(x), \dots, f^*(y_r)(x)) = f(x),$$

therefore $f = F \circ \varphi$.)

Since $F^* = f^*$, we have that $k[Z]$ is finite over $k[Y]$ via F^*, and since Y and Z are affine, F is finite in the affine sense. Then $F^{-1}D(g_i)$ are principal affine open subsets of Z and we have induced regular maps of affine sets $\varphi_i : D(f^*g_i) \to D(F^*g_i)$. Note that φ_i^* is the morphism in (5), and in particular is an isomorphism. This implies that the morphism φ_i between *affine* opens is an isomorphism. Then φ restricts to an isomorphism over a cover and therefore it is itself an isomorphism meaning that X is affine (isomorphic to Z). □

Corollary 3.19. *a) If $f : X \to Y$ is a finite map of quasiprojective sets, then* any *affine cover of Y has the properties of Definition 3.1.*

b) A composition of finite maps between quasiprojective sets is finite.

c) If $f : X \to Y$ is a finite map of quasiprojective sets, then $f|_X^Z : X \to Z$ is finite for any closed subset $Z \subset Y$ that contains $f(X)$.

Proof. a) is immediate from the previous proposition. Using a) one sees that if U_i is an affine cover of Z, then $g^{-1}U_i$ is an affine cover of Y satisfying the conditions of Definition 3.1 for f, hence U_i verify the same definition for $g \circ f$.

For c), it is enough to observe that f factors through Z and that an affine cover of Y restricts to an affine cover of Z, since $Z \subset Y$ is closed. □

4. Dimension

We have been speaking about projective spaces of a certain dimension, and have an intuitive understanding that curves should have dimension 1, and that hypersurfaces should have dimension one-less (also called codimension 1). Let's see what happens for arbitrary quasiprojective sets.

First of all, if X is reducible, then its "dimension" should be the maximal dimension among its irreducible components, so let's assume that X is a variety.

Definition/Theorem 4.1. Let $X \subset \mathbb{P}^n$ be a quasiprojective variety, let $\overline{X}$ be its closure, let $U \subset X$ be an affine open subset, and let $x \in X$. Then the following quantities are all equal:

i) $d \geq 0$ such that there exists a Noether normalization $\overline{X} \to \mathbb{P}^d$, i.e. a finite dominant map $\overline{X} \to \mathbb{P}^d$
ii) The *transcendence degree* of $k(X)$ over k, i.e. the maximal number of elements of $k(X)$ algebraically independent over k.
iii) The maximal *length* (i.e. the number of inclusions, or one less than the number of terms) of any chain $Y_0 \subsetneq \ldots \subsetneq Y_d = X$ of irreducible closed subsets of X.
iv) The *Krull dimension* of $k[U]$, i.e. the maximal length of any chain of proper (different from (1)) prime ideals $0 = p_d \subsetneq p_{d-1} \subset \ldots \subsetneq p_0$ in $k[U]$.
v) The Krull dimension of the ring of regular functions $\mathcal{O}_{X,x}$ at x.

The number resulting from any of the above is called the **dimension** of X, denoted $\dim(X)$. If $Y \subset X$ is a locally closed subset, then its **codimension** in X is $\operatorname{codim}(Y, X) = \dim X - \dim Y$.

If X is allowed to be reducible, then $\dim X$ is the maximal dimension among its components. If all components have the same dimension we say that X is **equidimensional**, or of **pure dimension**.

Proof. The equivalence between the last four statements is algebra. To relate the first with the second, note that a finite *dominant* map $\overline{X} \to \mathbb{P}^d$ induces a finite field extension $k(x_1, \ldots, x_d) \subset k(X)$. Then $\operatorname{trdeg}_k k(X) = \operatorname{trdeg}_k k(x_1, \ldots, x_d) = d$. In particular if $\overline{X} \to \mathbb{P}^d$ is a finite dominant map, then d is uniquely determined by X (changing the map doesn't change d). □

Some common sense properties of dimension are the following

Corollary 4.2.
- *If $Y \subset X$ is locally closed, then $\dim Y = \dim \overline{Y}$. In particular if $U \subset X$ is open and X is irreducible, then $\dim U = \dim X$.*
- *$\dim \mathbb{A}^d = \dim \mathbb{P}^d = d$. For linear subspaces of either of them, the new concept of dimension agrees with the one from linear algebra.*
- *If $f : X \to Y$ is a dominant map, then $\dim X \geq \dim Y$.*
- *If $f : X \to Y$ is a finite map, then $\dim X \leq \dim Y$. Equality holds if f is also dominant. In particular isomorphisms preserve dimension.*
- *If $Y \subsetneq X$ is a proper locally closed subset of X, then $\dim Y < \dim X$. In particular if X is irreducible, if $Y \subset X$ is closed of codimension 1 and if $Y \subsetneq Z \subset X$ is a chain with Z irreducible, then $Z = X$.*
- *$\dim(X \times Y) = \dim X + \dim Y$.*

Proof. For the second to last part note that a chain of irreducible closed subsets of Y can be extended by at least one more by adding X at the end.

For the last part, let's first treat the case $X = \mathbb{P}^n$ and $Y = \mathbb{P}^m$. Then $\mathbb{P}^n \times \mathbb{P}^m$ is irreducible and contains $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ as an open subset, therefore its dimension is $n + m$. In general, we can assume that X and Y are projective. If $X \to \mathbb{P}^n$ and $Y \to \mathbb{P}^m$ are Noether normalizations, then $X \times Y \to \mathbb{P}^n \times \mathbb{P}^m$ is finite and dominant: Clearly it is surjective and has finite fibers. It is also the product of two locally projective maps, and this is easily seen to be again locally projective as an application of the Segre embedding. Conclude by Lemma 3.9. But the corollary then says that $\dim(X \times Y) = \dim(\mathbb{P}^n \times \mathbb{P}^m) = n + m$. □

Example 4.3. The Grassmannian $G(d,n)$ has dimension $d(n-d)$. [5] (Recall that if $V = k^n$, and $W \subset V$ is a d-dimensional linear subspace, then the Plücker embedding sends $W \subset V$ to the line $\wedge^d W \subset \wedge^d V$, thus determining a point in $\mathbb{P}(\wedge^d V)$. The latter has homogeneous coordinates $z_{i_1,\dots,i_d}$ corresponding to simple wedges $e_{i_1} \wedge \dots \wedge e_{i_d}$ with $1 \le i_1 < \dots < i_d \le n$. If the coordinate $z_{1\dots,d}$ is nonzero, then we show that $G(d,n) \cap D(z_{1\dots,d}) \simeq \mathbb{A}^{d(n-d)}$: as we did in (4), for a point ω in the claimed open subset of $G(d,n)$, we can find a unique matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 & a_{1,1} & \dots & a_{1,n-d} \\ 0 & 1 & \dots & 0 & a_{2,1} & \dots & a_{2,n-d} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{d,1} & \dots & a_{d,n-d} \end{bmatrix}$$

such that $\omega = v_1 \wedge \dots \wedge v_d$, where v_i are the rows of the matrix (This is linear algebra: If $w_1, \dots, w_d$ is a basis for W, then after Gauss–Jordan elimination on the matrix whose rows are the w_i's, we get the the matrix whose rows are the v_i's). Conversely for any such matrix we can find a point in $G(d,n) \cap D(z_{1,\dots,d})$. Since there are $d(n-d)$ parameters in the matrix, this defines the claimed isomorphism.) □

4.1. Dimension of intersection with a hypersurface.

Theorem 4.4. *Let $X \subset \mathbb{P}^n$ be an irreducible closed subset. Then X has codimension 1 if and only if $\mathcal{I}(X) = (f)$ for some irreducible nonzero form f on $\mathbb{P}^n$, i.e. X is an irreducible hypersurface.*

Consequently, if X is reducible of codimension 1 and equidimensional, then $\mathcal{I}(X) = (f)$ for some (reducible) form f.

Proof. Let g be a nonzero form with $X \subset V(g)$. Such g exists by the Nullstellensatz, because $X \neq \mathbb{P}^n$. Let $g = g_1 \cdot \ldots \cdot g_r$ be a decomposition of g into irreducible (automatically homogeneous because g is) polynomials of positive degree in the UFD $k[X_0, \dots, X_n]$. Then $X \subset V(g) = \cup_i V(g_i)$. Then the closed sets $V(g_i) \cap X$ cover X, and by irreducibility $X \subset V(g_i)$ for some i. Put $f := g_i$.

We claim that $V(f)$ is irreducible. Indeed $(f) \trianglelefteq \mathrm{k}[\underline{X}]$ is a prime ideal because f is irreducible in the UFD $\mathrm{k}[\underline{X}]$. By the Nullstellensatz, $\mathcal{I}(V(f)) = \sqrt{(f)} = (f)$ is prime and the claim follows.

If X has codimension 1, since $X \subset V(f)$, the previous corollary shows that $X = V(f)$ and the previous paragraph shows $\mathcal{I}(X) = (f)$.

Conversely it is clear that if $\mathcal{I}(X) = (f)$, then $X = V(\mathcal{I}(X)) = V(f)$, so a hypersurface. Let's show that in this case $\dim X = n-1$. In any case, since $f \neq 0$, we have that $X = V(f) \neq \mathbb{P}^n$. Let $p \in D(f)$. By Corollary 3.10, if $\pi : X \to \mathbb{P}^{n-1}$ is the restriction to X of the stereographic projection from p, then π is finite. We claim that it is also dominant. Let $q \in \mathbb{P}^{n-1}$. The line ℓ through p and q is a closed projective infinite subset, therefore it intersects $V(f) = X$ by Corollary 2.55. If $x \in \ell \cap X$, then $\pi(x) = q$, proving that π is in fact surjective. Since $\pi : X \to \mathbb{P}^{n-1}$ is finite and dominant, it must follow that $\dim X = \dim \mathbb{P}^{n-1} = n-1$. □

Remark 4.5. There are analogous statements in $\mathbb{A}^n$ or $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$: The only equidimensional closed subsets of codimension one are the hypersurfaces (given by the vanishing of a polynomial equation in $\mathbb{A}^n$, or of a multi-homogeneous form on the product of projective spaces).

However in general if $X \subset \mathbb{P}^n$ is a projective variety, and $Y \subset X$ is an irreducible subset of codimension 1, then Y is not necessarily of form $V(f) \cap X$ where f is a form on $\mathbb{P}^n$:

For example if we look at $\mathbb{P}^1 \times \mathbb{P}^1$ as a quadratic surface $X := V(z_0z_3 - z_1z_2) \subset \mathbb{P}^3$ via the Segre embedding $([x_0 : x_1], [y_0, y_1]) \mapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$, then there is no form f on $\mathbb{P}^3$ such that $X \cap V(f)$ is the line $\ell := X \cap V(z_0, z_1)$ (this is the fiber of the first of $[0:1]$ via the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$). Indeed X also contains the line $\ell' := X \cap V(z_0, z_2)$ (the fiber over $[1:0]$), and $\ell \cap \ell' = \emptyset$. On the other hand, $V(f)$ must intersect ℓ' because it is an infinite projective set (cf. Corollary 2.55) inside $\mathbb{P}^3$, and therefore inside X as well, and them $\ell = V(f) \cap X$ would also meet ℓ', which is a contradiction.

Inside $\mathbb{P}^n$, or $\mathbb{A}^n$, or inside products of projective spaces, sets of form $V(f)$ have codimension 1. Let's see that this holds for all quasiprojective varieties.[6]

[5] Since $\mathbb{P}^n = G(1, n+1)$, we don't get a new formula for $\dim \mathbb{P}^n$.

[6] Remark 4.5 shows that the converse is not always true, even on products of projective spaces if we change the projective embedding.

Theorem 4.6. *Let $X \subset \mathbb{P}^n$ be a quasiprojective variety. Let f be a nonzero form on $\mathbb{P}^n$ and assume that $V(f) \cap X$ is a proper closed subset (not $\emptyset$ or X). Then $V(f) \cap X$ has codimension 1 in X. Moreover the same is true for any irreducible component of $V(f) \cap X$.*

Proof. Since dimension can be measured on open subsets, we may assume that X is projective. In this case the intersection is automatically nonempty by Corollary 2.55. Replacing $\mathbb{P}^n$ by a Veronese embedding, we reduce to the case when f is linear.

We do induction on $r := \operatorname{codim}(X, \mathbb{P}^n)$. The case $r = 0$, i.e. $X = \mathbb{P}^n$ is Theorem 4.4. If $r > 0$, then there exists $p \in V(f) \setminus X$ by the assumption that $V(f) \cap X \neq X$. By Corollary 3.10, stereographic projection $\varphi : \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1}$ then maps X finitely onto its image in $\mathbb{P}^{n-1}$. Denote the image by X'. Note that $\dim X' = \dim X$, in particular $\operatorname{codim}(X', \mathbb{P}^{n-1}) = r - 1$. Also observe that since $V(f)$ is a linear hyperplane passing through p, its image through φ is a linear hyperplane $V(g)$ in $\mathbb{P}^{n-1}$.

Again by Corollary 3.10, φ maps $V(f) \cap X$ finitely onto its image, hence $\dim(V(f) \cap X) = \dim(\varphi(V(f) \cap X))$. We claim that $\varphi(V(f) \cap X) = V(g) \cap X'$. The conclusion then follows by induction.

For the claim, the inclusion $\varphi(V(f) \cap X) \subset V(g) \cap X'$ is clear. Conversely, let $y \in V(g) \cap X'$. Then $V(f)$ contains y and p, and since it is linear, it contains the line ℓ passing through them. Since $y \in X'$, by the construction of the projection, $\ell \cap X$ is nonempty. If $x \in \ell \cap X$, then $x \in V(f) \cap X$ as well, because $\ell \subset V(f)$, and we see that $\varphi(x) = y$, which gives the inclusion $V(g) \cap X' \subset \varphi(V(f) \cap X)$.

The proof in [Sha] is also quite nice. □

Iterating the previous result we obtain:

Corollary 4.7. *Let $r \leq n$. Then any r hypersurfaces in $\mathbb{P}^n$ have nonempty intersection. More generally, if $X \subset \mathbb{P}^n$ is closed projective, and $r \leq \dim X$, then X has nonempty simultaneous intersection with any r hypersurfaces in $\mathbb{P}^n$. The case $r = 1$ is Corollary 2.55. In particular r homogeneous equations in $\mathbb{P}^n$ always have solutions as long as $r \leq n$.*

Corollary 4.8. *Let X and Y be closed subsets of $\mathbb{P}^n$. Then $\dim(X \cap Y) \geq \dim X + \dim Y - n$, equivalently $\operatorname{codim}(X \cap Y, \mathbb{P}^n) \leq \operatorname{codim}(X, \mathbb{P}^n) + \operatorname{codim}(Y, \mathbb{P}^n)$, and the same is true for any of its irreducible components.*

If X and Y are only quasiprojective, then the conclusion holds only for the nonempty components of $X \cap Y$ if any.

Proof. Observe that $X \cap Y$ is isomorphic to $(X \times Y) \cap \Delta$, where Δ is the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$, i.e. the graph of the identity morphism $\mathbb{P}^n \to \mathbb{P}^n$. Consider the Segre embedding $\mathbb{P}^n \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{(n+1)(n+1)-1}$. Denote the latter by $\mathbb{P}^N$. The diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ has n equations $x_0 y_i = x_i y_0$ for $1 \leq i \leq n$ (at least on the affine open subset $D(x_0) \cap D(y_0)$, but such subsets cover $\mathbb{P}^n \times \mathbb{P}^n$, and the question is local on X anyway) which being bihomogeneous of the same degree in the x's and the y's, are the restrictions to $\mathbb{P}^n \times \mathbb{P}^n$ of some forms f_i on $\mathbb{P}^N$. Then $X \cap Y \simeq (X \times Y) \cap \Delta = (X \times Y) \cap V(f_1) \cap \ldots \cap V(f_n)$, and by iterating Theorem 4.6, this has dimension at least $\dim(X \times Y) - n = \dim X + \dim Y - n$. Usually we have equality of dimensions, but the inquality can be strict if at any step in the iteration we deal with a nonproper intersection. □

This is a particular case of the following

Corollary 4.9. *Let $f : X \to \mathbb{P}^n$ be a regular map from a quasiprojective variety X. Let $Y \subset \mathbb{P}^n$ be a quasiprojective subset. Then if Z is a nonemtpy irreducible component of $f^{-1}Y$, then $\dim Z \geq \dim X + \dim Y - n$.*

Proof. Observe that $f^{-1}Y$ is isomorphic to $(X \times Y) \cap \Gamma_f$, where $\Gamma_f \subset X \times \mathbb{P}^n$ is the graph of f. The conclusion is local, so we may reduce to the affine case. In this case $f : X \to \mathbb{A}^n$ has that the graph $\Gamma_f \subset X \times \mathbb{A}^n$ is given by n equations: If f si given by n regular functions $f = (f_1, \ldots, f_n)$, then Γ_f is given by equations $y_i = f_i(x)$. Conclude again by the theorem. □

Remark 4.10. When $r > 1$, it is not true that every equidimensional (or even irreducible) closed subset $X \subset \mathbb{P}^n$ of codimension r is a **set theoretic complete intersection**, i.e. the intersection of r hypersurfaces, though simple examples are not that easy to construct and explain at this point. Many are known when $\dim X > 1$. It is an open question whether every irreducible projective curve in $\mathbb{P}^3$ is a set theoretic complete intersection.

4.2. **The dimension of the fibers of a regular map.** If $f : V \to W$ is a surjective linear map of vector spaces, then the fibers of f are all translations of $\ker f$, hence they are all of the same dimension, $\dim V - \dim W$. It is reasonable to expect at least a similar property from regular maps between algebraic varieties. And indeed we cannot hope that all the fibers have the same dimension in general: think about the blow-up.

The good news comes in the form of the following:

Theorem 4.11. *Let $f : X \to Y$ be a dominant regular map of quasiprojective varieties. Put $e = \dim X - \dim Y$. Then*

i) If $y \in Y$ and $X_y := f^{-1}\{y\}$ is nonempty, then every irreducible component of X_y has dimension at least e.
ii) There exists a nonempty open subset $U \subset Y$ such that $\dim X_y = e$ for all $y \in U$.
iii) If $x \in X$, denote by $X_{x,f}$ the union of the irreducible components of $f^{-1}\{f(x)\}$ that actually contain x. Let $X_h := \{x \in X \mid \dim X_{x,f} \geq h\}$. Then X_h is a closed subset of X.
iv) If f is locally projective, then $Y_h := \{y \in Y \mid \dim X_y \geq h\}$ is closed in Y.

Proof. For part *i*). The question is local on Y, so we are free to assume that it is affine. Let $g : Y \to \mathbb{A}^m$ be an affine Noether normalization with $m = \dim Y$. For any $y \in Y$, the fiber X_y is a union of irreducible components of $X_{g(y)}$, hence we may assume that $Y = \mathbb{A}^m$. Now apply Corollary 4.9 to conclude that since $y \in f(X) \subset Y$ has codimension m in $\mathbb{P}^m$, the fiber $f^{-1}\{y\}$ has codimension at most m in X.

For part *ii*). Write $X = \cup_i X_i$, where $X_i \subset X$ are affine open subsets. For each i we will produce a nonempty open $U_i \subset Y$ with the required property for the restricted map $X_i \to Y$, and then we put $U = \cap_i U_i$. (In this way we make sure that when working in U_i, components of X_y of dimension higher than e do not hide in some other U_j.) We have reduced to the affine case.

Then we do induction on e as in the lemma below. We use the notation in its proof. The case $e = 0$ follows by homework, since f is finite over a nonempty open subset of Y, and the fibers of a finite map are finite sets of points, hence of dimension 0. If $e > 0$, then induction takes care of g. If $F \in k[Y][t]$ and $V = D(F) \subset Y \times \mathbb{A}^1$ with $V \subset g(X)$, let $a \in k[Y]$ denote the leading term of F. Then as in the lemma $D(a) \subset f(X)$. Moreover, the fiber over $y \in D(a)$ is $D(F) \cap \{y\} \times \mathbb{A}^1$, which is open in $\mathbb{A}^1$, in particular of dimension 1.

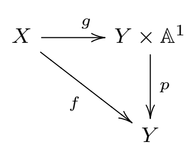
Here is our new setting: $g : X_y \to C \subset \mathbb{A}^1 = \{y\} \times \mathbb{A}^1$ is a surjection onto an open subset C of the affine line. We know by induction that the fibers of g are equidimensional of dimension $e-1$. We also know by part *i*) that every irreducible component of X_y has dimension at least e. We want to conclude that X_y is equidimensional of dimension e. To see this, first not that if X' is a component of X_y, then the conclusion $\dim X' \geq e$ implies that X' dominantes C, otherwise X' maps to a point on C, contradicting the findings on the dimension of fibers of g over C. Hence we may assume that X_y is irreducible. If $t_0 \in C \subset \mathbb{A}^1$, which we can see as an element of k, then the fiber of g over t_0 is $V(t - t_0) \cap X_y$, in particular a hypersurface. But as such it has codimension 1. The conclusion follows.

For part *iii*). The definition of X_h is local (which it wouldn't be if we just looked at the dimension of the full fiber). Then we can assume that X and Y are affine varieties. If $U \subset Y$ is as in part *ii*) and $Z = Y \setminus U$, then $X_h \subset f^{-1}Z$ for $h \geq e+1$. Let X' be an irreducible component of $f^{-1}Z$, and let Y' be the closure in Z of $f(X')$. Furthermore X_h is the union of all X'_h. The statement follows by induction on $\dim Y$, since $\dim Y' < \dim Y$, and the conclusion is trivial when Y is a point.

For part *iv*), observe that $Y_h = f(X_h)$, and f is closed since it is locally projective. □

Lemma 4.12. *Let $f : X \to Y$ be a dominant morphism of quasiprojective varieties. Then there exists $U \subseteq Y$ open with $U \subset f(X)$.*

Proof. We may assume that X and Y are affine varieties. Since $f : X \to Y$ is dominant, the pullback morphism of rational functions $f^* : k(Y) \to k(X)$ is well-defined and injective. We do induction in $e := \mathrm{trdeg}_{k(Y)} k(X)$. The case $e = 0$ was homework: you showed that there exists $U \in Y$ nonempty affine open such that f if finite over it. When $e > 0$, let $t \in k(X)$ be transcendental over $k(Y)$. We may assume $t \in k[X]$ since $k(X)$ is the fraction field of that. Then we have inclusions $k[Y] \subset k[Y][t] \subset k[X]$ and the composition is f^*. This means that f factors as



where p is the projection on the first component and $\mathbb{A}^1$ has coordinate t. Note that g is dominant and $\mathrm{trdeg}_{k(Y)(t)} k(X) = e-1$. By induction the lemma holds for g. Let $V \subset Y \times \mathbb{A}^1$ be an open subset contained in $g(X)$. We may assume that $V = D(F)$, with $F \in k[Y][t]$. Then $p(V) = \{y \in Y \mid F(y) \notin k^* \subset k[t]\}$. Looking at F as a polynomial in t with coefficients in $k[Y]$, let a be its leading coefficient. Then $D(a) \subset p(V) \subset p(g(X)) = f(X)$ is a nonempty open subset. □

Remark 4.13. • The "e" in the theorem is called the **relative dimension** of f. It is the *expected dimension* of the fibers of f.

- [Sha, Corollary on p76], stating that the sets Y_h are closed without the assumption that f is locally projective is incorrect. The correct statement in general is that Y_h is **constructible**, meaning a finite union of locally closed subsets of Y.

An important corollary of the theorem on the dimension of fibers is a criterion for irreducibility.

Corollary 4.14. *Let Y be a quasiprojective variety. Let $f : X \to Y$ be a locally projective map with X quasiprojective. If there exists $e \geq 0$ such that the fibers of f are irreducible of constant dimension e, then X is also irreducible.*

Proof. Let $X_i \subset X$ be the irreducible components of X. For each i, let $Y_{i,e} := \{y \in Y \mid \dim(X_i)_y \geq e\}$. Since f is locally projective, this is a closed subset of Y. Observe that $\cup_i Y_{i,e} = Y$. This is because for every $y \in Y$, we have $X_y = \cup_i (X_i)_y$, and since $\dim X_y = e$, at least one term in the union also has dimension e. By the irreducibility of Y, it follows that some component X_1 of X has all fibers of dimension at least e. But since X_y is irreducible and $(X_1)_y \subset X_y$, it follows that the fibers of X_1 and X over Y agree everywhere, meaning $X_1 = X$. □

Example 4.15. Let's give another proof for $\dim G(d,n) = d(n-d)$. $\big($We have a regular function $M_{n,d,d} \to G(d,n)$ that sends an $n \times d$ matrix of rank d to its image which is a d-dimensional subspace of k^n. The fibers are isomorphic to $GL_d(k)$, since we can use $d \times d$ invertible matrices to switch between bases of subspaces, and every invertible matrix corresponds to a different basis. Since $M_{n,d,d}$ is open in $M_{n,d}$, it follows that $d^2 = \dim GL_d(k) = \dim M_{n,d,d} - \dim G(d,n)$, therefore $\dim G(d,n) = \dim M_{n,d} - d^2 = nd - d^2 = d(n-d)$.$\big)$

4.3. **Lines on surfaces.**

Example 4.16. If f is a form of degree 3 on $\mathbb{P}^3_{\mathbb{C}}$, then $V(f)$ contains at least one line (linear subspace of $\mathbb{P}^3$ of dimension 1). $\big($A form of degree 3 on $\mathbb{P}^3$ has $\binom{6}{3} = 20$ coefficients. Then we can parameterize forms up to scaling with a $\mathbb{P}^{19}$ by looking at the 20 coefficients as homogeneous coordinates. On the other hand lines in $\mathbb{P}^3$ correspond to planes in k^4, so to points in $G(2,4)$.

Let $\mathbb{P}^3$ have coordinates $[z_0 : \ldots : z_4]$ and let $\mathbb{P}^{19}$ have coordinates s_{ijkl}, where $i+j+j+l = 3$ are exponents of monomials $z_0^i z_1^j z_2^k z_3^l$ of degree 3. Let $H \subset \mathbb{P}^3 \times \mathbb{P}^{19}$ be the universal hypersuface given by the vanishing of $F = \sum_{i+j+k+l=3} s_{ijkl} z_0^i z_1^j z_2^k z_3^l$. This is the universal hypersurface of $\mathbb{P}^3$ of degree 3, in that all $V(f)$ appear as fibers of the map $H \to \mathbb{P}^{19}$ (by assigning explicit values to the indeterminates s_{ijkl}).

Let $L \subset G(2,4) \times \mathbb{P}^3$ be the universal line. Let's find equations for it. Let $W \subset k^4$ be a 2-dimensional subspace and let $z = [z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3$. Then z is in the line in $\mathbb{P}^3$ determined by W if and only if z as a vector in k^4 is contained in W. An algebraic-looking condition that is equivalent to this is $u \wedge v \wedge z = 0$ if u, v is a basis for W. This is equivalent to asking that the determinants of the 3×3 minors in the matrix $\begin{bmatrix} u \\ v \\ z \end{bmatrix}$ vanish. The determinants of the 2×2 minors on the first two rows give the Plücker coordinates $(s_{ij})_{i<j}$ on $G(2,4)$ corresponding to $W \to \wedge^2 W$. By Laplace expansion using the last row, we can express the 3×3 determinants in terms of the z_i's and the s_{ij}'s. We get 4 polynomial equations bihomogneous of degree 1 in both the z_i's and the s_{ij}'s.

The set of inclusions $x \in \ell$ and $x \in V(f)$ is parameterized by $I := (G(2,4) \times H) \cap (L \times \mathbb{P}^{19})$ inside $G(2,4) \times \mathbb{P}^3 \times \mathbb{P}^{19}$.

We are looking for inclusions $x \in \ell \subset V(f)$. Here is how we can fix this: The fiber of the map $I \to G(2,4) \times \mathbb{P}^{19}$ over a point $(\ell, V(f))$ is precisely $\ell \cap V(f)$ as a subset of $\mathbb{P}^3$. The line ℓ is contained in $V(f)$ precisely when this fiber has positive dimension. But Theorem 4.11.*iv*) tells us that the locus in $G(2,4) \times \mathbb{P}^{19}$ over which all fibers have dimension ≥ 1 is closed. Denote this closed set by $J \subset G(2,4) \times \mathbb{P}^{19}$. Then $V(f)$ contains some line if and only if f is in the image of $J \to \mathbb{P}^{19}$.

Let's investigate J. The fibers of $J \to G(2,4)$ are hypersurfaces $V(f)$ that contain a fixed line. Up to changes of coordinates, we may focus on the line $z_0 = z_1 = 0$. Then $V(f)$ contains this, if and only if $f(0,0,z_2,z_3)$ is identically zero, which happens if and only if the coefficients $s_{0,0,k,l}$ with $k+l = 3$ all vanish. There are 4 such, and their vanishing is a linear condition on $\mathbb{P}^{19}$. Therefore for each $\ell \in G(2,4)$, the fiber J_ℓ is isomorphic to P^{15} (codimension 4 because of the 4 equations which are clearly independent). It follows that J is irreducible and of dimension $\dim G(2,4) + 15 = 19$.

If we can show that the map $J \to \mathbb{P}^{19}$ is generically finite (meaning that the fibers are finite over an open subset of the image), then by Theorem 4.11.*ii*), it follows that $J \to \mathbb{P}^{19}$ is dominant, hence surjective because J is projective. Again by Theorem 4.11.*ii*), it is enough to check that for one f the fiber J_f is finite nonempty. Let $f = z_0^3 + z_1^3 + z_2^3 + z_3^3$. If $\ell \subset V(f)$, then let $V(f) \ni [0 : a_1 : a_2 : a_3] := \ell \cap V(z_0)$ and $V(f) \ni [b_0 : 0 : b_1 : b_2] := \ell \cap V(z_1)$. Assume that these are two distinct points, so that any point on ℓ is of form $[tb_0 : sa_1 : sa_2 + tb_2 : sa_3 + tb_3]$ with s, t arbitrary. Since $\ell \subset V(f)$, all these points verify $f = 0$. Plugging in, this is equivalent to asking that the polynomial in s, t that we get is identically 0, and so all its coefficients are zero. The coefficients are

$$a_1^3 + a_2^3 + a_3^3 = 0 \qquad b_0^3 + b_2^3 + b_3^3 = 0 \qquad a_2^2 b_2 + a_3^2 b_3 = 0 \qquad a_2 b_2^2 + a_3 b_3^2 = 0$$

The conditions that the points $\underline{a}$ and $\underline{b}$ were distinct translates into $a_2 = b_3 = 0$ or $a_3 = b_2 = 0$ and the points are $[0 : 1 : 0 : \epsilon]$ and $[1 : 0 : \varepsilon : 0]$, or $[0 : 1 : \epsilon : 0]$ and $[1 : 0 : 0 : \varepsilon]$ as ϵ, ε range through third roots of -1. If the starting points coincide, intersect ℓ with $V(z_2)$ or $V(z_3)$ instead of $V(z_1)$. In any case we obtain only finitely many lines, the lines ℓ through such pairs of points.) □

Remark 4.17. In fact the Fermat cubic $V(z_0^3 + z_1^3 + z_2^3 + z_3^3)$ contains 27 lines. They are the 9 lines with parameterizations $[s : \epsilon s : t : \varepsilon t]$ as s, t range in $\mathbb{C}$, and ϵ, ε are cube-roots of -1, and the 18 conjugates under permutations of coordinates.

It can be shown that a *general* cubic surface contains exactly 27 lines.

Example 4.18. If f is a degree 4 form on $\mathbb{P}^3_{\mathbb{C}}$, then there exists an irreducible polynomial P in the coefficients of f seen as indeterminates such that $P(f) = 0$ if and only if $V(f)$ contains a line (linear subspace of dimension 1). (A form of degree 4 on $\mathbb{P}^3$ has $\binom{7}{3} = 35$ coefficients. Then we can parameterize forms f by $\mathbb{P}^{34}$. The lines in $\mathbb{P}^3$ are parameterized by $G(2, 4)$.

As before, let $J \subset G(2, 4) \times \mathbb{P}^{34}$ be the closed subset parameterizing inclusions $\ell \subset V(f)$. Then $V(f)$ contains some line if and only if f is in the image of $J \to \mathbb{P}^{34}$.

Let's investigate J. The fibers of $J \to G(2, 4)$ are hypersurfaces $V(f)$ that contain a fixed line. Up to changes of coordinates, we may focus on the line $z_0 = z_1 = 0$. Then $V(f)$ contains this, if and only if $f(0, 0, z_2, z_3)$ is identically zero, which happens if and only if the coefficients $s_{0,0,k,l}$ with $k + l = 4$ all vanish. There are 5 such, and their vanishing is a linear condition on $\mathbb{P}^{34}$. Therefore for each $\ell \in G(2, 4)$, the fiber J_ℓ is isomorphic to P^{29} (codimension 5 because of the 5 equations which are clearly independent). It follows that J is irreducible and of dimension $\dim G(2, 4) + 29 = 33$.

Let K be the image of J in $\mathbb{P}^{34}$. We will show that this also has dimension 33. Then it is $V(P)$ for some P, which is the desired polynomial.

Since J itself has dimension 33, it is enough to produce one quartic surface that contains only finitely many lines. Then the map $J \to K$ which is surjective by the definition of K, is also generically finite (finite fibers over an open subset) by Theorem 4.11. Let $f = z_0^4 + z_1^4 + z_2^4 + z_3^4$. Then one checks explicitly that $V(f)$ contains only finitely many lines: If $V(f)$ contains the line ℓ, then ℓ must intersect $V(z_0)$ and $V(z_1)$ say at points $[0 : a_1 : a_2 : a_3]$ and $[b_0 : 0 : b_2 : b_3]$. Assume for now that these are different points. The condition that $V(f) \supset \ell$ is that $f(tb_0, sa_1, sa_2 + tb_2, sa_3 + tb_3) = 0$ for all s, t, which after expanding and identifiying coefficients, leads to

$$a_1^4 + a_2^4 + a_3^4 = 0 \qquad b_0^4 + b_2^4 + b_3^4 = 0 \qquad a_2^3 b_2 + a_3^3 b_3 = 0 \qquad a_2^2 b_2^2 + a_3^2 b_3^2 = 0 \qquad a_2 b_2^3 + a_3 b_3^3 = 0$$

The solutions are the visible ones: the lines through $[0 : 1 : 0 : \epsilon]$ and $[1 : 0 : \varepsilon : 0]$ or through $[0 : 1 : \epsilon : 0]$ and $[1 : 0 : 0 : \varepsilon]$ as ϵ and ε range through fourth roots of -1. Other similar lines are obtained by looking at the cases when ℓ intersects $V(z_0)$ and $V(z_i)$ at two distinct points with $i = 2$ or $i = 3$.) □

Definition 4.19. Let C be a projective variety. A *family of quartic surfaces* in $\mathbb{P}^3$ over C intuitively is a projective set X and a map $\pi : X \to C$ such that the fibers of π are all isomorphic to quartics in $\mathbb{P}^3$.

Rigorously, by a family of quartics we mean that there exists $\varphi : C \to \mathbb{P}^{34}$ inducing a cartesian diagram

$$\begin{array}{ccc} C \times_{\mathbb{P}^{34}} H & \longrightarrow & H \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle p_2} \\ C & \xrightarrow[\varphi]{} & \mathbb{P}^{34} \end{array}$$

where $H \subset \mathbb{P}^3 \times \mathbb{P}^{34}$ is the universal quartic surface in $\mathbb{P}^3$ and

$$\begin{aligned} C \times_{\mathbb{P}^{34}} H &:= \{(c, (x, V(f))) \mid \varphi(c) = p_2(x, V(f)) = V(f)\} \\ &= p_{13}((\Gamma_\varphi \times H) \cap (C \times \Gamma_{p_2})) \subset C \times H, \end{aligned}$$

where $p_{13} : C \times \mathbb{P}^{34} \times H \to C \times H$ is the projection on the first and last component, and $\Gamma_\varphi \subset C \times \mathbb{P}^{34}$ is the graph of φ.

We say that $\pi : C \times \mathbb{P}^{34} \times H$ is **family of quartic surfaces** in $\mathbb{P}^3$.

Observe that $\pi^{-1}\{c\} = p_2^{-1}\{\varphi(c)\}$. The fibers of p_2 are quartic surfaces in $\mathbb{P}^3$, so the terminology is justified.

We say that π is a **nontrivial family** if $\dim \varphi(C) > 0$.

Corollary 4.20. *If $\pi : X \to C$ is a nontrivial family of quartic surfaces in $\mathbb{P}^3$, then there exists $c_0 \in C$ such that $X_{c_0} = \pi^{-1}\{c_0\}$ contains at leas one line in $\mathbb{P}^3$.*

Proof. Since $\dim \varphi(C) > 0$, the intersection $\varphi(C) \cap V(P)$ is nonempty in $\mathbb{P}^{34}$, where P is the polynomial in the previous example. Let $c_0 \in C$ such that $\varphi(c_0) \in \varphi(C) \cap V(P)$. Then the example tells us that $X_{c_0} = H_{\varphi(c_0)}$ contains at least one line. $\square$

5. Nonsingular varieties

Nonsingular (or smooth) varieties are the analogue of manifolds from differential geometry. At every point they have a tangent space of the same dimension as the variety. On the other hand singularity vs. nonsingularity gives us another way to distinguish between varieties up to isomorphism, although just like with dimension this is not a perfect way.

5.1. **Tangent space.** Let $X \subset \mathbb{P}^n$ be a quasiprojective set. Let $x \in X$. Assume that $\mathcal{I}(X)$ is generated by $(f_1, \dots, f_m)$. By abuse we use the same notation in the affine or the projective setting.

5.1.1. *Geometric definition.* Assume $X \subset A^n$ with ideal $\mathcal{I}(X) = (f_1, \dots, f_m)$ is closed and that coordinates on $\mathbb{A}^n$ are such that $x = (0, \dots, 0)$. Among the lines in $\mathbb{A}^n$ through x, some are more "attached" to X at x than others. Let's make this more formal.

Definition 5.1. If $\ell = \{(ta_1, \dots, ta_n) \mid t \in k\}$ is the line through the origin (which is x) and through $(a_1, \dots, a_n)$ (not all $a_i = 0$, so that we get a well-defined line), then the **intersection multiplicity** at x of $\ell \cap X$ is the minimal (among all i) multiplicity of 0 as a root of $f_i(ta_1, \dots, ta_n)$ seen as polynomials in t for $1 \leq i \leq m$. We denote this multiplicity by $i_x(X \cap \ell)$.

Since $x \in X$, we always have $i_x(X \cap \ell) \geq 1$. If $i_x(X \cap \ell) \geq 2$, we say that ℓ is **tangent** to X at x.

The **tangent space** $T_{X,x}$ is the union of all lines in $\mathbb{A}^n$ tangent to X at x.

When $X \subset \mathbb{P}^n$, we choose coordinates $[z_0 : \dots, z_n]$ such that $x = [1 : 0 : \dots : 0]$. Then we define $T_{X,x}$ using the definition in $X \cap D(z_0)$ after dehomogenization (setting $z_0 = 1$).

We may also define the **projective tangent space** $\mathbb{T}_{X,x}$, which is the union of lines in $\mathbb{P}^n$ that have multiplicity of intersection with X at least 2 at x. The multiplicity of intersection is of course computed on $D(z_0)$.

We have $T_{X,x} = \mathbb{T}_{X,x} \cap D(z_0)$.

Example 5.2.
- $T_{\mathbb{A}^n,x} = \mathbb{A}^n$. (The ideal in this case is (0), and 0 as a root of the zero polynomial has infinite multiplicity.)
- $T_{\mathbb{P}^n,x} = \mathbb{A}^n$.
- $\mathbb{T}_{\mathbb{P}^n,x} = \mathbb{P}^n$.

Remark 5.3. It is clear from the definition that $T_{X,x}$ does not depend on the choice of generators for $\mathcal{I}(X)$, however **it is important to work with *generators* and not just any equations.**

For example $(0, 0)$ has ideal (x, y). For any (a, b), we have that the order of vanishing of ta and tb at $t = 0$ is 1, so no line is tangent. We say that $T_{X,(0,0)} = 0$.

But $(0, 0)$ can be given by equations (x, y^2), although these do not generate $\mathcal{I}((0, 0)) = (x, y)$. However ta and t^2b^2 give a multiplicity of 2 whenever $a = 0$. This means that $V(x)$ would be the tangent space in this case. If we work with the equations (x^2, y^2), then we would get that $\mathbb{A}^2$ is the tangent space. Working with actual generators for $\mathcal{I}(X)$ instead of arbitrary equations removes this inconsistency.

Here is a practical way of computing the tangent space:

Remark 5.4. Since $x = (0, \dots, 0) \in V(f_i)$ for all i, the polynomials $f_i(\underline{x})$ have no free term. We can write them as $f_i = \sum_{j \geq 1} f_{i,j}$, where $f_{i,j}$ is the homogeneous part of f_i of degree j.

If ℓ is the line through $(a_1, \dots, a_n)$, then $f_i(t\underline{a}) = \sum_{j\geq 1} f_{i,j}(t\underline{a}) = \sum_{j\geq 1} t^j f_{i,j}(\underline{a})$. The multiplicity $i_x(X \cap \ell) \geq 2$ precisely when $f_{i,1}(\underline{a}) = 0$. Therefore

$$\boxed{T_{X,x} = V((f_{i,1})_i)}$$

Recall that we have the formula coming from Taylor expansions, or by checking directly:

$$f_{i,1}(\underline{X}) = \sum_{j=1}^{n} X_j \frac{\partial f_i}{\partial x_j}(0, \dots, 0).$$

Here and everywhere else, X_i or Z_i are coordinate functions, so indeterminates, whereas x_i or z_i are coordinates of a point x, while ∂x_i are notations for directional derivatives. The equations that describe tangent spaces are always equations in X_i's.

Remark 5.5. Since $f_{i,1}$ are all forms of degree 1, the tangent space $T_{X,x}$ is a linear subspace of $\mathbb{A}^n$.

Remark 5.6. When we don't want to change coordinates to make $x = (0, \dots, 0)$, then $x = (x_1, \dots, x_n)$ and then we write $f_i = \sum_{j\geq 1} f_{i,j,x}$, where $f_{i,j,x}$ is the parts of degree j in $f((X - x_1) + x_1, \dots, (X - x_n) + x_n)$. A formula with partial derivatives:

$$f_{i,1,x}(X_1, \dots, X_n) = \sum_{j=1}^{n} (X_j - x_j) \frac{\partial f_i}{\partial x_j}(x_1, \dots, x_n).$$

This is by definition the **differential** $d_x f_i(X)$.

Caution: We look at $d_x f_i(X)$ as linear in the indeterminates $X_i - x_i$, not in X_i. This is to support looking at $(x_1, \dots, x_n) \in \mathbb{A}^n$ as the origin of $T_{X,x}$.

The differential has the known properties

$$\begin{aligned} d_x(rf) &= r d_x f \\ d_x(f + g) &= d_x f + d_x g \\ (d_x(fg))(X) &= (d_x f)(X) \cdot g(x) + f(x) \cdot (d_x g)(X) \end{aligned}$$

where f, g are polynomials and $r \in k$ is a scalar. Then

$$\boxed{T_{X,x} = V((d_x f_i)_i)}$$

Example 5.7.
i) Let $X = V(y - f(x))$ with $f(0) = 0$. Then $T_{X,(0,0)}$ is the line of equation $0 = X\frac{\partial(y-f(x))}{\partial x}(0,0) + Y\frac{\partial(y-f(x))}{\partial y}(0,0) = -Xf'(0) + Y$, which is the line of slope $f'(0)$ through $(0,0)$, meaning what Calculus had us expect it would be.
ii) Let $X = V(x^3 - y^2)$. Then $T_{X,(0,0)} = \mathbb{A}^2$. This is because both partials of $x^3 - y^2$ vanish at $(0,0)$.
iii) Same with $X = V(x^3 - x^2 - y^2)$.
iv) However the tangent space at the origin of $V(x^3 - x - y^2)$ has equation $X = 0$, so it is just a line.
v) If $X = V(x^3 - y^2, z) \subset \mathbb{A}^3$, then $T_{X,(0,0,0)}$ is given by the equation $Z\frac{\partial z}{\partial z}(0,0,0) = 0$, i.e. $Z = 0$. So the tangent space remembers that X was inside the plane $V(z)$.

Remark 5.8 (Projective case)**.** If $X \subset \mathbb{P}^n$ is quasiprojective, and $x \in X$, then each equation f_i for X is homogeneous. After homogenizing the previous equations, we get that $\mathbb{T}_{X,x}$ is given by the vanishing of all

$$\sum_{j=1}^{n}(Z_j - x_j Z_0)\frac{\partial f_i}{\partial z_j}(1, x_1, \dots, x_n),$$

where where x is the point in $D(z_0)$ with homogeneous coordinates $[1 : x_1 : \dots : x_n]$. This still has a hint of affineness to it, due to the presence of the x_i's, but we can make it entirely projective. In fact a formula of Euler says that if f is **homogeneous**, then

$$(\deg f) \cdot f = \sum_{i=0}^{n} z_i \frac{\partial f}{\partial z_i}.$$

In particular if $[x_0 : \dots : x_n] \in V(f)$, then $\sum_{i=0}^{n} x_i \frac{\partial f}{\partial z_i}(x_0, \dots, x_n) = 0$. Plugging this into the formula for the tangent space, we get that

$$\boxed{\mathbb{T}_{X,x} \text{ is given by the vanishing of all } \sum_{j=0}^{n} Z_j \frac{\partial f_i}{\partial z_j}(z_0, \dots, z_n)}$$

where now $x = [z_0 : \dots : z_n]$ doesn't require an affine chart.

5.1.2. *Intrinsic nature of the tangent space.* We have defined the tangent space $T_{X,x}$ in terms of an embedding X in $\mathbb{P}^n$ or $\mathbb{A}^n$. However we show that $T_{X,x}$ is intrinsic to X. In particular it does not depend on the projective or affine embedding. We need the following

Theorem 5.9. *Let $X \subset \mathbb{P}^n$ be quasiprojective, and let $x \in X$. Let $x \in V \subset X$ be an open neighborhood of x with inclusion morphism $\imath : V \to X$. Assume there exists $\varphi : V \to U$ an isomorphism with $U \subset \mathbb{P}^m$ quasiprojective. Then*

- *i) $\imath^* : \mathcal{O}_{X,x} \to \mathcal{O}_{V,x}$ is an isomorphism, where $\mathcal{O}_{X,x}$ is the ring of regular functions at x (meaning in neighborhoods of x).*
- *ii) $\varphi^* : \mathcal{O}_{U,\varphi(x)} \to \mathcal{O}_{V,x}$ is an isomorphism.*
- *iii) If $U \subset \mathbb{A}^m$ is affine (closed), then $\mathcal{O}_{U,\varphi(x)} \simeq k[U]_{\mathfrak{m}}$, where $\mathfrak{m}$ is the maximal ideal of $k[U]$ corresponding to $\varphi(x)$, and $k[U]_{\mathfrak{m}}$ is the localization of $k[U]$ at the multiplicative system $k[U] \setminus \mathfrak{m}$.*
- *iv) $\mathcal{O}_{X,x}$ is isomorphic to the* ring of germs at *x, meaning the equivalence classes of pairs (U, f) of a neighborhood of x and a regular function on U modulo the relation $(U, f) \sim (U', f')$ if there exists $x \in V \subset U \cap U'$ another neighborhood of x such that $f|_V = f'|_V$.*

Proof. i) If f is a function regular in a neighborhood of x that is contained in V, then this neighborhood is also contained in X, and it follows that $\imath^*$ is surjective. If a regular function at x vanishes in a neighborhood of x contained in U, then it also vanishes in a neighborhood of x in X. Hence $\imath^*$ is also injective.

ii) This is clear.

iii) We have a map $e : k[U] \to \mathcal{O}_{U,\varphi(x)}$ that sends a regular function on U to itself on U seen as a neighborhood of x. If $g \in k[U] \setminus \mathfrak{m}$, then $g(\varphi(x)) \neq 0$, hence $\frac{1}{g}$ is also regular in a neighborhood of x. In particular $e(g)$ is invertible for every $g \in k[U] \setminus \mathfrak{m}$. The universality property of the fraction ring says that there exists a well-defined induced map $E : k[U]_{\mathfrak{m}} \to \mathcal{O}_{U,\varphi(x)}$ such that $E(\frac{f}{g}) = \frac{e(f)}{e(g)}$.

Assume $E(\frac{f}{g}) = 0 \in \mathcal{O}_{U,\varphi(x)}$. Then $e(f) = 0 \in \mathcal{O}_{U,\varphi(x)}$, meaning that f vanishes in an open neighborhood $\varphi(x) \in V \subseteq U$. Let $W = U \setminus V$. This is closed in U and does not contain $\varphi(x)$. Then there exists $h \in k[U]$ a regular function with $h|_W = 0$ and $h(\varphi(x)) \neq 0$. In particular $h \in k[U] \setminus \mathfrak{m}$. The function $hf \in k[U]$ vanishes everywhere on U, therefore $hf = 0 \in k[U]$. This precisely means $\frac{f}{1} = 0 \in k[U]_{\mathfrak{m}}$, and hence $\frac{f}{g} = 0$ as well. We have the injectivity of E.

For surjectivity, recall that a regular function at $\varphi(x)$ is a ratio of a priori forms on $\mathbb{P}^m$, but after dehomogenization, a ratio $\frac{f}{g}$ of polynomials on $\mathbb{A}^m$ with $g(\varphi(x)) \neq 0$. Replacing f and g by their restrictions to U, we also recover the surjectivity of E.

iv) This is also clear.

□

Corollary 5.10. *The ring $\mathcal{O}_{X,x}$ of regular functions at x is independent of the embedding of X, or of a neighborhood of x in a projective space.*

Proof. Use *i*) and *ii*). ☐

Corollary 5.11. *The ring $\mathcal{O}_{X,x}$ is Noetherian and* **local**, *meaning it has a unique maximal ideal $\mathfrak{m}$, which is the ideal of the regular functions at x that vanish at x. Furthermore $\mathcal{O}_{X,x}/\mathfrak{m} \simeq k$.*

Proof. Any localization $k[U]_\mathfrak{m}$ at a prime ideal is a local ring. And a localization of a Noetherian ring is again Noetherian. The quotient isomorphism is obtained by the remark that we have an evaluation morphism $f \mapsto f(x) : \mathcal{O}_{X,x} \to k$ that is clearly onto because $k \subset \mathcal{O}_{X,x}$, and whose kernel is by definition $\mathfrak{m}$. ☐

Theorem 5.12. *Let X be quasiprojective and $x \in X$. Then we have a natural linear isomorphism*

$$d_x : \mathfrak{m}/\mathfrak{m}^2 \to T^\vee_{X,x},$$

where $\mathfrak{m} \trianglelefteq \mathcal{O}_{X,x}$ is the maximal ideal, and where the RHS is the dual vector space $\mathrm{Hom}(T_{X,x}, k)$. *Here the vector space structure on $\mathfrak{m}/\mathfrak{m}^2$ comes from the isomorphism $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}$ which induces an $k \simeq \mathcal{O}_{X,x}/\mathfrak{m}$-module structure on $\mathfrak{m}/\mathfrak{m}^2$.*

Proof. Let U be an affine neighborhood of x in X. By abuse use use $\mathfrak{m}$ as notation for the maximal ideal in $k[U]$ corresponding to x, as well as for the maximal ideal in the local ring $k[U]_\mathfrak{m} \simeq \mathcal{O}_{X,x}$. Then it is easy to show that as k-vector spaces

$$\mathfrak{m}k[U]_\mathfrak{m}/\mathfrak{m}^2 k[U]_\mathfrak{m} \simeq \mathfrak{m}k[U]/\mathfrak{m}^2 k[U].$$

In particular we can work in $k[U]$, instead of $\mathcal{O}_{X,x}$. We have $U \subset \mathbb{A}^n$ closed for some n. In this setting we have constructed

$$(d_x f)(X_1, \ldots, X_n) = \sum_{j=1}^n (X_j - x_j) \frac{\partial f}{\partial x_j}(x_1, \ldots, x_n)$$

for any polynomial $f \in k[X_1, \ldots, X_n]$, where $x = (x_1, \ldots, x_n)$. It is convenient to assume $x = (0, \ldots, 0)$, so that we can work with indeterminates X_i instead of $X_i - x_i$, and so that we get less confused.

The differential is a linear form on k^n, and since $T_{X,0} \subset k^n$ is a linear subspace, $d_0 f$ restricts to a linear form on $T_{X,0}$. We have a map that we continue to denote

$$d_0 : k[X_1, \ldots, X_n] \to T^\vee_{X,0}.$$

Let $f_1, \ldots, f_r$ be a set of generators for $\mathcal{I}(X) \trianglelefteq k[X_1, \ldots, X_n]$. Then

$$T_{X,0} = V(d_0 f_1, \ldots, d_0 f_r).$$

If $f \in \mathcal{I}(X)$, then $f = \sum_{i=1}^r f_i g_i$ for some $g_i \in \mathrm{k}[\underline{\mathrm{X}}]$. Then using the product rule, for any $\underline{v} \in T_{X,0}$ we have

$$(d_0 f)(\underline{v}) = \sum_i d_0 f_i(\underline{v}) \cdot g_i(x) + \sum_i f_i(x) \cdot d_0 g_i(\underline{v}).$$

But $(d_0 f_i)(\underline{v}) = 0$ for all i because $\underline{v} \in T_{X,0} = V((d_0 f_i)_i)$, and $f_i(x) = 0$ for all i because $0 \in X \subseteq V(\mathcal{I}(X)) = V((f_i)_i)$. Therefore $d_0 f = 0 \in T^\vee_{X,0}$ for all $f \in \mathcal{I}(X)$, meaning $\mathcal{I}(X) \subset \ker d_0$. We get an induced map from the universality property of quotients that we continue to denote

$$d_0 : \mathrm{k}[\underline{\mathrm{X}}]/\mathcal{I}(X) = k[U] \to T^\vee_{X,0}.$$

Restricting to $\mathfrak{m} \subset k[U]$ we get

$$d_0 : \mathfrak{m} \to T^\vee_{X,0}.$$

Note that $\mathfrak{m} \trianglelefteq k[U]$ is generated as an ideal by X_i for all i, these seen as regular functions on X by restriction from $\mathbb{A}^n$. Using this and the product rule again, it is clear that $\mathfrak{m}^2 \subset \ker d_0$. The universality property for quotients leads to a linear map

$$\boxed{d_0 : \mathfrak{m}/\mathfrak{m}^2 \to T^\vee_{X,0}}$$

If the map is not surjective, then it must be contained in some hyperplane in $T^\vee_{X,0}$, meaning there exists $0 \neq \underline{v} \in (T^\vee_{X,0})^\vee = T_{X,0}$ such that $\mathrm{Im}(d_0) \subset \ker(\underline{v})$, i.e. $(d_0 f)(\underline{v}) = 0$ for all $f \in \mathfrak{m} \trianglelefteq k[U]$. In particular for each i, we have $0 = (d_0 X_i)(\underline{v}) = v_i$, leading to $v_i = 0$ for all i, i.e. $\underline{v} = 0 \in T^\vee_{X,0}$, which contradicts the assumption. Therefore d_0 is surjective.

We check injectivity. Recall that $\mathfrak{m}$ is generated as an ideal (or $k[U]$-module) by the classes of X_i modulo $\mathcal{I}(X)$ for all i, hence $\mathfrak{m}/\mathfrak{m}^2$ is generated as a $k = k[U]/\mathfrak{m}$-vector space by the classes modulo $\mathcal{I}(X) + \mathfrak{m}^2$ of the same elements. Assume $d_0(\sum_i a_i X_i) = 0 \in T^\vee_{X,0}$ for some $a_i \in k$. Then for every $\underline{v} \in T_{X,0}$

$$0 = \left(d_0(\sum_i a_i X_i) \right)(\underline{v}) = \left(\sum_i a_i X_i \right)(\underline{v}).$$

This means that $\sum_{i=1}^n a_i X_i$ vanishes on $T_{X,0}$. By linear algebra, or by the Nullstellensatz, it must be a linear combination of any given set of linear equations for $T_{X,0}$. In particular

$$\sum_i a_i X_i \in \mathrm{Span}_k(d_0 f_1, \dots, d_0 f_r).$$

Since $0 \in X = V(f_1, \dots, f_r)$, we have $f_i(0) = 0$ for all i and then $f_i - d_0 f_i = \sum_{j \geq 2} f_{i,j} \in \mathrm{k}[\underline{\mathrm{X}}]$, where $f_{i,j}$ are homogeneous of degree j in $X_1, \dots, X_n$. In particular

$$f_i = d_0 f_i \bmod (X_1, \dots, X_n)^2,$$

leading to

$$\sum_i a_i X_i = 0 \bmod \mathcal{I} + \mathfrak{m}^2.$$

Hence $\ker d_0 = 0$ for $d_0 : \mathfrak{m}/\mathfrak{m}^2 \to T^\vee_{X,0}$. Since it is injective, surjective, and linear, d_0 is an isomorphism. □

Corollary 5.13. *$T_{X,x}$ is intrinsic to X, and does not change if we replace X by a neighborhood of x.*

Definition 5.14. The vector space $\mathfrak{m}/\mathfrak{m}^2 \simeq T^\vee_{X,x}$ is called the **cotangent** space of X at x. It is denoted $\Omega_{X,x}$.

5.1.3. *Differentials for regular maps.* Just like we defined differentials for regular functions, we can also define them for maps.

Definition 5.15. Let $f : X \to Y$ be a regular map of quasiprojective varieties, and let $x \in X$. Put $y = f(x)$. Then the morphism $f^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is local, i.e. $f^*(\mathfrak{n}) \subset \mathfrak{m}$, where $\mathfrak{n}$ and $\mathfrak{m}$ are the unique maximal ideals in $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ respectively (The inclusion holds true because f^* sends functions that vanish at y to functions that vanish on x). Clearly $f^*(\mathfrak{n}^2) \subset \mathfrak{m}^2$ as well, therefore it induces

$$f^* : \mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2.$$

The **differential of** f is the dual of this map, which by the theorem above can be written as

$$d_x f : T_{X,x} \to T_{Y,y}.$$

We can also write this in coordinates. There exist affine open neighborhoods $x \in U \subset X$ and $y \in V \subset Y$ with $f(U) \subset V$. If $V \subset \mathbb{A}^m$ closed, then $f = (f_1, \dots, f_m)$ for some regular functions $f_i \in k[U]$. Then

$$d_x f = (d_x f_1, \dots, d_x f_m).$$

Example 5.16. If X is affine and f is a regular function on it, we can see it as a regular map $f : X \to \mathbb{A}^1$. Then $d_x f$ should be a linear map from $T_{X,x}$ to $T_{\mathbb{A}^1, f(x)} \simeq k$. But a linear map $T_{X,x} \to k$ is the same as an element of $T^\vee_{X,x}$, which is good because that is what $d_x f$ was before: an element of $T^\vee_{X,x}$.

Remark 5.17. The differential is linear and functorial, i.e. if $f : X \to Y$ is regular, and $g : Y \to Z$ is regular, and $x \in X$, then we have the chain rule:

$$(d_x(g \circ f))(v) = (d_{f(x)} g)(d_x f(v)) = (d_{f(x)} g \circ d_x f)(v),$$

which is basically $d(g \circ f) = dg \circ df$.

Example 5.18. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be the function $f : (x, y) = (x + y, (x + y)^2)$, and let's compute $d_0 f$, where $0 = (0, 0)$. The formula above says

$$(d_0 f)(a, b) = ((d_0(x + y))(a, b), (d_0(x + y)^2)(a, b)) = (a + b, 0).$$

Observe that $\mathrm{Im}(d_0 f) = V(y) = T_{V(y - x^2),(0,0)}$. In other words $d_0 f$ notices that the image of f is contained in the parabola $y = x^2$, which is what functoriality promised: f factors through the inclusion of the parabola in $\mathbb{A}^2$.

Corollary 5.19. *If $X \subset Y$, and $x \in X$, then $T_{X,x} \subset T_{Y,x}$.*

Proof. We can assume that $X \subset Y \subset \mathbb{A}^n$. By functoriality, we get maps $T_{X,x} \to T_{Y,x} \to T_{\mathbb{A}^n,x}$. The composition is injective, because we have initially defined $T_{X,x}$ as a subspace of $T_{\mathbb{A}^n,x} = k^n$. Then the first map is automatically also injective. □

Corollary 5.20. *Let $f : X \to Y$ be a regular map, and let $x \in X$. Put $y = f(x)$ and let $X_y = f^{-1}y$. Then*

$$T_{X_y,x} \subset \ker d_x f \subset T_{X,x}.$$

Proof. We can factor

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle\subseteq} & & \uparrow{\scriptstyle\subset} \\ X_y & \xrightarrow[f|_{X_y}]{} & \{y\} \end{array}$$

Then $d_x f(T_{X_y,x}) \subset T_{y,y} = 0$. □

Example 5.21. However if $f : X \to Y$ is surjective, it does **not** follow that $d_x f : T_{X,x} \to T_{Y,f(x)}$ is surjective for every $x \in X$, although it is surjective for almost all x. Consider for example $(x, y) \mapsto y : V(y - x^2) \xrightarrow{f} \mathbb{A}^1$. This is a finite 2-to-1 map. Given that points on $V(y - x^2)$ all look like (x, x^2), we have

$$T_{V(y-x^2),(x,x^2)} = V(d_{(x,x^2)}(y - x^2)) = V((X - x) \cdot (-2x) + (Y - x^2) \cdot 1)$$

$$(X, Y) \mapsto Y - x^2 \;:\; T_{V(y-x^2),(x,x^2)} \overset{d_{(x,x^2)}f}{\to} k$$

The map $d_{(x,x^2)}f$ is going to be surjective, whenever it is not identically zero (since the target is k), meaning whenever $Y - x^2$ is not identically zero for all $(X, Y) \in V((X-x)\cdot(-2x)+(Y-x^2)\cdot 1)$. This happens precisely when $x \neq 0$, i.e. $(x, y) = (x, x^2) \neq (0, 0)$.

This respects the geometric intuition: The projection $(x, y) \mapsto y$ is linear, so it can be identified with its differential. The only lines it contracts are those parallel to the x-axis. The only time one of these lines is tangent to the parabola $y = x^2$, is at the origin.

We can also check with algebra. The map $f^* : k[\mathbb{A}^1] \to k[V(y - x^2)]$ is the inclusion map $k[y] \to k[x, y]/(y - x^2)$ determined by sending y to y. The maximal ideals we work with are $(y) \trianglelefteq k[y]$ for $0 = f(0, 0) \in \mathbb{A}^1$, and $(x, y) \trianglelefteq k[x, y]/(y - x^2)$ for $(0, 0) \in V(y - x^2)$. The induced map

$$f^* : (y)/(y^2) \to (x, y)/(x^2, xy, y^2)$$

sends y to (y mod (x^2, xy, y^2)). However, $x^2 = y$ on $V(y - x^2)$, so we get that y is sent to zero. On the other hand, y generates $(y)/(y^2)$ as a k-vector space, so we get that the map f^* is zero at the level of cotangent spaces, so its dual $d_{(0,0)}f$ is also the zero map, in particular not surjective.

Example 5.22. Let $V = k^{n+1}$ as a vector space, so that $\mathbb{P}^n = \mathbb{P}(V)$, i.e. the set of lines in V through the origin. We have seen that $T_{\mathbb{P}^n,x}$ is isomorphic to k^n, but how does this relate to V and x?

The point $x \in \mathbb{P}(V)$ corresponds to a line $\ell \subset V$ through the origin by the definition of the projective space. Let $x = [x_0, \ldots, x_n]$. Then $(x_0, \ldots, x_n) \in \ell$. Let $f : V \setminus \{(0, \ldots, 0)\} \to \mathbb{P}(V)$ be the map $(X_0, \ldots, X_n) \mapsto [X_0 : \ldots : X_n]$. This is regular (because the X_i's are regular on the source and don't have common zeros).

The differential at $(x_0, \ldots, x_n)$ is

$$d_{(x_0,\ldots,x_n)}f : T_{V,(x_0,\ldots,x_n)} \to T_{\mathbb{P}(V),(x_0,\ldots,x_n)}.$$

To compute it, we work in an affine $D(z_i)$ that contains $[x_0 : \ldots : x_n]$. Assume $x_0 \neq 0$, so that $x \in D(z_0)$. On $D(z_0) \simeq \mathbb{A}^n$ we have coordinates $\frac{z_i}{z_0}$ for all $1 \leq i \leq n$. Over $D(z_0)$, the function f is $(x_0, \ldots, x_n) \mapsto (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$. Given that $T_{V,(x_0,\ldots,x_n)}$ is V, we get that for $v = (v_0, \ldots, v_n) \in V$,

$$\begin{aligned}(d_{(x_0,\ldots,x_n)}f)(v) &= ((d_{(x_0,\ldots,x_n)}\frac{x_1}{x_0})(v)), \ldots, (d_{(x_0,\ldots,x_n)}\frac{x_n}{x_0})(v))) \\ &= (-(v_0 - x_0)\frac{x_1}{x_0^2} + (v_1 - x_1)\frac{1}{x_0}, \ldots, -(v_0 - x_0)\frac{x_n}{x_0^2} + (v_n - x_n)\frac{1}{x_0}) \\ &= (\frac{v_1x_0 - v_0x_1}{x_0^2}, \ldots, \frac{v_nx_0 - v_0x_n}{x_0^2})\end{aligned}$$

The kernel of this map is those v's such that $v_ix_j = v_jx_i$ for all i, j. This is equivalent to asking that the matrix with columns $\underline{v}$ and $\underline{x}$ has rank 1, i.e. v and x are proportional, i.e. $v \in \ell$. So $\ker d_{(x_0,\ldots,x_n)}f = \ell$. For dimension reasons and because of the fundamental isomorphism theorem, it follows that

$$\boxed{T_{\mathbb{P}^n,x} = V/_{\ell} = \mathrm{Hom}_k(\ell, V/_{\ell})}$$

□

Remark 5.23. A similar, but of course more involved argument shows that if $V = k^n$ and $G(d, n)$ is the Grassmannian of d-dimensional subspaces of V, then for every $x \in G(d, n)$ corresponding to a d-dimensional subspace $W \subset V$, we have a natural identification

$$T_{G(d,n),x} = \mathrm{Hom}_k(W, V/_{W}).$$

5.2. **Singular points.**

Definition 5.24. Let $X \subset \mathbb{P}^n$ be quasiprojective and let $x \in X$. We define the **local dimension** at x denoted $\dim_x X$ or $\dim(X, x)$ as any of the following equal quantities:

i) The maximal dimension among the irreducible components of X *that contain* x.
ii) The minimal dimension among all neighborhoods $x \in U \subset X$.
iii) The Krull dimension (i.e. max length of ascending chain of prime ideals) of $\mathcal{O}_{X,x}$.

Finally we say that $x \in X$ is **singular** if $\dim T_{X,x} > \dim(X, x)$, and **nonsingular** or **smooth** otherwise. The **singular locus** of X is the set of all singular points $x \in X$. It is denoted $\mathrm{Sing}(X)$. Its complement is called the **nonsingular** or **smooth** locus and denoted X_{nonsing} or X_{sm}.

Lemma 5.25. *Let $X_1, \ldots, X_r$ be the irreducible components of X that contain x. Then*

i) $T_{X,x} \supseteq T_{X_i,x}$ for all i. In particular $T_{X,x} \supseteq \sum_i T_{X_i,x}$, the sum taking place inside $T_{\mathbb{P}^n,x} \simeq \mathbb{A}^n$.
ii) There exists $d_i \geq 0$ such that $U_i := \{x \in X_i \mid \dim T_{X_i,x} = d_i\}$ is open nonempty and $\dim T_{X_i,x} > d_i$ when $x \in X_i \setminus U_i$.
iii) $d_i = \dim X_i$ for all i. In particular $\dim T_{X,x} \geq \dim(X, x)$, and x is nonsingular if and only if equality holds.

Proof. i) This is a consequence of $i_x(\ell \cap X) \geq i_x(\ell \cap X_i)$ for all i.
ii) We may assume that $X = X_1 \subset \mathbb{A}^n$ is irreducible closed and we look at $\mathbb{A}^n$ as $D(z_0) \subset \mathbb{P}^n$. If the ideal of X in $\mathbb{P}^n$ is generated by homogeneous forms f_i, then the set

$$\mathbb{T}_X := \{(x, v) \in X \times \mathbb{P}^n \mid v \in \mathbb{T}_{X,x}\}$$

is closed, given by equations $\sum_{j=0}^n Z_i \frac{\partial f_i}{\partial z_i}(x) = 0$ for all i, which are bihomogeneous. The fibers of the first projection $p : \mathbb{T}_X \to X$ are precisely the tangent spaces $T_{X,x}$. The map p is projective. The result follows by Theorem 4.11.*iv*). (With the notation there, we have $U_i = X_{d_i} \setminus X_{d_i+1}$ for $d_i = \dim \mathbb{T}_X - \dim X$.)

iii) When $X = X_i$ is an irreducible (nonempty) hypersurface $V(f) \subset \mathbb{P}^m$, where we can assume that $f \neq 0$ is irreducible, then $\mathbb{T}_X$ is given by the equation $F = 0$ where

$$F = \sum_{i=0}^m Z_m \frac{\partial f}{\partial z_i}(x)$$

on $X \times \mathbb{P}^m$. As such it has dimension $\dim X_i + m - 1$ when $F \neq 0$ and $\dim X + m$ when $F = 0$.

- But $F = 0$ if and only if $\frac{\partial f}{\partial z_i}$ vanish everywhere on X. This means that $X \subset V(\frac{\partial f}{\partial z_i})$, hence $f | \frac{\partial f}{\partial z_i}$ for all i. Since the degree on the right is less than that of f, it follows that $\frac{\partial f}{\partial z_i} = 0$ for all i everywhere on $\mathbb{P}^m$ (not just on $X = V(f)$). Euler's formula gives

$$(\deg f) \cdot f(x_0, \dots, x_n) = \sum_{i=0}^m x_i \frac{\partial f}{\partial z_i}(x_0, \dots, x_n),$$

and since every partial is zero, if follows that $f = 0$ or $\deg f = 0$. Both contradict assumptions on the hypersurface X.
- When $F \neq 0$, then the first projection $p : \mathbb{T}_X \to X$ is clearly dominant, and then the general fiber has dimension $\dim \mathbb{T}_X - \dim X = m - 1 = \dim X$. But the fibers are precisely the tangent spaces $T_{X,x}$.

In the general case, X_i is birational to a hypersurface, hence up to passing to an open subset, we reduce to the previous case.

□

Remark 5.26. The singular locus $\mathrm{Sing}(X)$ is closed in X. (If the ideal of X in $\mathbb{A}^n$ is generated by f_i, then inside X, the singular locus is the locus where the matrix $\frac{\partial f_i}{\partial x_i}$ has bigger kernel (lower rank) than $\dim X$. This is described by the vanishing of the $\dim X \times \dim X$ minors.)

Example 5.27. a) If $\mathcal{I}(X) = (f)$ on $\mathbb{A}^n$ and X is closed, then

$$\mathrm{Sing}(X) = V_{\mathbb{A}^n}(\underbrace{f}_{f \text{ is here}}, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}).$$

(The vanishing of f at $x \in \mathbb{A}^n$ tells us that $x \in X$. The vanishing of all the partial derivatives tells us that $d_x f$ is zero at x, i.e. that $k^n = \ker d_x f = T_{X,x}$, i.e. that $\dim T_{X,x} = n > n - 1 = \dim X$, so x is singular.)

b) If $\mathcal{I}(X) = (f)$ on $\mathbb{P}^n$, and X is closed, then

$$\mathrm{Sing}(X) = V_{\mathbb{P}^n}(\underbrace{\quad}_{\text{no } f \text{ here}} \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}).$$

(This is same as before, but with the added bonus that when we work projectively, then Euler's formula tells us that the vanishing of the partials of f at x also implies the vanishing of f at x, at least when $\deg f > 0$, which is the case here.)

5.3. **Codimension one subvarieties.** Nonsingular varieties admit an analogue of Theorem 4.4, at least locally.

Definition 5.28. Let X be a quasiprojective variety, and let $Y \subset X$ be a closed subset and let $x \in Y \subset X$. We say that $f_1, \dots, f_r$ are **local equations** for Y at x if there exists an affine open neighborhood $x \in U \subset X$ such that

- f_i are defined on U for all i, and
- They generate $\mathcal{I}_U(Y \cap U) \trianglelefteq k[U]$.

Remark 5.29. The definition above seems to depend on the affine neighborhood of x. However if f_i are local equations for Y on U, then they are also local equations on any other affine neighborhood V with $x \in V \subset U$.

Let $\mathcal{I}_{X,x}(Y) \subset \mathcal{O}_{X,x}$ be the ideal generated by regular functions f at x that in a neighborhood U_f of x vanish along $Y \cap U_f$. Then $\mathcal{I}_{X,x}(Y)$ is the ideal generated by the image of $\mathcal{I}_U(Y \cap U) \subset k[U] \to k[U]_{\mathfrak{m}_x} \simeq \mathcal{O}_{X,x}$ for any affine neighborhood $x \in U \subset X$.

Then any generators $f_1, \dots, f_r$ of $\mathcal{I}_{X,x}(Y) \trianglelefteq \mathcal{O}_{X,x}$ give local equations for Y on some neighborhood of x where they are all defined.

Theorem 5.30. *If X is a quasiprojective variety and $Y \subset X$ is an equidimensional closed subset of codimension 1, then for any $x \in X$ nonsingular point, $\mathcal{I}_{X,x}(Y) = (f_x) \trianglelefteq \mathcal{O}_{X,x}$ for some $f_x \in \mathcal{O}_{X,x}$. In other words, locally around any $x \in X$, the subset Y is given by one (local) equation (depending on x).*

Proof. The proof is analogous to the easier implication in Theorem 4.4. All we need to know is that $\mathcal{O}_{X,x}$ is a UFD. This is true because $\mathcal{O}_{X,x}$ is a *regular local ring*, and then one applies the Auslander–Buchsbaum Theorem. □

Corollary 5.31. *Let X be a nonsingular quasiprojective variety and let $\varphi : X \dashrightarrow \mathbb{P}^m$ be a rational map. Then the complement of the domain of φ is a closed subset of X of codimension at least two.*

Proof. Let $x \in X$. In a neighborhood U of x, the rational function φ is given by a formula $(f_0 : \dots : f_m)$, where f_i are regular functions on U. Since $\mathcal{O}_{X,x}$ is an UFD, we can divide by any common factor of the f_i's, and get the same rational map. Thus we can assume that the f_i's don't have a common factor.

On U, the map φ is not defined where f_i all vanish. If $V(f_0, \dots, f_m) \cap U$ contains an irreducible closed subset $Y \subset U$ of codimension 1, then Y has a local equation g_x around x, so that $\mathcal{I}_{X,x}(Y) = (g_x) \trianglelefteq \mathcal{O}_{X,x}$. But then $Y \subset V(f_i)$ implies $g_x | f_i$ in $\mathcal{O}_{X,x}$ for all i. But we arranged so that the f_i's have no common factor. □

Corollary 5.32. *Any two birational projective nonsingular curves are isomorphic.*

Proof. The previous corollary tells us that a rational map from a nonsingular curve to a projective set is defined except in codimension 2, hence defined everywhere. If $\varphi : C \to C'$ is a birational map of projective nonsingular curve, we apply this for φ and φ^{-1}. □

Example 5.33. The nonsingularity of $x \in X$ is important in Theorem thrm:smloceq.

- The cusp $C = V(y^2 - x^3)$, the origin $(0,0)$ is a subset of codimension 1. It has ideal (x,y) in $\mathcal{O}_{C,(0,0)}$, and this is not principal. This is even though $(0,0)$ can be described by one equation: either x or y will do. But no equation will generate the ideal (x,y).
- Inside the cone over the quadric surface in $\mathbb{P}^3$, i.e. the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, consider the cone over the line $[1:0] \times \mathbb{P}^1$. Around the vertex of the cone, this cannot even be described by one equation. In particular its ideal is also not generated by one element.

5.4. **Nonsingular subvarieties of nonsingular varieties.** It is natural to wonder what happens in codimension bigger than 1. It turns out that we are in good shape if we add some nonsingularity condition on Y as well.

Theorem 5.34. *Let X be a quasiprojective variety and let $Y \subset X$ be a closed subvariety of codimension d. Then for any $x \in X$ that is nonsingular for both X and Y, the ideal $\mathcal{I}_{X,x}(Y)$ is generated by d elements.*

Proof.

- *Case $x = Y$.* Then $d = \dim X$. By the nonsingularity assumption, the cotangent space ${\mathfrak{m}}/{\mathfrak{m}^2}$ is a k-vector space of dimension $d = \dim X$, where $\mathfrak{m} = \mathfrak{m}_x$ is the ideal of functions that vanish at x. Choose generators for this vector space, represented mod $\mathfrak{m}^2$ by $f_1, \dots, f_d \in \mathfrak{m}$. We show that they generate $\mathfrak{m}$ as an ideal of $\mathcal{O}_{X,x}$. The f_i's are called a **system of parameters** at x (for X). If $\mathfrak{a} = (f_1, \dots, f_r)$, then the short exact sequence
$$0 \to \mathfrak{a} \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{a} \to 0$$
gives after tensoring with $k \simeq \mathcal{O}_{X,x}/\mathfrak{m}$ as a $\mathcal{O}_{X,x}$-module the exact sequence:
$$\mathfrak{a}/\mathfrak{m}\mathfrak{a} \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{a} \otimes_{\mathcal{O}_{X,x}} k \to 0$$

by the right-exactness of tensor products. The map on the left is surjective, therefore

$$\mathfrak{m}/_{\mathfrak{a}} \otimes_{\mathcal{O}_{X,x}} k = 0.$$

Nakayama's Lemma says that this only happens if $\mathfrak{m}/_{\mathfrak{a}} = 0$, i.e. $\mathfrak{a} = \mathfrak{m}$, i.e. the f_i's generate $\mathfrak{m}$.

- In the general case, the inclusion $T_{Y,x} \subset T_{X,x}$ comes from a surjection

$$\imath^* : \mathfrak{m}/_{\mathfrak{m}^2} \to \mathfrak{n}/_{\mathfrak{n}^2},$$

where $\mathfrak{m}$ is the ideal of x in $\mathcal{O}_{X,x}$, where $\mathfrak{n}$ is the ideal of x in $\mathcal{O}_{Y,x}$, and where $\imath : Y \to X$ is the inclusion of Y. Up to changing bases we can then pick $f_1, \ldots, f_{n+d} \in \mathfrak{m} \trianglelefteq \mathcal{O}_{X,x}$ a system of parameters at x for X such that $\imath^* f_{1+d}, \ldots, \imath^* f_{n+d} \in \mathfrak{n} \trianglelefteq \mathcal{O}_{Y,x}$ is also a system of parameters at x, but for Y, and such that $f_1, \ldots, f_d$ generate $\ker \imath^* = \mathcal{I}_{X,x}(Y)$ mod $\mathfrak{m}^2$.

We show that there exist local equations $g_1, \ldots, g_d$ for $Y \subset X$ at x, i.e. they generate $\mathcal{I}_{X,x}(Y)$, such that $g_i = f_i$ mod $\mathfrak{m}^2$.

Start by choosing $g_i \in (f_i + \mathfrak{m}^2) \cap \mathcal{I}_{\mathcal{O}_{X,x}}(Y)$. This is possible by the assumption that $f_1, \ldots, f_d$ generate $\mathcal{I}_{X,x}(Y)$ mod $\mathfrak{m}^2$.

Let $\mathfrak{a} \trianglelefteq \mathcal{O}_{X,x}$ be the ideal generated by the g_i's, and let Y' be the closed subset of X defined by these equations (in a neighborhood of x where all functions involved are regular). Then we have $\sqrt{\mathfrak{a}} = \mathcal{I}_{X,x}(Y')$ by the Nullstellensatz and $Y \subseteq Y'$ since $\mathfrak{a} \subset \mathcal{I}_{X,x}(Y) \trianglelefteq \mathcal{O}_{X,x}(Y)$. Let h_j be generators for $\mathcal{I}_{X,x}(Y')$. Looking in $T_{X,x}$, we have

$$T_{Y,x} \subseteq T_{Y',x} = \ker(d_x h_j)_j \subseteq \ker(d_x g_i)_i = \ker(d_x f_i)_{1 \le i \le d} = T_{Y,x}.$$

Therefore $T_{Y,x} = T_{Y',x}$. Since $\dim T_{Y',x} \geq \dim(Y', x) \geq \dim(Y, x) = \dim T_{Y,x}$, it follows that $\dim(Y', x) = \dim(Y, x)$ and x is a nonsingular point of Y'. Since Y is an irreducible subset of Y' and they have the same dimension (around x), it follows that Y is an irreducible component of Y'.

If $Y \subsetneq Y'$, then Y' must have another irreducible component through x, and this implies that $\mathcal{O}_{Y',x}$ is not a domain. But if $x \in Y'$ is a nonsingular point, then $\mathcal{O}_{Y',x}$ is a *regular local ring*, and as such is a domain. Therefore $Y = Y'$.

We are not quite done. We still should check that g_i generate the ideal of $Y' = Y$ around x, i.e. $\mathfrak{a} = \mathcal{I}_{X,x}(Y')$, and not just that they are equations for Y. This follows with a bit of work from the equality $\ker(d_x h_j)_j = \ker(d_x g_i)_i$. Intuitively, if instead of working with generators, we work with equations, then we get a larger kernel for the differentials as in Remark 5.3.

□

6. Blow-ups

The blow-up of a subvariety (or subscheme) is the most important construction of birational maps in algebraic geometry. It is used to "resolve" singularities, or maps.

6.1. The blow-up of $\mathbb{P}^2$ at one point.

This is the map pictured on the cover of the textbook.

Definition 6.1 (Definition by equations)**.** Let $\mathrm{Bl}_p\mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the closed subset of equation

$$z_1x_2 - z_2x_1 = 0,$$

where $[z_0 : z_1 : z_2]$ are coordinates on $\mathbb{P}^2$ and $[x_1 : x_2]$ on $\mathbb{P}^1$, and where $p = [1 : 0 : 0]$ is the origin in $D(z_0) \simeq \mathbb{A}^2$. The restriction of the first projection induces a regular map

$$\sigma : \mathrm{Bl}_p\mathbb{P}^2 \to \mathbb{P}^2.$$

This is called the **blow-up of $\mathbb{P}^2$ at** $[1 : 0 : 0]$.

Remark 6.2. The blow-up at the point $[1 : a : b]$ has equation $(z_1 - az_0)x_2 - (z_2 - bz_0)x_1 = 0$.

Proposition 6.3. *The blow-up map $\sigma : \mathrm{Bl}_p\mathbb{P}^2 \to \mathbb{P}^2$ has the following properties:*

i) We can see $\mathbb{P}^1$ as a parameter space for lines in $\mathbb{P}^2$ through p. To a point $[u : v] \in \mathbb{P}^1$ we can associate bijectively the line

$$\ell_{uv} := V(z_1v - z_2u)$$

which clearly contains $p = [1 : 0 : 0]$. Then

$$\mathrm{Bl}_p\mathbb{P}^2 = \{(x, \ell) \mid \{x, p\} \subset \ell\}.$$

ii) $\mathrm{Bl}_p\mathbb{P}^2$ is the closure of the graph of the stereographic projection

$$[z_0 : z_1 : z_2] \mapsto [z_1 : z_2] : \mathbb{P}^2 \overset{\varphi_p}{\dashrightarrow} \mathbb{P}^1.$$

We say that the blow-up at p resolves the indeterminacies *of the rational map φ_p, in that it replaces φ_p with the second projection $\pi : \mathrm{Bl}_p\mathbb{P}^2 \to \mathbb{P}^1$ which is defined everywhere.*

iii) Then second projection $\pi : \mathrm{Bl}_p\mathbb{P}^2 \to \mathbb{P}^1$ has the property that

$$\pi^{-1}\{[u : v]\} = \bar{\ell}_{uv} = \ell \times [u : v] \subset \mathbb{P}^2 \times \mathbb{P}^1.$$

iv) If $[u : v] \neq [u' : v']$, then $\bar{\ell}_{uv}$ and $\bar{\ell}_{u'v'}$ are disjoint in $\mathrm{Bl}_p\mathbb{P}^2$. We say that the blow-up of p "separates" the lines ℓ_{uv} through p.

v) $\mathrm{Bl}_p\mathbb{P}^2$ is nonsingular.

vi) The blow-up map σ is birational and an isomorphism away from p.

vii) $\sigma^{-1}\{p\} = \{p\} \times \mathbb{P}^1$. Denote this curve by E. It is called **exceptional***.*

viii) $\sigma^{-1}\ell_{uv} = E \cup \bar{\ell}_{uv}$, where $\bar{\ell}_{uv} = \ell \times [u : v] \subset \mathbb{P}^2 \times \mathbb{P}^1$.

ix) $E \cap \bar{\ell}_{uv} = \{p\} \times \{[u : v]\}$. This means that we have a point on E for every line $\ell_{uv} \subset \mathbb{P}^2$ through p.

x) If f is a reduced form (i.e. no multiple factors) of degree d on $\mathbb{P}^2$ and $C = V(f)$, then

$$\sigma^{-1}C = \begin{cases} E \cup \bar{C}, & \text{if } C \text{ passes through } p \\ \bar{C}, & \text{otherise} \end{cases}$$

where $\bar{C}$ is the closure of $\sigma^{-1}(C \setminus p) \subset \mathbb{P}^2 \setminus \{p\} \times \mathbb{P}^1$ in $\mathbb{P}^2 \times \mathbb{P}^1$. The curve $\bar{C}$ is called the **strict transform** *of C.*

xi) If after dehomogenization in $\mathbb{A}^2 \simeq \mathbb{D}(z_0)$, the form f is written as $f_k + f_{k+1} + \ldots$, where f_i are forms in the coordinates on $\mathbb{A}^2$ and f_k is the first nonzero one that appears, then

$$E \cap \bar{C} = \{(p, [u : v]) \mid f_k(u, v) = 0\}.$$

In other words we have a point in $E \cap \bar{C}$ for each line through p in $\mathbb{P}^2$ that has higher multiplicity of intersection with C at p than "usual".

Example 6.4. Consider the cusp $C = V(z_2^2 z_0 - z_1^3) \subset \mathbb{P}^2_{\mathbb{C}}$. The singularities are given by the vanishing of the partials of the equations, which means

$$\mathrm{Sing}(C) = V(z_2^2, 3z_1^2, 2z_2z_0) = \{[1 : 0 : 0]\}.$$

Let's see that $\bar{C}$ is smooth.

Since the blow-up of $\mathbb{P}^2$ at $[1 : 0 : 0]$ is isomorphic to $\mathbb{P}^2$ outside p, we only care about what happens at or around p. This is captured by working over $D(z_0) \simeq \mathbb{A}^2$.

$\mathrm{Bl}_p\mathbb{P}^2$ is covered by $D(x_1)$ and $D(x_2)$ in $\mathbb{P}^2 \times \mathbb{P}^1$.

Let's work in $D(x_1)$. Put $u = \frac{z_1}{z_0}$, $v = \frac{z_2}{z_0}$ and $x = \frac{x_2}{x_1}$. Then

- $C \cap D(z_0)$ is given by the equation $v^2 - u^3 = 0$ in $D(z_0)$
- $\mathrm{Bl}_p\mathbb{P}^2$ inside $\mathbb{A}^2 \times \mathbb{A}^1$ is given by $v = ux$
- $\sigma : D(z_0x_1) \to D(z_0)$ given by the formula $\sigma(u, v, x) = (u, v)$ can be identified with
$$(u, x) \mapsto (u, ux) : \mathbb{A}^2 \to \mathbb{A}^2$$
- With respect to this map, $\sigma^{-1}C$ is the curve of equation
$$0 = \sigma^*(v^2 - u^3) = (ux)^2 - u^3 = u^2(x^2 - 1) = u^2(x^2 - u)$$
- Also with respect to this map, the exceptional E is $V(u)$.
- Then $\bar{C} \cap D(x_1)$ is given by the equation $x^2 - u = 0$ inside $D(z_0x_1) \simeq \mathbb{A}^2$ with coordinates u, x. But $V(x^2 - u)$ is a smooth parabola in $\mathbb{A}^2$.

Similarly in $D(x_2)$, by putting $y = \frac{x_1}{x_2}$, we get that $\bar{C} \cap D(x_2)$ has equation $1 - y^3v = 0$, which is again a smooth curve. □

7. Divisors and Class Group

Let $X \subset \mathbb{P}^m$ be a smooth projective variety of dimension n.

Definition 7.1. A **divisor** on X is an element of the free abelian group generated by irreducible subvarieties of X of codimension 1. Such an irreducible subvariety is also called a **prime divisor**. Denote this group by $Z_{n-1}(X)$.

Example 7.2. If $X = \mathbb{P}^1$, then $Z_0(X)$ is just the set of all possible combinations $\sum_{i=1}^r a_i x_i$, where $a_i \in \mathbb{Z}$ and $r \geq 0$. The sum is purely formal, it is not supposed to return a point in $\mathbb{P}^1$ as the result of the linear combination.

Example 7.3. If $X \subset \mathbb{P}^n$ and f is a form that does not vanish on all of X, then $V(f) \cap X$ is a divisor on X.

A very important family of examples of divisors are constructed from forms or from rational functions on X.

Definition 7.4. Let f be a form on $\mathbb{P}^n$ that does not vanish on all of X. Then we constrcut a divisor

$$\operatorname{div}(f) := \sum_i \operatorname{ord}_{D_i}(f),$$

where the D_i's range through the irreducible components of $V_X(f)$, and the **order** $\operatorname{ord}_{D_i}(f)$ is computed as follows: Let x_i be an arbitrary (but fixed) point on D_i. Then there exists $f_i \in \mathcal{O}_{X,x}$ such that $\mathcal{I}_{X,x}(D_i) = (f_i)$ and $D_i = V(f_i)$ in a neighborhood of x (see Theorem 5.30). Then $\operatorname{ord}_{D_i}(f)$ is the largest $r_i \geq 0$ such that $\frac{f}{f_i^{r_i}} \in \mathcal{O}_{X,x}$.

If $f = \frac{g}{h}$ is a nonzero rational function on X, i.e. a ratio of forms of the same degree that do not vanish on all of X, then define

$$\operatorname{div}(f) := \operatorname{div}(g) - \operatorname{div}(h).$$

In this case when f is a rational function, we say that $\operatorname{div}(f)$ is a **principal divisor**.

Remark 7.5. The principal divisors on X form a subgroup. In fact the map

$$f \mapsto \operatorname{div}(f) : (k(X)^*, \cdot) \to (Z_{n-1}, +)$$

is a group morphism (turns multiplication into addition). All this uses is that $\mathcal{O}_{X,x}$ is an UFD for all $x \in X$, which is true because X is nonsingular/smooth.

Definition 7.6. The **class group** of X is the quotient

$$Cl(X) := Z_{n-1}(X) \big/ \text{principal divisors}.$$

We say that divisors D and D' are **linearly equivalent** and write $D \sim D'$ if they have the same class in $Cl(X)$. Equivalently if $D - D' = \operatorname{div}(f)$ for some nonzero rational function on X.

This is typically a huge group. Maybe infinitely generated.

Example 7.7. $Cl(\mathbb{P}^n) \simeq \mathbb{Z}$ and a generator is the class of a (any) hyperplane. (Let H be the class of the hyperplane $V(z_0)$ in $Cl(\mathbb{P}^n)$. It is enough to show that if D is any divisor, then $D \sim mH$ for some m. Write $D = \sum_i a_i D_i$. Since we work on $\mathbb{P}^n$, by Theorem 4.4 we know that $D_i = V(f_i)$ for some irreducible form f_i of degree d_i. Put $m = \sum_i a_i d_i$. Then $D - mH = \operatorname{div}(z_0^{-m} \cdot \prod_i f_i^{a_i})$, and the latter is readily seen as a ratio of forms of the same degree if we separate the negative coefficients from the positive ones.) □

Example 7.8. If C is a smooth projective curve, then $\mathrm{Cl}(C) \simeq \mathbb{Z} \times \mathrm{Jac}(C)$, where $\mathrm{Jac}(C)$ is a variety of dimension equal to the genus $g(C)$. It is actually an **abelian variety** (projective variety with a group structure), called the **Jacobian** of C. It can be identified with the kernel of the **degree morphism** $(Cl(C), +) \to (Z, +)$ which sends $D = \sum_i a_i x_i$ to the integer $\deg(D) = \sum_i a_i$.

Hidden here is the nontrivial result that $\deg(D) = 0$ for all principal divisors on the curve C, so that deg is well defined modulo principal divisors. For example on a Riemann surface, this says that a meromorphic function has as many zeros as it has poles (when counting multiplicities/orders). □

Remark 7.9. The constructions so far can be adapted for singular quasiprojective varieties as long as $\mathrm{Sing}(X)$ has codimenion two or more in X, i.e. it contains no divisors of X. This happens for example for the so called *normal* varieties. Then for each D_i we can pick $x_i \in D_i$ that is nonsingular in X and repeat the definition of orders.

The only result mentioned so far that may fail if quasiprojective replaces projective is that $\mathrm{div}(f) = 0$ if $f \in k(C)^*$, where C is a curve.

With some work, $\mathrm{div}(f)$ can be defined even when $\mathrm{Sing}(X)$ contains divisors.

Example 7.10. $Cl(\mathbb{A}^1) = 0$. This is because every point $x \in \mathbb{A}^1$ is $V(f)$ for some rational function $f \in k(\mathbb{A}^1)$: Just put $f(X) = X - x_0$.

Somewhat similarly one can show that $Cl(\mathbb{A}^n) = 0$ for any $n \geq 1$.

Example 7.11. Put $X = \mathrm{Bl}_p\mathbb{P}^2$. Then $Cl(X) = \mathbb{Z}^2$, where $p = [1 : 0 : 0]$. A set of generators is given by H and E, where $H = \sigma^{-1}\ell$ is the inverse image (not just the strict transform usually, but for this particular line they are the same) of the line $\ell = V(z_0) \subset \mathbb{P}^2$ via $\sigma : X \to \mathbb{P}^2$, and E is the exceptional divisor of the blow-up. (Let's first show that if T is an irreducible curve inside the blow-up, then

$$T \sim dH + mE$$

for some integers d and E. Look at $\sigma(T)$. This is an irreducible curve inside $\mathbb{P}^2$, hence $T = V_{\mathbb{P}^2}(g)$, where g is a form of some degree d. Put $f = \frac{g}{z_0^d}$. This is a rational function, since it is a ratio of forms of the same degree. We have

$$\mathrm{div}_{\mathbb{P}^2}(f) = \sigma(T) - d\ell.$$

Since σ is birational, $k(X) = k(\mathbb{P}^2)$. Therefore f can also be seen as a rational function on X via σ^*. Let's compute $\mathrm{div}_X(f)$. Since $\sigma : X \setminus E \to \mathbb{P}^2 \setminus \{p\}$ is an isomorphism, the divisors $\mathrm{div}_X(f)$ and $\mathrm{div}_{\mathbb{P}^2} f$ are equal except possibly over p. But over p we only have the divisor E, therefore

$$\mathrm{div}_X(f) = \overline{\sigma(T)} - d\bar{\ell} - mE = T - d\bar{l} - mE$$

for some integer m, where over-lines denote strict transforms, since the irreducibility of T implied $T = \overline{\sigma(T)}$. Finally $T - \mathrm{div}_X(f) = dH + mE$.

Next we show that H and E are linearly independent in $Cl(X)$. Assume $\mathrm{div}_X(f) = aH + bE$ for some integers a, b not both zero and some nonzero rational function f. Since $k(X) = k(\mathbb{P}^2)$, we can also see f as rational on $\mathbb{P}^2$. It is not hard to show that $\mathrm{div}_{\mathbb{P}^2}(f) = a\sigma(H)$, since $\sigma(E) = p \subset \mathbb{P}^2$ is not a divisor. Therefore $a\sigma(H)$ is principal on $\mathbb{P}^2$, and this only happens if $a = 0$. Now look inside the copy of $\mathbb{A}^2 \subset X$ which $D(z_0x_1)$ from last time is isomorphic to. Recall that the coordinates on $\mathbb{A}^2$ where (u, x) with $u = \frac{z_1}{z_0}$ and $x = \frac{x_2}{x_1}$. And $E \cap \mathbb{A}^2 = V_{\mathbb{A}^2}(u)$. From this and the condition $\mathrm{div}_X(f) = bE$, we get that as rational functions on $\mathbb{A}^2$ which is open in X, we have $f = cu^b$, where c is a nonzero constant. Therefore

$$f = c\frac{z_1^b}{z_0^b}$$

on X and/or $\mathbb{P}^2$. But $\mathrm{div}_X(c\frac{z_1^b}{z_0^b}) = b(V_X(z_1) - V_X(z_0)) = b(\bar{\ell}_{01} + E - \bar{\ell}) \neq bE$. □)

8. Bézout's Theorem

Maybe the simplest form of Bézout's Theorem (BT) is that a linear polynomial has exactly one solution. It has a famous generalization, the Fundamental Theorem of Algebra (FTA) a particular case of which says that *most* polynomials of degree d have exactly d zeroes. What distinguishes *most* polynomials from any polynomial is that they have nonzero discriminant, meaning that all their zeros have multiplicity one. If we want to obtain a general statement, we count multiplicities, and indeed the FTA says that if we sum the multiplicities of each zero, we recover the degree of the polynomial.

Let's see some of these as geometric statements:

Example 8.1.

- To intersect $V(y)$ with $V(y - ax - b)$ in $\mathbb{A}^2$, we solve the system $\begin{cases} y = 0 \\ y = ax + b \end{cases}$ By substitution, this is the same as solving $ax + b = 0$.
- Similarly, if $f(x)$ is a polynomial of degree d, then solving $f(x) = 0$ amounts to intersecting $V(y)$ with $V(y - f(x))$ in $\mathbb{A}^2$. Note that if x_0 is a zero of f, then its multiplicity as a zero of f is equal to the multiplicity of intersection $i_{(x_0,0)}(\ell \cap C)$ defined in 5.1, where $\ell = V(y)$ and $C = V(y - f(x))$.

In both cases, we intersect a curve of degree 1 with a curve of degree d (given by one equation of degree d), and the result is d points in $\mathbb{A}^2$, possibly after counting them with their multiplicities. We also have that the lines is not an irreducible component of the curve.

- If we intersect the two curves $V(y)$ and $V(y-1)$ in $\mathbb{A}^2$, then we get no point, because the lines are parallel, or because $y = y-1 = 0$ is impossible. This seems to contradict the principle above that when I intersect two curves of degree one I get one point. The culprit for this pathology is not the principle, but the ambient space. We know intuitively that parallel lines intersect "at infinity" and this can be made precise **by working in the projective space.**

  Indeed, the projective equations of the two lines are y and $y - z$, which intersect in the point $[1 : 0 : 0]$ in $\mathbb{P}^2$, which is a point in the line at infinity $V(z)$.

By changing coordinates, the second example says that when we intersect any line with any curve of degree d in $\mathbb{P}^2$, we get d points when counting multiplicities, at least when the line is not a component of the curve. □

So how about when we intersect a curve of degree d with a curve of degree e? And what can we do when we want to intersect curves that have common components?

Let's recall how we can justify multiplicities of intersection that are higher than 1. For example the line $V(y)$ has multiplicity two of intersection with the parabola $y = x^2$ at $(0, 0)$, because it is the limit of the lines ℓ_ϵ through $(0, 0)$ and (ϵ, ϵ^2) as $\epsilon \to 0$. Every line ℓ_ϵ intersects the parabola at two points (near $(0, 0)$). We keep this "two" when we pass to the limit, remembering that the one point we see is actually two points collapsed into one. We get the following:

Principle: The *number of* points counted with multiplicity of two curves does not change when we deform one or both curves (making sure we don't get common components).

Example 8.2. We can even use the principle to "intersect" the curve $V(y)$ with itself. Set theoretically, we get the curve, instead of the expected one point. But if instead of intersecting $V(y)$ and $V(y)$ we intersect $V(y)$ and $V(y - \epsilon x)$, then we get one point for all ("small") nonzero epsilon, so we can say that the curves intersect at one point. □

Example 8.3. More importantly we can use the principle to intersect curves C and C' of degrees d and e respectively by deforming C' until it becomes a union of e distinct lines none of which is a component C.

Specifically, if the ideal of C' is generated by g, a form of degree e, and $f_1, \ldots, f_e$ are distinct and nonproportional forms of degree 1, then if we put

$$g_\epsilon := (1-\epsilon)g + \epsilon \cdot f_1 \cdot \ldots \cdot f_e,$$

then for each $\epsilon \in k$, we have that g_ϵ is a form of degree e. When $\epsilon = 0$, we get C', and when ϵ approaches 1, we approach $\cup_{i=1}^e V(f_i)$, a union of e distinct lines. We add together all the multiplicities when we intersect C with each of them, and get de points. □

Remark 8.4. It may seem like we could do the g_ϵ trick even when we use more or less than e linear forms. However in that case g_ϵ itself is no longer a form, so that won't do.

Still, like we did when we intersected a curve with a line and the set theoretic intersection was points, we should be able to assign multiplicities of intersection when we intersect curves of arbitrary degrees and get only points set theoretically.

Definition 8.5. Let C and D be projective plane curves without common components. Let $x \in C \cap D$. Then the **multiplicity of intersection at** x is

$$i_x(C \cap D) = \dim_k \mathcal{O}_{\mathbb{P}^2,x} / \mathcal{I}_{\mathbb{P}^2,x}(C) + \mathcal{I}_{\mathbb{P}^2,x}(D).$$

Example 8.6. Assume that $f \in k[x,y]$ with $f(0,0) = 0$ and f irreducible. Let ℓ_a be the line $y = ax$. Let C be the projective closure of $V(f)$ and $\bar{\ell}_a$ the projective closure of ℓ_a. Then for $x = (0,0) = [0:0:1]$ belonging to both curves, we get

$$i_x(C \cap \bar{\ell}_a) = \dim_k \mathcal{O}_{\mathbb{P}^2,x} / (f, y-ax) = \dim_k k[x,y]_{(x,y)} / (f, y-ax).$$

Recall that $k[x,y]_{(x,y)}$ is the fraction ring of $k[x,y]$ with multiplicative system $k[x,y] \setminus (x,y)$: every polynomial f not in (x,y), i.e. with $f(0,0) \neq 0$ is invertible in this ring.

We want to see that this multiplicity of intersection agrees with Definition 5.1.

Write $f = f_1 + f_2 + \ldots$, where f_i is the homogeneous part of degree i. The old multiplicity of intersection is the multiplicity of $x = 0$ as a zero of $f(x, ax) = \sum_{i \geq 0} f_i(x, ax)$. This is m if $f_m(x, ax)$ is the first nonzero term in the sum. Then we can write $f(x, ax) = x^m g_a(x)$, with $g_a(0) \neq 0$.

Now use the isomorphisms

$$k[x,y]_{(x,y)} / (f, y-ax) \simeq k[x]_{(x)} / (f(x,ax)) \simeq k[x]_{(x)} / (x^m g_a(x)) \simeq k[x]_{(x)} / (x^m),$$

where the latter is true since $g_a(x)$ is invertible in $k[x]_{(x)}$, because it is not in (x). Now the last quotient is of dimension m, generated by $1, x, \ldots, x^{m-1}$. So the new multiplicity agrees with the old one (when we intersect with lines). □

Let's also see a new example

Example 8.7. Let's intersect the cusp with the node at $(0,0)$. The ideals are generated by $y^2 - x^3$ for the cusp and by $y^2 - x^3 - x^2$ for the node. Then we compute

$$k[x,y]_{(x,y)} / (y^2 - x^3, y^2 - x^3 - x^2) = k[x,y]_{(x,y)} / (y^2 - x^3, x^2) = k[x,y]_{(x,y)} / (y^2, x^2).$$

This is generated by $1, x, y, xy$, so that the multiplicity of intersection there is equal to 4.

The closures in $\mathbb{P}^2$ are of equations $y^2z - x^3 = 0$ and $y^2z - x^3 - x^2z = 0$ give another solution $[0:1:0]$, which lives on the line at infinity, sitting as the origin in the affine set

$D(y)$. In this open subset, the equations are $z - x^3$ and $z - x^3 - x^2 z = 0$. To compute the multiplicity of intersection at $(0,0) = [0:1:0]$, we look at

$$k[x,z]_{(x,z)}\big/(z - x^3, z - x^3 - x^2 z) = k[x,z]_{(x,z)}\big/(z - x^3, x^2 z) = k[x]_{(x)}\big/(x^5).$$

This is generated by $1, x, \ldots, x^4$, so it has dimension 5.

So adding the multiplicities at the two points gives $4 + 5 = 3 \cdot 3$, so still the products of degrees. □

Remark 8.8. The reason why we work in the local ring at x is that this preserves the "local picture". For example if we look at the node given by the equation $y^2 - x^3 - x^2$, then in the local ring of $\mathbb{P}^2$ around $[0:0:1]$, this equation is $y^2 = x^3 + x^2 = x^2(1+x)$. Since $1 + x$ does not vanish at $(0,0) = [0:0:1]$, it is invertible in the ring of regular functions around $(0,0)$. It is not trivial why we can do this, but we can pretend that it is 1. Then the equation looks like $y^2 = x^2$, which is a union of distinct lines $y = x$ and $y = -x$.

Actually we can compute the multiplicity of intersection between the cusp and the node at $(0,0)$ this way. The cusp has multiplicity of intersection 2 with each of the curves $y = \pm x$, so a total of 4, which is the number we got before.

If you want to make it precise why you can replace $y^2 = x^2(1+x)$ by $y^2 = x^2$, the trick is to see that the multiplicity of intersection gives the same result whether computed in $\mathcal{O}_{\mathbb{P}^2,[0:0:1]}$ or in its *completion*, which in this case is just the ring of power series $k[[x,y]]$. Then $1 + x$ admits a square root $r(x)$ that is invertible and $\begin{cases} x \mapsto x \\ y \mapsto \frac{y}{r(x)} \end{cases}$ is an automorphism of $k[[x,y]]$ that turns the node into the union of two lines through $(0,0)$.

Remark 8.9. As long as both C and D pass through x, we have that $I_{\mathbb{P}^2,x}(C) + \mathcal{I}_{\mathbb{P}^2,x}(D) \subset \mathfrak{m}_x \trianglelefteq \mathcal{O}_{\mathbb{P}^2,x}$. In particular the class of 1 is nonzero in $\mathcal{O}_{\mathbb{P}^2,x}\big/\mathcal{I}_{\mathbb{P}^2,x}(C) + \mathcal{I}_{\mathbb{P}^2,x}(D)$, so the multiplicity of intersection is at least one.

Remark 8.10. If C is singular at x, and $x \in C \cap D$, then $i_x(C \cap D) \geq 2$. (If f and g are the equations of C and D, then the singularity of x implies $f \in \mathfrak{m}_x^2$. Choose an affine neighborhood of x and coordinates so that $x = (0,0)$. Let g_1 be the linear term of g.

- If $g_1 = 0$, then the classes $1, x, y$ are independent in $\mathcal{O}_{\mathbb{P}^2,x}\big/\mathcal{I}_{\mathbb{P}^2,x}(C) + \mathcal{I}_{\mathbb{P}^2,x}(D) = k[x,y]_{(x,y)}\big/(f,g)$, because the denominator is included in m_x^2. In this case the multiplicity is at least 3.
- If $g_1 \neq 0$, then by changing coordinates, we make $g_1 = y$. Then the classes of $1, x$ are independent in the quotient, so we get a dimension at least 2, meaning a multiplicity of intersection of at least 2.)

Theorem 8.11 (Plane Bézout)**.** *If C and D are curves without common components, then the sum of the multiplicities of intersection is* $\deg C \cdot \deg D$.

Corollary 8.12. *Any irreducible conic in $\mathbb{P}^2$ is nonsingular.*

Proof. Let f be an irreducible form of degree 2 and $C = V(f)$. Assume $p \in C$ is a singular point. Let $q \in C$ be another point. Let ℓ be the line through p and q. We have $\ell \subsetneq C$, or else f is reducible. By BT, we get two points of intersection between ℓ and C when we count multiplicities. The points p and q are in this intersection, so the multiplicity is at least 1 at each of them, but the line ℓ must have multiplicity at least 2 at p, because it is singular, so $\mathbb{T}_p C = \mathbb{P}^2 \supset \ell$. We get the contradiction $2 \geq 3$. □

Corollary 8.13. *A singular irreducible cubic in $\mathbb{P}^2$ is rational.*

Proof. Let C be a cubic (curve of degree 3) and let p be a singular point. Fix a line ℓ in $\mathbb{P}^2$ that does not pass through p. For each $x \in \ell$, let ℓ_x be the line through p and x. By BT it meets C at 3 points when counting multiplicities. As in the previous corollary, it meets it at p at least twice, because it is a singular point. Then there is at most one more points in $C \cap \ell_x$, and we denote this by q_x. This defines an actually regular map $\mathbb{P}^1 \to C$. It is of course not an isomorphism, because C is singular, and $\mathbb{P}^1$ is not. But it has a birational inverse given by $q \mapsto \overline{pq} \cap \ell : C \setminus \{p\} \to \mathbb{P}^1$, where $\overline{pq}$ is the line through p and q. □

Corollary 8.14. *An irreducible plane quartic (curve of degree 4 in $\mathbb{P}^2$) cannot have four singular points.*

Proof. Assume C is a quartic and that $p_1, \dots, p_4$ are distinct singular points on it. Choose $p_5 \in C$ different from the previous ones. We show that there exists a conic Q that passes through all of them. Then $Q \cap C$ contains the 4 singular points $p_1, \dots, p_4$ with multiplicity at least 2 at each by Remark 8.10 and p_5 with multiplicity at least 1, which by BT leads to the contradiction $4 \cdot 2 \geq 2 + 2 + 2 + 2 + 1$.

Let's prove the claim. Assume first that 3 of the singular points are on a line ℓ. Then let ℓ' be the line through the other two. We can put $Q = \ell \cup \ell'$. Assume now that no three singular points are on a line. The space of conics in $\mathbb{P}^2$ is parameterized by a $\mathbb{P}^5$ by looking at the coefficients of the form. The condition that a conic passes through a point $p \in \mathbb{P}^2$ is linear in the coefficients of the equation of the conic. This means that conics in $\mathbb{P}^2$ through a fixed point correspond to a hyperplane in $\mathbb{P}^5$. If we have 5 points, no three on the same line, then in $\mathbb{P}^5$ we'll see 5 hyperplanes that meet at one point. (This may require some thinking). This says actually that through any 5 general points in $\mathbb{P}^2$ there passes *exactly* one (automatically smooth) conic. □

8.1. **Arbitrary smooth projective surfaces.** Let X be a smooth projective surface.

Definition 8.15. If C and D are curves on X that intersect in just points (no common curve components), then by analogy with Definition 8.5, define

$$C \cdot D = \sum_{x \in C \cap D} i_x(C \cap D)$$

where $i_x(C \cap D) = \dim_k \mathcal{O}_{X,x}/(f,g)$, where f, g are *local equations* for C and D around x.

The formula is easily seen to be linear in both C and D (so bilinear), thus allowing to assume that C and D are divisors (with possibly negative coefficients).

Assume that $D = \operatorname{div}_X(f)$, where f is a nonzero rational function on X. So D is a *principal divisor*. If C is an irreducible curve in X that does not appear among the components of D, then we can write $f = \frac{g}{h}$, where g, h are forms on some $\mathbb{P}^n$ containing X such that C is not a component of $V(g)$ or $V(h)$. This precisely means that f restricts to a nonzero rational function on C. It can be shown that for each $x \in C \cap (V(g) \cup V(h))$, we have $i_x(C \cap V(g)) - i_x(C \cap V(h) = \operatorname{ord}_x(f|_C)$. Then

$$\deg(\operatorname{div}(f|_C)) = \sum_x \operatorname{ord}_x(f|_C) = \sum_x i_x(C \cap D) = C \cdot D.$$

By abuse $C \cap D$ denotes the set theoretic intersection between the union of components of C and curves appearing in D with positive or negative coefficient. However last time we agreed that the degree of a principal divisor on a curve is zero. Therefore $C \cdot \operatorname{div}_X(f) = 0$.

From the previous paragraph it follows that for any two divisors C and D, the intersection $C \cdot D$ depends only on their classes in $Cl(X)$, which is divisors modulo principal divisors. We have obtained the following:

Theorem 8.16 (Surface Bézout). *If X is a smooth projective surface, then there exists a bilinear pairing called an* **intersection product**

$$(C, D) \mapsto C \cdot D : Cl(X) \times Cl(X) \to \mathbb{Z}$$

such that if $C \cap D$ is just points, then $C \cdot D$ is the sum of the multiplicities at these points.

Remark 8.17. The intersection product is intrinsic to X. In particular it is preserved by isomorphisms.

Example 8.18. If $X = \mathbb{P}^2$, then $C \cdot D = \deg(C) \cdot \deg(E)$. This also works if C and D have components in common, because then we can replace one of them by a linearly equivalent divisor such that we only get points in the intersection.

For example if $C = D = V(z_0)$, then $D \sim V(z_1)$, because $V(z_1) - V(z_0) = \mathrm{div}_{\mathbb{P}^2}(\frac{z_1}{z_0})$. Then $C \cdot D = V(z_0) \cdot V(z_1) = 1 \cdot 1 = 1$. □

Example 8.19. If X is the blow-up of $\mathbb{P}^2$ at one point $p = [0 : 0 : 1]$, and $H \in Cl(X)$ is the class of the strict transform of a line not passing through p, and $E \in Cl(X)$ is the class of the exceptional divisor of the blow-up, then the intersection product on X is determined by the formulas.

$$H \cdot H = 1$$
$$H \cdot E = 0$$
$$E \cdot E = -1$$

(The three numbers determine the product because $Cl(X)$ is the free abelian group of rank 2 generated by the classes of H and E.

The first is true because using divisors on $\mathbb{P}^2$ and the fact that $X \setminus E \simeq \mathbb{P}^2 \setminus \{p\}$, one shows that the strict transform of *any* line in $\mathbb{P}^2$ not passing through p is H. The multiplicity of intersection is local, so we can compute it on $\mathbb{P}^2$ away from p, which means it is one.

The second relation is true because the strict transform of a line not through p and the exceptional divisor do not intersect, so there is no point to compute an intersection of multiplicity at.

The third relation may seem unexpected. After all the exceptional divisor is a $\mathbb{P}^1$, but we are not in $\mathbb{P}^2$ anymore. Indeed consider the rational function $f = \frac{z_2}{z_0}$ on $\mathbb{P}^2$, which is birational to X, so f is also rational on X. Then $\mathrm{div}_X(f) = V(z_2) - V(z_0) = H - (\ell + E)$, where ℓ is the strict transform of $V(z_0)$.

Then using the bilinearity and that principal divisors don't contribute to intersection numbers, we get

$$E \cdot E = (E + \mathrm{div}_X(f)) \cdot E = (H - \ell) \cdot E = H \cdot E - \ell \cdot E = -\ell \cdot E.$$

We show that $\ell \cdot E = 1$, which gives $E \cdot E = -\ell \cdot E = -1$. In X we have an open subset $\mathbb{A}^2$ such that the blow-up map $X \to \mathbb{P}^2$ looks like $(x, y) \mapsto (xy, x) : \mathbb{A}^2 \to \mathbb{A}^2$. Then $V_{\mathbb{P}^2}(z_0)$ corresponds to $V_{\mathbb{A}^2 \subset \mathbb{P}^2}(x)$, so its inverse image in X is $V_X(xy) = V_X(x) \cup V_X(y)$. The first part, $V_X(x)$ is the inverse image of $(0, 0)$, so it is the exceptional divisor E. The second part is the strict transform ℓ by definition. Furthermore ℓ and E meet at $(0, 0)$ with multiplicity 1, because so do $V_{\mathbb{A}^2 \subset X}(y)$ and $V_{\mathbb{A}^2 \subset X}(x)$.) □

Example 8.20. Perhaps a simpler example is $X = \mathbb{P}^1 \times \mathbb{P}^1$. Here we also have two special curve classes: f_1 and f_2 the classes of the fibers of each projection. Then

i) $Cl(X)$ is isomorphic to $\mathbb{Z}^2$, generated by f_1 and f_2.
ii) The intersection product is determined from bilinearity by the following intersection relations:
$$f_i \cdot f_j = 1 - \delta_{ij},$$
with δ_{ij} the Kronecker delta's.

(Put coordinates $[x_0 : x_1]$ on the first copy and $[z_0 : z_1]$ on the second copy of $\mathbb{P}^1$ in the product. Then $V_X(x_0) + \operatorname{div}_X(\frac{x_1}{x_0}) = V_X(x_1)$, and this argument can be adjusted to show that any two fibers of the first projection are linearly equivalent. Then $f_1^2 = V_X(x_0)^2 = V_X(x_0) \cdot V_X(x_1) = 0$, because they do not intersect. Analogously $f_2^2 = 0$.

To compute $f_1 \cdot f_2$, we look at $V(x_1) \cap V(z_1) = \{([1 : 0], [1 : 0])\} \in \mathbb{P}^1 \times \mathbb{P}^1$. This is one point and we want to check the multiplicity of intersection. It lives in the affine open $D(x_0 z_0) \simeq \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ with coordinates $x = \frac{x_1}{x_0}$ and $z = \frac{z_1}{z_0}$. In this chart $V(x_1) = V(x)$ and $V(z_1) = V(z)$. These are the axes of $\mathbb{A}^2$, so they meet with multiplicity 1, because they have degree 1.

We show that if $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve given by an equation $g(x_0, x_1, z_0, z_1)$ bihomogeneous of bidegree (n, m), then $C \sim nf_1 + mf_2$. This shows that f_1, f_2 generate $Cl(X)$. Their linear independence is a consequence of the intersection relations: If $af_1 = bf_2$, then $0 = af_1^2 = bf_1 f_2 = b$ and $a = af_1 f_2 = bf_2^2 = 0$, so $a = b = 0$.

Consider $\frac{g}{x_0^n z_0^m}$. This is bihomogeneous of bidegree $(0, 0)$, so a rational function on $\mathbb{P}^1 \times \mathbb{P}^1$. Then its associated principal divisor has class 0 in $Cl(X)$, but $\operatorname{div}_X(\frac{g}{x_0^n z_0^m}) = V_X(g) - nV(x_0) - mV(z_0) = C - nf_1 - mf_2$, gives us what we want.) □

Remark 8.21. With some work, you can use the previous two examples to show that $\mathrm{Bl}_p\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are not isomorphic, because the second contains no irreducible curve of self intersection -1.

9. Appendix

9.1. **Classical algebraic structures.** Let $(G, \cdot)$ be a set with a function $\cdot : G \times G \to G$. We think of this function as an "operation" on G and denote it $a \cdot b := \cdot(a, b)$, or just ab.

- The operation is **commutative** if $ab = ba$ for all $(a, b) \in G^2$. We also say that G is **commutative** when the operation is clear in context.
- $(G, \cdot)$ is a **semigroup** if the operation is associative: $a(bc) = (ab)c$ for all $a, b, c \in G$.
- $(G, \cdot)$ is a **monoid** if it is a semigroup and it contains a *unit/neutral element*: An element $e \in G$ such that $ea = ae = a$ for all $a \in G$. Sometimes, especially if G is commutative, this is denoted by 1.
  N.B. Oftentimes people use semigroup for monoid.
- $(G, \cdot)$ is a **group** if it is a monoid, and *every element has an inverse*: There exists a function ${}^{-1} : G \to G$ such that $a(a^{-1}) = a^{-1}a = e$ for every $a \in G$. Often, $(G, \cdot)$ is said to be a *multiplicative* group.
- When G is a commutative group, the notation $(G, +)$ is sometimes preferred and we say that G is an *additive group*. In this case the neutral element is denoted 0 and the inverse of a is $-a$.
- As far as notation goes, for $n \in \mathbb{N}$ and $a \in G$, denote $na = \underbrace{a + \ldots + a}_{n \text{ times}}$ for additive structures and $a^n = a \cdot \ldots \cdot a$ for multiplicative structures. For groups, $(-n)a := -(na)$ and $a^{-n} := (a^n)^{-1}$.
- A subset $H \subset G$ is a subsemigroup if $H \cdot H \subset H$ (where $H \cdot H = \{ab \mid (a, b) \in H \times H\}$ and the operation $\cdot$ is the one on G). If G is a monoid and H a subsemigroup that also contains the unit element, we say that H is a subgroup. If G is a group and H is a submonoid such that $H^{-1} \subset H$, then H is a subgroup of G.
- A **morphism** $f : H \to G$ is a function that *preserves* operations and properties: For semigroups we want $f(ab) = f(a)f(b)$. For monoids we also want $f(1) = 1$, for groups we also want $f(a^{-1}) = (f(a))^{-1}$.
- If $H = G$, so that f is a self-map $f : G \to G$, we say that f is an **endomorphism**.
- A **monomorphism** is an injective (one-to-one) morphism. An **epimorphism** is a surjective (onto) morphism.
- An **isomorphism** is a bijective (one-to-one and onto) morphism. Its inverse is then also a morphism.
- If $f : H \to G$ is a morphism, then the **kernel** of f is $\ker f = \{h \in H \mid f(h) = 1\}$. This is a subgroup of H.

Assume now that $(R, +, \cdot)$ is a set R with two operations (*addition* and *multiplication*) on it.

- $(R, +, \cdot)$ is a **semiring** if $(R, +)$ is commutative and multiplication is *distributive* with respect to addition: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$. It is **commutative** if $(R, \cdot)$ is also commutative.
- $(R, +, \cdot)$ is a **ring** if it is a semiring, if $(R, +)$ is an additive group, and if $(R, \cdot)$ is a monoid (it has a unit 1). In this case $a0 = 0a = 0$ for all $a \in R$. **N.B.** Sometimes people don't ask that rings contain multiplicative units, and mark the distinction by saying *ring with unit.*
- $(R, +, \cdot)$ is a **division ring** if it is a ring, and if $(R^*, \cdot)$ is a group, where $R^* := R \setminus \{0\}$.

- $(R, +, \cdot)$ is a **field** if it is a commutative division ring. **N.B.** Sometimes people use field for division ring and *non-commutative field* when it is a division ring, but not a field. We have clear notions of **subsemiring, subrings, subdivision rings, and subfields**. We also have clear notions of **morphisms and isomorphisms**.

9.2. **Commutative algebra.** Let $(R, +, \cdot)$ be a commutative ring.

9.2.1. *Ideals.*

- A subset $I \subset R$ is an **ideal** of R if:
  (i) I is closed under addition: $I + I \subset I$. (i.e. $a + b \in I$ for all $a, b \in I$)
  (ii) I is closed under taking opposites: $-I \subset I$. (i.e. $-a \in I$ for all $a \in I$)
  (iii) I is closed under multiplication with elements of R: $RI \subset I$. (i.e. $ra \in I$ for all $r \in R$ and $a \in I$)

  In this case, we denote $I \trianglelefteq R$.

  If $\{x_i\}_{i\in I}$ is a family of elements of R, we denote by $(x_i)_i$ the **ideal generated by** $\{x_i\}_{i\in I}$.

  If $I = (x)$ for some $x \in R$, we say that $I \trianglelefteq R$ is a **principal** ideal.

  Example:
  - An ideal $I \trianglelefteq R$ is equal to R iff $1 \in I$.
  - Even integers form the ideal $2\mathbb{Z} \trianglelefteq \mathbb{Z}$.
  - If R^X denotes the ring (pointwise operations) of functions $f : X \to R$, where R is a ring, and if $Y \subset X$ is any subset, then the set of functions $f \in R^X$ such that $f(y) = 0$ for all $y \in Y$ forms an ideal in R^X.

9.2.2. *Modules.*

- An additive group $(M, +)$ (different + than in R) is an R-**module** if there exists a function $R \times M \to M$ that we denote $(r, m) \to rm$ called *the module action* of R on M such that
  (i) $r(m + m') = rm + rm'$. In particular $r0 = 0$ and $r(-m) = (-r)m$.
  (ii) $(r + r')m = rm + r'm$. In particular $0m = 0$.
  (iii) $rs(m) = r(sm)$.
  (iv) $1m = m$.

  N.B. One can define left ideals and left modules with the exact same definitions when R is non-commutative. And right ideals or modules are defined by writing the action of R on the right, and in particular asking $m(rs) = (mr)s$. When R is commutative, the distinction is not important.
- The **free module of rank n** is R^n with componentwise R-module action. We denote by e_i the usual "*basis*" elements. They are the analogue of a basis for vector spaces: every element of R^n can be written *uniquely* as an R-combination of them.
- A subset $N \subset M$ of an R-module is a **submodule** if
  (i) N is closed under the addition on M.
  (ii) N is closed under the R-module action.

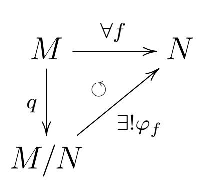
  Example: The ideals of R are the only submodules of R.
- A **morphism of modules** is an additive group morphism $f : (N, +) \to (M, +)$ that preserves the R-module action: $f(rn) = rf(n)$. We have clear notions of monomorphism, epimorphism, and isomorphism.
- An R-module M is **finitely generated** if there exists a surjective morphism $f : R^n \to M$ of R-modules. This is the same as saying that there exist finitely many elements

$x_i \in M$ (they are $f(e_i)$) such that every $x \in M$ can be written as an R-combination of them (though not necessarily unique). We say that the $\{x_i\}$ **generate** M.

- If $f : N \to M$ is a surjective morphism and N is finitely generated, then so is M.
- More generally we say that $\{x_i\}_{i \in I}$ **generate** M if every $x \in M$ can be written as an R-combination of finitely many of the $\{x_i\}_{i \in I}$.
- If N and M are R-modules, we can form the **direct sum** $N \oplus M$. This is the set $N \times M$ with componentwise operations.
- If $(N_i)_{i \in I}$ is a family of R-modules, then the **direct product** $\times_i N_i$ is the product of sets with componentwise R-module action.
- If $(N_i)_{i \in I}$ is a family of R-modules, then the **direct product** $\oplus_i N_i$ is the subset of $\times_i N_i$ consisting of the elements of *finite support* (i.e. only finitely many components are nonzero for each element.)
- $R^{\times\infty}$ is a ring, but $R^{\oplus\infty}$ is not, because there is no unit element (the only viable one is $(1, 1, 1, \dots)$, and this does not have finite support.)

9.2.3. *Kernels, Quotients, Isomorphism Theorems, Snake Lemma.*

- If $f : N \to M$ is a morphism of R-modules, then the **kernel**
$$\ker(f) := \{n \in N \mid f(n) = 0\}$$
is a submodule of N. And the **image**
$$\mathrm{Im}(f) := \{m \in M \mid m = f(n) \text{ for some } n \in N\}$$
is a submodule of M.
  Example: If $f : R \to S$ is a morphism of rings, then $\ker(f) \trianglelefteq R$ is an ideal, while $\mathrm{Im}(f) \subseteq S$ is a subring.
- If $N \subset M$ is a submodule, then
$$x \sim y \Leftrightarrow x - y \in N$$
determines an *equivalence relation* on M. We may write $x = y(mod\ N)$. The **equivalence class** of $x \in M$ is the set of all $y \in M$ such that $x - y \in N$. We may identify it with $x + N$.
  The set of all equivalence classes is denoted M/N.
  It is the **quotient** of M by N. It inherits the structure of an R-module with the action given by $r(x + N) = rx + N$.
- The **cokernel** of $f : N \to M$ is
$$\mathrm{coker}(f) := M/\mathrm{Im}(f).$$
- **The fundamental homomorphism theorem**: If $N \subset M$ is an R-submodule, then the function $q : M \to M/N$ given by $q(m) = m + N$ is an epimorphism of R-modules, which is *universal* in the following sense:
  For every morphism $f : M \to P$ of R-modules such that $N \subseteq \ker f$ (so $f(N) = 0$), there exists a unique morphism $\varphi_f : M/N \to P$ such that $\varphi_f \circ q = f$. With diagrams:



- **The first isomorphism theorem**: If $f : N \to M$ is a morphism of R-modules, then
$$\mathrm{Im}(f) \simeq N/\ker(f).$$
The sign $\simeq$ means *isomorphic*.
- **The second isomorphism theorem**: If $N, P \subseteq M$ are submodules, then
$$\frac{N+P}{N} \simeq \frac{P}{N \cap P}$$
as R-modules.
- **The third isomorphism theorem**: If $P \subseteq N \subseteq M$ is a sequence of R-submodules, then as R-modules,
$$\frac{M}{N} \simeq \frac{M/P}{N/P}.$$
- A sequence of morphisms
$$\ldots \to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \to \ldots$$
is called a **complex** if
$$\ker(f_{i+1}) \subseteq \mathrm{Im}(f_i)$$
for all i. If equality holds at M_{i+1}, we say that the complex (or sequence) is **exact** there. If it is exact everywhere, it is said to be exact.
- If $f : N \to M$ is a morphism of R-modules, then we have an exact sequence:
$$0 \to \ker(f) \to N \xrightarrow{f} M \to \mathrm{coker}(f) \to 0.$$
- If $\varphi : M \to M/N$ is a quotient map, and if $P \subset M$ is a submodule, then $P + N$ is the smallest submodule of M containing P and N, and $\varphi(P) \simeq \frac{P+N}{N}$.
If $Q \subset M/N$ is a submodule, then $\varphi^{-1}Q \subset M$ is a submodule of M containing N and $\varphi(\varphi^{-1}Q) = Q$.
We have a one-to-one correspondence between submodules of M/N and submodules of M containing N.
- **Snake Lemma**: Consider a diagram with exact rows and commutative squares
$$\begin{array}{ccccccc} & & N_1 & \longrightarrow & M_1 & \longrightarrow & P_1 & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & N_2 & \longrightarrow & M_2 & \longrightarrow & P_2 & & \end{array}$$
Then there exists a *connecting morphism*
$$\delta : \ker(h) \to \mathrm{coker}(f)$$
that fits into an induced exact sequence
$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\delta} \mathrm{coker}(f) \to \mathrm{coker}(g) \to \mathrm{coker}(h).$$
If one can add a 0 to the left of the first row, or the end of the second row, then the induced exact sequence also gets a 0 to the left, or to the right respectively.

9.2.4. *Algebras.*

- If $f : R \to S$ is a morphism of rings, we say that S is an R-**algebra**.
- S is **finite algebra** over R if S is finitely generated as an R-module.
- A **morphism of R-algebras** $f : S \to S'$ is a morphism of rings and R-modules.
- If $f : R \to S$ is a morphism of rings, we say that S is an R-**algebra of finite type** if there exists a surjective morphism $g : R[X_1, \ldots, X_n] \to S$ of R-algebras. In this case there exist elements $x_i \in S$ (they are $g(X_i)$) such that every element of S can be written as a polynomial in the x_i, but maybe not uniquely.
- If $\mathfrak{a} \trianglelefteq R$, then the ideal generated by $f(\mathfrak{a})$ in S is denoted $\mathfrak{a}S$ and called the **extension** of $\mathfrak{a}$ in S.
- If $\mathfrak{b} \trianglelefteq S$, then the ideal $f^{-1}\mathfrak{b} \trianglelefteq R$, also denoted $\mathfrak{b} \cap R$ (even if f is not injective), is the **restriction** of $\mathfrak{b}$ to R.
- The restriction of a prime ideal is prime. (If $\mathfrak{p} \trianglelefteq S$ is prime, and $xy \in f^{-1}(\mathfrak{p})$, then $f(x)f(y) \in \mathfrak{p}$, so $f(x) \in \mathfrak{p}$ or $f(y) \in \mathfrak{p}$, i.e. $x \in f^{-1}(\mathfrak{p})$ or $y \in f^{-1}(\mathfrak{p})$.)

9.2.5. *Noetherianity.*

- An R-module M is called **Noetherian** if it satisfies the ACC condition: Every ascending chain of submodules $N_1 \subset N_2 \subset N_3 \subset \ldots$ becomes constant (there exists i such that $N_i = N_{i+1} = \ldots$.
- A ring R is Noetherian if it is Noetherian as a module over itself.
- If $N \subset M$ is a submodule, then N and M/N are both Noetherian iff M is Noetherian.

  If M is Noetherian, then every chain in N is a chain in M, so N is Noetherian.

  Still if M is Noetherian, then every chain in M/N corresponds naturally to a chain of submodules of M, all containing N. Also, two submodules of M containing N are equal iff their images in M/N are equal. Therefore M/N is Noetherian.

  Conversely, if both N and M/N are Noetherian, let $N_1 \subset N_2 \subset \ldots$ be an ascending chain in M. There exists i such that $N_i \cap N = N_{i+1} \cap N = \ldots$ and $(N_i + N)/N = (N_{i+1} + N)/N = \ldots$. The latter is equivalent to $N_i + N = N_{i+1} + N = \ldots$. Say $x \in N_{i+1}$. Then $x + n = y + n'$ for some $n, n' \in N$ and $y \in N_i$. Then $y - x \in N$ and $y - x \in N_{i+1}$, so $y - x \in N_{i+1} \cap N = N_i \cap N$, so $x \in N_i$. This shows $N_{i+1} \subset N_i$, and the other inclusion was there by construction.
- If $f : N \to M$ is a morphism of Noetherian modules, then $\ker(f)$, $\mathrm{Im}(f)$, and $\mathrm{coker}(f)$ are all Noetherian.
- A Noetherian module is finitely generated. (Construct a sequence $x_i \in M$ inductively as follows: x_1 is nonzero in M. Then x_{i+1} is an element of $M \setminus \sum_{j=1}^{i} Rx_j$. Since the ascending sequence $\sum_{j=1}^{i} Rx_i$ must stabilize, the construction ends after finitely many steps.)

  Consequently all submodules of a Noetherian module are finitely generated.
- A module is Noetherian iff all its submodules are finitely generated. (One implication is handled by the above. Assume that all submodules are finitely generated. Let $N_1 \subset N_2 \subset \ldots$ be an ascending sequence, and let $N = \cup_i N_i$. This is a submodule of M, hence finitely generated. Let $x_1, \ldots, x_r$ be a finite set of generators. Then there exists $i > 0$ such that N_i contains them all. Then $N_i = N_{i+1} = \ldots = N$.)
- Any finitely generated R-module is Noetherian iff R is a Noetherian ring.
- R is Noetherian iff all its ideals are finitely generated.
- **Hilbert Basis Theorem:** If R is a Noetherian ring, then $R[X]$ is also a Noetherian ring. Consequently $\mathrm{k}[\underline{X}]$ is Noetherian for any field k.

  (Let $a \trianglelefteq R[X]$, and let a_i be the set of leading coefficients of polynomials in a of degree at most i. Observe that $a_0 \subset a_1 \subset \ldots$ is an ascending sequence of ideals of R. Let d be such that $a_d = a_{d+1} = \ldots$, and let $f_1, \ldots, f_r$ be a set of elements of $a \trianglelefteq R[X]$ of degrees $d_i \leq d$ whose leading coefficients generated a_d. If f is an arbitrary element of $a \trianglelefteq R$, of degree $e \geq d$, then by the definition of $a_d = a_e$ there exist $s_i \in R$ such that $f' := f - \sum_{i=1}^{r} s_i X^{e-d_i} f_i \in a$ and

$\deg f' < \deg f$. We deduce that a is generated by the f_i's and by the polynomials in a of degree at most $d-1$. The latter set can be seen as a sub R-module of R^d. But the Noetherianity of R implies that this is finitely generated. A finite set of generators of this R-module and the f_i's together form a finite set of generators of a as an $R[X]$-module.)

9.2.6. *More commutative ring and ideal theory.* Throughout $I \trianglelefteq R$ is an ideal.

- $x \in R \setminus \{0\}$ is a **zero divisor** if $xy = 0$ for some $y \in R \setminus \{0\}$, i.e. $f : R \to R$ defined by $f(y) = xy$ is not injective.
- R is a **domain** if it has no zero divisors.
- An ideal I is said to be **prime** if R/I is a domain. This is the same as $xy \in I$ iff $x \in I$ or $y \in I$. (If R/I is a domain, then $xy \in I$ means $xy = 0(mod\ I)$, so $x = 0(mod\ I)$ or $y = 0(mod\ I)$. This means precisely $x \in I$ or $y \in I$. The converse is similar.)

  A typical example is $p\mathbb{Z} \trianglelefteq \mathbb{Z}$ when p is prime, hence the name.
- An ideal I is called **proper** if it is not $R = (1)$.
- A proper ideal I is **maximal** if it is maximal among the proper ideals of R, i.e. if $J \trianglelefteq R$ and $I \subset J$, then $I = J$ or $J = R$. Equivalently, I is maximal if R/I is a field. (If R/I is a field, then its only ideals are 0 and (1). This means that the only ideals of R containing I are I and (1). The converse is similar.)
- If $I \trianglelefteq R$, then the correspondence between ideals of R/I and ideals of R containing I respects prime and maximal ideals.
- Every proper ideal $I \trianglelefteq R$ is contained in some maximal ideal $m \trianglelefteq R$. (Replacing R by R/I, we can assume $I = 0$. One uses Zorn's Lemma to show that the set of ideals of R, ordered by inclusion admits a maximal element. Details can be found here.)
- A ring R is a domain iff 0 is a prime ideal. It is a field iff 0 is maximal.
- The **radical** of I is $\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n > 0\}$.
- I is called **radical** if $\sqrt{I} = I$.
- If R is Noetherian, then for any two ideals $a, b \trianglelefteq R$ such that $b \subset \sqrt{a}$, there exists $n > 0$ such that $b^n \subset a$. (Let $x_1, \ldots, x_r$ be a finite set of generators for b. Since $b \subset \sqrt{a}$, there exist $n_i > 0$ such that $x_i^n = x_i^{n_i} \cdot x_i^{n-n_i} \in a$ for all $n \geq n_i$. Let $n' = \max_i\{n_i\}$. Let's show that $x^{rn'} \in a$ for all $x \in b$. Any element of b is of the form $x = p_1x_1 + \ldots + p_rx_r$ for some $p_i \in R$. Then $x^{rn'}$ is a linear combination of multiples of $x_1^{m_1} \cdot \ldots \cdot x_r^{m_r}$ with $\sum_i m_i = rn'$. By the pidgeon-hole principle, for each r-tuple $(m_1, \ldots, m_r)$ with $\sum_i m_i = rn'$, at least one of the $m_i \geq n' \geq n_i$, and consequently $x^{rn'} \in a$ because a is an ideal.)
- $f \in R$ is called **nilpotent** if $f^n = 0$ for some $n > 0$.
- $\mathcal{N} := \sqrt{0}$ is the **nilradical** of R. It is the set of nilpotents in R.
- If $\sqrt{0} = 0$, then R is said to be **reduced**. R/I is reduced iff I is a radical ideal.
- $\mathcal{N}$ is the intersection of all prime ideals of R. (If $x \in \mathcal{N}$, then $x^n = 0$ for some n, and then it is easy to see that x is in every prime ideal. For the converse, one shows that an ideal maximal among those that do not contain $\{1, x, x^2, \ldots\}$ is prime. This is again an application of Zorn's Lemma and details can be found here.)
- If $I \trianglelefteq R$, then $\sqrt{I}$ is the intersection of the prime ideals of R that contain I. (The nilradical of R/I is $\sqrt{I}/I$. The primes in R/I correspond to the primes in R that contain I.)
- If $a, b \trianglelefteq R$, then $\sqrt{a \cdot b} = \sqrt{a \cap b} = \sqrt{a} \cap \sqrt{b}$. (The only interesting things to prove here are $\sqrt{a \cap b} \subset \sqrt{ab}$ and $\sqrt{a} \cap \sqrt{b} \subset \sqrt{a \cap b}$. If $x \in \sqrt{a \cap b}$, then $x^n \in a \cap b$ for some n, and then $x^{2n} \in ab$ shows $x \in \sqrt{ab}$. The other inclusion is easier.)

9.2.7. *Primary ideal decomposition for radical ideals.*

- An ideal $Q \trianglelefteq R$ is called **primary** if $xy \in Q$ implies $x \in Q$, or $y^n \in Q$ for some n.
- The radical of a **primary** ideal is prime (If $xy \in \sqrt{Q}$, then $x^ny^n \in Q$ for some n, so $x^n \in Q$ or $y^{nm} \in Q$ for some m. In both cases, either x or y are in $\sqrt{Q}$.) If $P := \sqrt{Q}$, we say that Q is P-**primary**.

Look here for an example of an ideal that is not primary, but its radical is prime.

- If $P \trianglelefteq R$ is prime, then it is P-primary.
- A strong result that we won't use, is that in a Noetherian ring every ideal admits a **primary decomposition**, i.e. $I = \cap_{i=1}^{r} Q_i$ for some r and some primary ideals Q_i. But the case of radical ideals is easier and this is what we'll use.
- Let $I \trianglelefteq R$. Then $\sqrt{I}$ is the intersection of the prime ideals that contain it (not finitely many yet).
- An ideal $p \supset I$ is **minimal prime associated to** I if it is minimal (containing no other) among the prime ideals containing I. Minimal primes exist by the Zorn Lemma (the set of primes containing I is nonempty because I is contained in some maximal ideal, and it is ordered by inclusion, though in the other direction now). The minimal primes associated to (0) are called simply **minimal primes**, or the **minimal primes associated to** R. The minimal primes of R/I correspond to the minimal primes of I.
- Let $I \trianglelefteq R$. Then $\sqrt{I}$ is the intersection of the minimal primes that contain it.
- If R is Noetherian, and $I \trianglelefteq R$, then $\sqrt{I}$ is the intersection of finitely many primes. (Let $\mathcal{I}$ be the set of radical ideals that cannot be written as an intersection of finitely many primes, and let $\mathcal{R}$ be the set of radical ideals of R. If $\mathcal{R} \subsetneq \mathcal{I}$, let b be a maximal element (exists because R is Noetherian). We show that b is prime, which provides a contradiction. Let $xy \in b$. Then $b = \sqrt{b} = \sqrt{(b + Rx) \cdot (b + Ry)} = \sqrt{b + Rx} \cap \sqrt{c + Rx}$. The maximality of b implies that $\sqrt{b + Rx}$ and $\sqrt{b + Ry}$ are intersections of finitely many primes, hence the same is true of the intersection, unless $x \in b$ or $y \in b$, i.e. b is prime. The conclusion is $\mathcal{R} = \mathcal{I}$.)
- If R is Noetherian, and $I \trianglelefteq R$, then there are only finitely many prime ideals $p_1, \ldots, p_r$ associated to I, and $\sqrt{I} = p_1 \cap \ldots \cap p_r$ is the only way (up to permutation) of writing $\sqrt{I}$ as a finite intersection of *minimal* primes associated to I. (Let $\sqrt{I} = q_1 \cap \ldots \cap q_s$ be a way of expressing $\sqrt{I}$ as an intersection of finitely many primes. Let p be a minimal prime associated to I, in particular $p \supseteq \cap_i q_i$. The next result implies $p \supset q_i$ for some i, hence $p = q_i$ by minimality. There may be some q_i's that are not minimal, but they are now seen to be redundant, and can be removed from the intersection without changing it.)
- If $p \trianglelefteq R$ is prime and $p \supset \cap_{i=1}^{s} q_i$ for some *ideals* $q_i \trianglelefteq R$, then $p \supset q_i$ for some i. (If not true, then we can pick $x_i \in q_i \setminus p$ for all i. Then $\prod_i x_i \in \cap_i q_i \subset p$ is impossible.)
- If $\sqrt{I} = I \trianglelefteq R$ is a radical ideal of a Noetherian ring R, and p_i are the minimal primes associated to I, then $I = \cap_i p_i$ is called the **primary ideal decomposition** of I.
- An element $x \in R$ is a **zero divisor** if $xy = 0$ for some $y \neq 0$.
- Every element of a minimal prime $p \trianglelefteq R$ is a zero divisor. (The solution is form here. Let S be the multiplicative monoid generated by the complement of p and of the zero-divisors of R. Note that $0 \notin S$. Use Zorn's Lemma to expand (0) to a maximal ideal I among those that do not meet S. We prove that I is prime, and then the minimality of p shows $p = I \subset R \setminus S$ is a subset of the zero-divisors of R. If $xy \in I$, but $x, y \notin I$, then by the maximality of I we must have $x, y \in S$. But then $xy \in I \cap S$ is a contradiction. Therefore I is prime.)

9.2.8. *Special types of rings.* Throughout, R is a commutative ring.

PID's.

- An ideal $\mathcal{I} \trianglelefteq R$ is called **principal** if $\mathcal{I}$ is generated by one element, i.e. there exists $x \in \mathcal{I}$ such that $\mathcal{I} = Rx$.
- If R is a domain whose ideals are all principal, we say that R is a **PID** (short for *principal ideal domain*).
- $\mathbb{Z}$ and $k[X]$, where k is a field are the easiest examples of PID's.
- Any quotient by a prime ideal and any localization of a PID is again a PID.
- Every PID is Noetherian. (Every ideal is finitely generated, actually generated by just one element).

- In a PID, every nonzero prime ideal is maximal. (If $p = (x)$ is a nonzero prime ideal, and if $p \subsetneq \mathcal{I} = (y)$ for some $y \in R$, then $x = yz$ for some $z \in R$. Then necessarily $z \in (x)$, hence $z = xt$ for some $t \in R$. Therefore $x = yxt$ and $1 = yt$, so y is a unit, and in particular $(y) = (1)$. So p is maximal.)

UFD's.

- An element $x \in R$ is called a **unit**, if it is invertible with respect to multiplication.
- An element $x \in R$ is called **prime** if $(x) \trianglelefteq R$ is a prime ideal.
- An element $x \in R$ is called **irreducible** if it is not a product of non-units.
- If R is a domain, then any prime element is irreducible. (If $p \in R$ is prime, and $p = xy$, then $x \in (p)$, or $y \in (p)$. Say $x = pq$ for some $q \in R$. Then $p = pqy$ implies $1 = qy$, so y is a unit.)
- There exist domains R and irreducible elements $x \in R$ such that x is not prime. For example 3 in $R = Z[\sqrt{5}]$ is irreducible, but not prime. A few more details are here.
- A domain R is an **UFD** (short for *unique factorization domain*) if every element $x \in R$ can be written as a product of primes, and the decomposition is unique up to units and reordering.
- $\mathbb{Z}$ is an UFD. This is actually the Fundamental Theorem of Arithmetic.
- If R is an UFD, then $x \in R$ is prime iff it is irreducible. (One implication is always true. Conversely, if x is irreducible, then by the definition of UFD it can be written uniquely up to units as a product of primes. By irreducibility there cannot be more than two primes in this product.)
- If x, y are elements of an UFD, then $\text{lcm}(x, y)$ and $\gcd(x, y)$ are well-defined in R up to units by factorizing and picking the maximum and minimum respectively between exponents of primes.
- In fact any PID is an UFD. (We first show that every x in the PID R can be written as a product of finitely many irreducibles. If this were not true, by Noetherianity we could choose $x \in R$ such that (x) is maximal among the ideals of R generated by elements without finite decomposition as product of irreducibles. By its choice, x cannot be irreducible. So we can write it as $x = yz$ with y, z both non-units. Consequently $(x) \subsetneq (y)$ and $(x) \subsetneq (z)$. By the maximality of (x), it follows that y and z are products of finitely many irreducibles, hence so is x. This proves the first claim.

  Second, we show that in a PID every irreducible is prime: Let x be irreducible, and let $yz \in (x)$. Since x is irreducible, the ideal (x, y) is either (x), or (1). This is because if $(y, x) = (s)$, then $x = ss'$ for some $s' \in R$, so either s or s' is a unit by irreducibility. If $(x, y) = (x)$, then $y \in (x)$. If $(x, y) = 1$, then there exist $u, v \in R$ such that $ux + vy = 1$. Multiplying by z, we get $z = (uz) \cdot x + v \cdot (yz) \in (x)$. Therefore (x) is a prime ideal, and x is a prime element.

  Finally, we observe that the uniqueness of the decomposition is a consequence of the primality of the factors.)
- If R is a PID, then $\gcd(x, y)$ is a generator of the ideal (x, y) and $\text{lcm}(x, y)$ is a generator of $(x) \cap (y)$. In a UFD, we only know for sure that $(x, y) \subset (\gcd(x, y))$ and $\text{lcm}(x, y) \subset (x) \cap (y)$.
- If R is an UFD, then $R[X]$ is also an UFD. Consequently $k[\underline{X}]$ is an UFD. (Since R is a domain, $R[X]$ is also a domain, and we have a well-defined notion of degree of polynomials which satisfies $\deg(fg) = \deg(f) + \deg(g)$.

  First, observe that a nonconstant (meaning of positive degree) element $P \in R[X]$ is irreducible precisely when it cannot be written as a product of polynomials of smaller degree *and* when it is *primitive*, i.e. no non-unit element of R divides the coefficients of P.

  Then it is clear that every element of $R[X]$ is the product of finitely many polynomials as above and an element of R which is the product of finitely many prime elements of R, which are still prime in R. (If $\mathcal{I} \trianglelefteq R$, then $R[X]/_{\mathcal{I}R[X]} \simeq R/_{\mathcal{I}}[X]$.)

  It remains to show that an irreducible nonconstant polynomial is prime. Then primality implies the uniqueness of the factorization up to units. Let P be an irreducible polynomial, and assume $FG \in (P)$, i.e. $FG = PQ$ for some polynomials F, G, Q. Let $c(F)$ denote the gcd in R of the coefficients of F. Observe that $\frac{F}{c(F)}$ is still in $R[X]$, and it is primitive. Let's show that $c(F \cdot G) = c(F) \cdot c(G)$. It is easy to see that the right side divides the left. Then we can assume that F and G are primitive, and show that $F \cdot G$ is also primitive. Say a prime element $p \in R$ divides all coefficients of $F \cdot G$. Then $F \cdot G = 0$ in $R/_{(p)}[X]$ which is a domain, therefore either F or G is 0 modulo p, and this is a contradiction. Then in the equality $FG = PQ$ we can assume that all terms are primitive nonconstant polynomials.

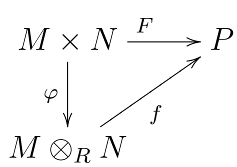
The equality $FG = PQ$ also holds in $K[X]$, where K is the fraction ring of R. But $K[X]$ is a PID, hence also an UFD. We check that P is still irreducible in $K[X]$ which implies that $(P) \trianglelefteq K[X]$ is prime, hence $F \in (P)$ or $G \in (P)$. Say $F = PF_1$ for some $F_1 \in K[X]$. Clearing denominators we write $rF = PF'$ for some $r \in R$ and $F' \in R[X]$. Since P and F are primitive, necessarily $c(F') = r$ up to units in R. Then we can divide F' by it to get $F = PF''$ where now everything is in $R[X]$. This shows that P is prime in $R[X]$.

See also Gauss's Lemma $\square$)

9.2.9. *Tensor products.* Let R be a commutative ring and let M, N and P be R-modules.

- A function $f : M \times N \to P$ is R-**bilinear** if for each $m \in M$, the map $n \mapsto f(m,n) : N \xrightarrow{f_m} P$ is R-linear, and similarly $f_n : M \to P$ is R-linear for all $n \in N$.
- Observe that a linear function $\varphi : M \times N \to P$ from the *direct product module* $M \times N$ is not bilinear unless $M = 0$ or $N = 0$. This is because $\varphi(r(m,n)) = \varphi(rm, rn) = \varphi_{rn}(rm) = r\varphi_{rn}(m) = r\varphi(m, rn) = r\varphi_m(rn) = r^2\varphi_m(n) = \varphi(m,n)$.
- If $f : R \times R \to R$ is the function $f(x,y) = ax^2 + bxy + cy^2$ for some $a,b,c \in R$ and all $x,y \in R$, then f is R-bilinear.
- Bilinear maps $f : M \times N \to P$ are **balanced** in that $f(r \cdot m, n) = f(m, r \cdot n) = r \cdot f(m,n)$.
- The **tensor product** of the R-modules M and N is an R-module $M \otimes_R N$ with an R-bilinear map $\varphi : M \times N \to M \otimes_R N$ that verifies the following universal property:

  For each R-bilinear map $F : M \times N \to P$ there exists a unique R-module morphism $f : M \otimes_R N \to P$ such that the following diagram commutes:



- The tensor product is unique up to isomorphism. (Follows from the universality property.)
- The tensor product exists. ($M \otimes_R N \simeq R^{M\times N}/\mathcal{B}$, where $R^{M\times N}$ is the R-free module on the set $M \times N$ (much larger than the set $M \times N$ itself), and $\mathcal{B}$ is the submodule of this free group generated by the elements of the form $(m+m',n) - (m,n) - (m',n)$, $(m, n+n') - (m,n) - (m,n')$, $(rm,n) = r(m,n)$, and $(m,rn) = r(m,n)$ for each $(m,m',n,n',r) \in M \times M \times N \times N \times R$.

  One checks that $R^{M\times N}/\mathcal{B}$ verifies the universality property of the tensor product with the morphism $\varphi(m,n) = (m,n) \bmod \mathcal{B} \in R^{M\times N}/\mathcal{B}$.)
- If $m \in M$ and $n \in N$, we denote $\varphi(m,n) = m \otimes n$.
- A general element of $M \otimes_R N$ looks like $\sum_{i=1}^s m_i \otimes n_i$ for some s. In general it is not possible to write every element of $M \otimes_R N$ as $m \otimes n$. So φ is not usually surjective.
- $M \otimes_R N \simeq N \otimes_R M$ (They both verify the same universality property.)
- However if $m, m' \in M$, then $m \otimes m' \neq m' \otimes m$ in $M \otimes_R M$. (So the previous isomorphism is not the identity when $M = N$.)
- $M \otimes_R R \simeq R \otimes_R M \simeq M$.
- $(M \otimes_R N) \otimes_R P \simeq M \otimes_R (N \otimes_R P)$.
- If S is an R-algebra, then apart from an R-module structure, $M \otimes_R S$ has the structure of an S-module with S-action induced by $s(m \otimes s') = m \otimes ss'$.
- If S is an R-algebra, then $S \otimes_R R[X] \simeq S[X]$.
- If $\mathcal{I} \trianglelefteq R$, then $M \otimes_R R/\mathcal{I} \simeq M/\mathcal{I}M$ as $R/\mathcal{I}$-modules.
- If $\mathcal{I}, \mathcal{J} \trianglelefteq R$, then

$$\mathcal{J} \otimes_R R/\mathcal{I} \simeq \mathcal{J}/\mathcal{I}\mathcal{J} \quad \text{and} \quad \mathcal{I} \otimes_R R/\mathcal{I} \simeq \mathcal{I}/\mathcal{I}^2 \text{ as } R/\mathcal{I}\text{-modules}$$

$$R/\mathcal{J} \otimes_R R/\mathcal{I} \simeq R/\mathcal{I}+\mathcal{J} \text{ as } R/\mathcal{I},\ R/\mathcal{J},\ \text{or } R-\text{modules}$$

The multiplication map $(x, y) \mapsto xy : \mathcal{I} \times \mathcal{J} \to \mathcal{I}\mathcal{J}$ is bilinear and hence induces a morphism of R-modules $\mathcal{I} \otimes_R \mathcal{J} \to \mathcal{I}\mathcal{J} \trianglelefteq R$, but while this is surjective (anything in the product $\mathcal{I}\mathcal{J}$ looks like $\sum_{i=1}^{s} r_i x_i y_y$ for some s and some $(r_i, x_i, y_i) \in R \times \mathcal{I} \times \mathcal{J}$, and this is the image of $\sum_{i=1}^{s} r_i x_i \otimes y_i$), it is usually not an isomorphism.

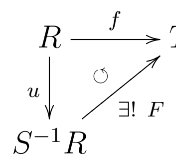
- The tensor product is a covariant right-exact function from the category of R-modules to itself. Right-exactness means that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of R-modules and N is an R-module, then
$$M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N \to 0$$
is an exact sequence, but the map on the left may fail to be injective.
- Tensoring the exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ by $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$ produces the sequence
$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{=} \mathbb{Z}/2\mathbb{Z} \to 0$$
which is not exact on the left.

9.2.10. *Localization.* Let R be a commutative ring.

- A subset $S \subset R$ is a **multiplicative subset** if $1 \in S$ and $ab \in S$ whenever $a, b \in S$. The basic examples are
  - If $\mathfrak{p} \trianglelefteq R$ is a *prime ideal*, then $S = R \setminus \mathfrak{p}$ is a multiplicative set, because $a, b \notin \mathfrak{p} \Rightarrow ab \notin \mathfrak{p}$.
    A particular case of this is when R is a domain and $\mathfrak{p} = (0)$. Then $S = R^*$.
  - If $f \in R$, then $S = \{1, f, f^2, f^3, \ldots\}$ is clearly a multiplicative set
  - If $S \subset R$ is the set of all nonzero divisors, then S is a multiplicative set.
- On the set $R \times S$ we put the equivalence relation $(r, s) \sim (r', s')$ iff there exists $t \in S$ such that $t(rs' - r's) = 0$. The "t" in the definition is that $\sim$ is a transitive relation. The equivalence class of (r, s) is denoted as expected by $\frac{r}{s}$.
- The **localization** or R at S is the set of equivalence classes $S^{-1}R := R \times S/\sim$. It has a ring structure by copying the fraction operations from $\mathbb{Q}$.
- There is a natural map $r \mapsto \frac{r}{1} : R \xrightarrow{u} S^{-1}R$. This map is injective if and only if S contains no zero divisors. In this case we can remove the "t" from the definition of $\sim$.
- The classical examples are
  - The **localization at a prime ideal** $R_{\mathfrak{p}}$ when $S = R \setminus \mathfrak{p}$ and $\mathfrak{p} \trianglelefteq R$ is prime. In particular, if R is a domain, then $R_{(0)}$ is the **fraction field** of R.
  - The **localization at an element** R_f when $S = \{1, f, f^2, \ldots\}$.
  - The **total fraction ring** of R is what we get when S is the set of all nonzero divisors. In particular we recover the fraction field if R is a domain.

  These are relevant to Algebraic Geometry in the following way:
  - If X is an affine variety, and $f \in k[X]$ is a regular function, then $k[X]_f = k[D(f)]$, i.e. the set of regular functions on the open subset $X \setminus V(f)$.
  - If X is an affine variety, and $Y \subset X$ is a subvariety corresponding to a prime ideal $\mathfrak{p} = \mathcal{I}_X(Y) \trianglelefteq k[X]$, then $k[X]_{\mathfrak{p}}$ is the set of "germs of rational functions on X, regular in a neighborhood of Y", by which we mean
$$k[X]_{\mathfrak{p}} = \{f \in k(X) \mid \text{Domain}(f) \cap Y \neq \emptyset\}.$$

  - When X is only a closed affine set, then the total fraction ring of $k[X]$ can be taken as a definition for the set of rational functions on X.
- The localization verifies the following universality property: Let $S \subset R$ be a multiplicative set. Then for every ring morphism $f : R \to T$ such that $f(s)$ is invertible in T for all $s \in S$, there exists a unique $F : S^{-1}R \to T$ that makes the following diagram commute:



The map F is then defined as $F(\frac{r}{s}) = \frac{f(r)}{f(s)}$. It makes sense to divide by $f(s)$ precisely because we imposed the condition that it is a unit.
- If M is an R-module, and $S \subset R$ is a multiplicative subset, we can define $S^{-1}M$ as $S \times M/_{\sim}$, where $\sim$ is defined just like on $S \times M$. The localization $S^{-1}M$ is an $S^{-1}R$-module, and
$$S^{-1}M \simeq S^{-1}R \otimes_R M.$$
- The ideals of $S^{-1}R$ are all extensions (via u) of ideals of R. Moreover for any $\mathfrak{b} \trianglelefteq S^{-1}R$ we have $\mathfrak{b} = (\mathfrak{b} \cap R)S^{-1}R$, i.e. $\mathfrak{b}$ is the extension of its restriction. (If $\frac{r}{s} \in \mathfrak{b}$, then $\mathfrak{b} \ni \frac{s}{1}\frac{r}{s} = \frac{r}{1} \Rightarrow r \in \mathfrak{b} \cap R$. Conversely if $r \in \mathfrak{b} \cap R$, then $\frac{r}{1} \in \mathfrak{b}$, so $\mathfrak{b} \ni \frac{1}{s}\frac{r}{1} = \frac{r}{s}$.)
- If $\mathfrak{a} \trianglelefteq R$, then the extension $\mathfrak{a}S^{-1}R = \{\frac{x}{s} \mid x \in \mathfrak{a},\ s \in S\}$. (An element in the LHS is of form $\sum_i x_i \frac{r_i}{s_i}$, with $x_i \in \mathfrak{a}$, $r_i \in R$, and $s_i \in S$. After operations with fractions, using that $\mathfrak{a}$ is an ideal, we can write this in the required form. Clearly an element of the RHS is in the LHS.)
- If $\mathfrak{a} \trianglelefteq R$, then the extension $\mathfrak{a}S^{-1}R = (1)$ if and only if $\mathfrak{a} \cap S \neq \emptyset$.
- If $\mathfrak{p} \trianglelefteq R$ is a proper (i.e. not (1)) prime ideal and $\mathfrak{p} \cap S = \emptyset$, then $\mathfrak{p}S^{-1}R$ is also a proper prime ideal of $S^{-1}R$. (If $\frac{r}{s}\frac{r'}{s'} \in \mathfrak{p}S^{-1}R$, then $\frac{ss'}{1}\frac{rr'}{ss'} \in \mathfrak{p}S^{-1}R$, so $\frac{r}{1}\frac{r'}{1} \in \mathfrak{p}S^{-1}R$, therefore $\frac{rr'}{1} = \frac{x}{s}$, with $x \in \mathfrak{p}$ and $s \in S$, so $t(srr - x) = 0$ for some $t \in S$. Since $0 \in \mathfrak{p}$ and $\mathfrak{p} \cap S = \emptyset$, using the primality of $\mathfrak{p}$ this implies $srr - x \in \mathfrak{p}$, so $srr' \in \mathfrak{p}$. Using again that $\mathfrak{p}$ is prime and $\mathfrak{p} \cap S = \emptyset$, we get $rr' \in \mathfrak{p}$, hence $r \in \mathfrak{p}$ or $r' \in \mathfrak{p}$ and $\frac{r}{s} = \frac{r}{1}\frac{1}{s} \in \mathfrak{p}S^{-1}R$ or $\frac{r'}{s'} = \frac{r'}{1}\frac{1}{s'} \in \mathfrak{p}S^{-1}R$.)
- There is an order preserving correspondence between proper prime ideals in $S^{-1}R$ and proper prime ideals in R that do not intersect S.
- For example
  - The prime ideals of $R_{\mathfrak{p}}$ correspond to the prime ideals $\mathfrak{q} \trianglelefteq R$ with $\mathfrak{q} \subseteq \mathfrak{p}$. The ring $R_{\mathfrak{p}}$ is **local**, i.e. it has only one maximal ideal, which is $\mathfrak{p}R_{\mathfrak{p}}$.
  - The prime ideals of R_f correspond to the prime ideals of R that do not contain f.

9.2.11. *Integrality.* Let $f : R \to S$ be a morphism of rings, not necessarily injective.
- We see S as an R-module with the operation $r \cdot s := f(r)s \in S$ for any $r \in R$.
- We say that $s \in S$ is **integral** over R if there exists a *monic* polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_n$ in $R[x]$ such that $p(s) = 0$.
- Examples:
  - $x \in k[x]/_{x^2+1}$ is integral over k.
  - $x \in \mathbb{Z}[x]/_{2x+1}$ is not integral over $\mathbb{Z}$.
- We can always reduce to the case when f is injective, by replacing R with the image $f(R)$.
- The following are equivalent:

– s is integral over R.
– $R[s]$, the sub-R-algebra of S generated by s is a finite module over R.
– There exists $M \subset S$ a finitely generated R-module and such that $sM \subset M$ and $\mathrm{Ann}_S(M) = 0$, i.e. there exists no $x \in S \setminus 0$ such that $xm = 0$ for all $m \in M$.

(If s is integral over R and $p(x)$ is a monic polynomial in $R[x]$ of degree n with $p(s) = 0$, then $R[s]$ is generated by $1, s, \ldots, s^{n-1}$ as an R-module: we can write s^n as a combination of the ones below using $p(s) = 0$, and that p is monic. Then we continue by induction, because $x^k p(x)$ is also monic for all $k \geq 0$.

If $R[s]$ is a finite R-module, we can put $M = R[s]$. Clearly $sM \subset M$. Moreover $\mathrm{Ann}_S M = 0$, because $1 \in R \subset R[s] = M$.

If $M \subset S$ is a finitely generated R-module such that $sM \subset M$ and $\mathrm{Ann}_S(M) = 0$, let $x_1, \ldots, x_n$ be a finite set of generators of M as an R-module. The function $y \mapsto sy : M \xrightarrow{f} M$ is an R-linear endomorphism of M. Write $f(x_i) = \sum_j a_{ij} x_j$ with $a_{ij} \in R$, using that the x_i's generate M as an R-module. Then the $n \times n$ matrix $A = (a_{ij})_{ij}$ satisfies $s(\sum_i r_i x_i) = f(\sum_i r_i x_i) = A \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$. The matrix A verifies its characteristic polynomial by Hamilton–Cayley (this proof works over rings too), i.e. there exists $p(x) \in R[x]$ monic of degree n such that $p(x) = \det(xI_n - A)$ and $p(A) = 0_n$. This is equivalent to $p(A)$ is the zero endomorphism of M. But as an endomorphism of M, $p(A)$ is multiplication (in S) by $p(s)$. Since $\mathrm{Ann}_S(M) = 0$, it follows that $p(s) = 0$, i.e. s is integral over R.)

- The set of elements in R that are integral over S is a subring, i.e. closed under addition and multiplication. We denote it $\overline{R}$ and call it the **integral closure** of R in S. (Follows from the equivalence above: If $R[s]$ and $R[t]$ are finite modules over R, then so is $R[s,t]$. We have $(r+s)R[s,t] \subset R[s,t]$ and $rsR[s,t] \subset R[s,t]$.)
- If $\overline{R} = S$, we say that S **is integral over** R.
- If $\overline{R} = R$, we say that R is **integrally closed** in S.
- If R is a domain and S is not specified, we say that R is **integrally closed** or **normal**, if it is integrally closed in its fraction field.
- If S is an R-*algebra* of finite type, then S is integral over R if and only if S is a finitely generated R-*module*. In this case we say that S is **finite** over R. (This is an immediate consequence of the equivalence above.)

9.3. **Topology.** A **topological space** is a set X together with $\tau \subset \mathcal{P}(X)$ a subset of subsets of X called **open subsets** of X with the following properties:

- $\emptyset$ and X are both in τ.
- If $U, V \in \tau$, then $U \cap V \in \tau$.
- If $(V_i)_{i \in I}$ is a family indexed by an arbitrary set (maybe uncountable) I of elements of τ, then $\cup_i V_i \in \tau$.

Remark 9.1. The Zariski topology that we work with is quite different from the Euclidian topology you may be more familiar with. This topology is not given by any norm.

If $U \subset X$ is an open set, then we say that $V := X \setminus U$ is **closed**. We can also define the topology on X in terms of closed sets by asking:

- $\emptyset$ and X are both closed.
- The union of two closed sets is closed.
- The intersection of arbitrarily many closed subsets is closed.

If $Y \subset X$ is a subset, the **closure** $\overline{Y}$ of Y in X is the intersection of all the closed subsets $Z \subset X$ that contain Y. Equivalently, it is the "smallest" closed subset containing Y. We say that Y is **dense** if $\overline{Y} = X$.

A function $f : X \to Y$ between topological spaces (X, τ) and (Y, σ) is **continuous** if $f^{-1}(U) \in \tau$ for any open set $U \in \sigma$. Similarly, f is continuous if the preimage through f of any closed subset of Y is closed in X.

We say that a continuous function $f : X \to Y$ is a **homeomorphism** if f is bijective and f^{-1} is also continuous.

Caution. It is not usually true that continuous functions send closed sets to closed sets, and similarly for opens.

For example the projection on the first component $\mathbb{A}^2 \to \mathbb{A}^1$ sends the closed subset which is the hyperbola $xy = 1$ to the open nonclosed subset $\mathbb{A}^1 \setminus \{0\}$.

It is also not true that a bijective continuous function is a homeomorphism. For example, let X be the topological space with is $\mathbb{R}$ with the Euclidean topology, and let Y be the same $\mathbb{R}$, but with the trivial topology (only Y and $\emptyset$ are open). Then the identity $f := \mathrm{id}_{\mathbb{R}} : X \to Y$ which sends $x \to x$ for all $x \in \mathbb{R}$ is continuous, but its inverse which is still $\mathrm{id}_{\mathbb{R}}$ is not continuous.

If $Y \subset X$ is a subset, the **induced** topology on Y is that whose open subsets are of form $U \cap Y$ where $U \in \tau$ is open on X.

If τ, τ' are different topologies on X, we say that τ is **coarser** than τ' if $\tau \subset \tau'$, i.e. every open set on τ is also open on τ'. Note that in this case it is the identity morphism $f : (X, \tau') \to (X, \tau)$ that is continuous, and not the other way around.

For example the Zariski topology is coarser than the Euclidian topology on $\mathbb{A}^n_{\mathbb{C}}$.

If $x \in X$ and U is an open subset of X with $x \in U$, we say that U is a **neighborhood** of x in X.

9.4. **Categories.**

Definition 9.2. A **category** $\mathcal{C}$ is a pair $(\mathcal{O}b, \mathcal{M}or)$ of **objects** and **morphisms** (also called *arrows*). An object is a set, and a morphism is a function between objects of $\mathcal{C}$. We ask two conditions of $\mathcal{M}or$:

- For any object A, the identity function $x \mapsto x : A \xrightarrow{1_A} A$ is a morphism in $\mathcal{M}or$.
- If $f : A \to B$ and $g : B \to C$ are morphisms between objects, then $g \circ f$ is also a morphism.

Example 9.3.

- Set is the category of all sets with all functions as morphisms. The objects of Set do not form a set themselves (see here). Actually this is one of the reasons why categories are studied.
- Gp is the category of groups as objects with group morphisms as morphisms.
- $Ring$ is the category of rings.
- Mod_R is the category of modules over a ring R.
- Var_k is the category of algebraic varieties over k with morphisms given by regular maps.
- Alg_k is the category of k-algebras.

The appropriate equivalent of a function between categories is the notion of functor. Given that we also have morphisms in the categories to be concerned about, there are two types of functors:

Definition 9.4.

- A **covariant functor** $F : \mathcal{C} \to \mathcal{C}'$ associates to each object $A \in \mathcal{O}b$ and object $F(A) \in \mathcal{O}b'$, and to each morphism $f : A \to B$ from $\mathcal{M}or$ a morphism $F(f) : F(A) \to F(B)$ from $\mathcal{M}or'$ that respects the composition of morphisms: $F(f \circ g) = F(f) \circ F(g)$ and preserves identity functions: $F(1_A) = 1_{F(A)}$.
- A **contravariant functor** $F : \mathcal{C} \to \mathcal{C}'$ similarly associates A from $\mathcal{O}b$ to $F(A)$ from $\mathcal{O}b'$, but "reverses arrows": for each morphism $f : A \to B$, it associates a morphism $F(f) : F(B) \to F(A)$ that respects composition in the sense that $F(f \circ g) = F(g) \circ F(f)$ and preserves identity functions.

Example 9.5. The following are covariant functors:

- The forgetful functor $Gp \to Set$ takes a group $(G, \cdot)$ to the set G, and a morphism to the function that it is. All this does is "forget" the group structure, whence the name. We have forgetful functors to *Set* from all the other categories.
- Sending a set S to the free abelian group $\mathbb{Z}^S$ (can be seen as functions $S \to \mathbb{Z}$) is a functor from *Set* to *Gp*.
- Sending a group G to its group algebra $\mathbb{Z}[G]$.
- Taking the fundamental group of a pointed topological space is a covariant functor from pointed topological spaces to groups.
- The homology groups are functors of topological spaces to abelian groups.

Example 9.6. The following are contravariant functors:

- The cohomology groups from topological spaces to abelian groups.
- Taking duals is a functor from vector spaces to vector spaces.
- Sending a closed algebraic subset of an affine space to its ring of regular functions is a functor from the category of such algebraic subsets to the category of reduced algebras of finite type.
- Sending a topological space to the ring of real valued continuous functions is a functor to *Ring*.

References

[Harr] J. Harris, *Algebraic Geometry: A first course*
[Mil] J.S. Milne, *Algebraic Geometry* here
[Rei] M. Reid, *Undergraduate Algebraic Geometry*
[Sha] I.R. Shafarevich, *Basic Algebraic Geometry 1*

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*
[Har] R. Hartshorne, *Algebraic Geometry*
[Mum] D. Mumford, *The Red Book of Varieties and Schemes*
[Vak] R. Vakil, *Foundations of Algebraic Geometry* here

[AM] M. Atiyah and I.G. MacDonald, *Introduction to Commutative Algebra*
[Eis] D. Eisenbud, *Commutative Algebra: With a view towards Algebraic Geometry*
[Mat] H. Matsumura, *Commutative Ring Theory*