# Math 5521 Homework 3

Due Spring 2018

In all problems, let $\Omega$ be an open, bounded, simply-connected subset of $\mathbb{R}^2$ with a polygonal boundary. Also, $\tau_h$ is a regular triangulation of $\Omega$ (actually using triangles), $K$ is a constant, SPD matrix, and $\Gamma_D$, $\Gamma_N$ are boundary components as defined in class for the imposition of Dirichlet and Neumann boundary conditions, respectively. Lastly, all other notation and references below are to be interpreted as in class for the discussion of DG methods.

1. Let $\sigma_e^0 > 0$ for all $e \in \Gamma_h^i \cup \Gamma_h^D$. Prove that

$$\langle v, w \rangle_a \equiv \sum_{E \in \tau_h} \int_E (K\nabla v) \cdot \nabla w \, dx + J_0(v, w)$$

defines an inner-product on $H^1(\tau_h)$, if $\Gamma_D$ is not empty, or on $X$ if $\Gamma_D$ is empty. Here,

$$X \equiv \left\{ v \in H^1(\tau_h) \mid \int_\Omega v \, dx = 0 \right\}.$$

2. Prove there exists $C \in \mathbb{R}^+$ depending on $k$ only such that for any $E \in \tau_h$, any edge $e \subset \partial E$ and any $v \in \mathcal{D}_k$, the following inequalities hold for $v|_E$:

$$\|v\|_{L^2(e)} \leq C|e|^{1/2}|E|^{-1/2}\|v\|_{L^2(E)}, \tag{1}$$

$$\|v\|_{L^2(e)} \leq Ch_E^{-1/2}\|v\|_{L^2(E)}, \tag{2}$$

$$\|\nabla v \cdot \hat{n}_e\|_{L^2(e)} \leq C|e|^{1/2}|E|^{-1/2} \|\nabla v\|_{L^2(E)}, \tag{3}$$

$$\|\nabla v \cdot \hat{n}_e\|_{L^2(e)} \leq Ch_E^{-1/2} \|\nabla v\|_{L^2(E)}. \tag{4}$$

I suggest following the procedure below.

- Prove (1) first.
  - Map to the reference element.
  - Use the standard trace inequality at first.
  - Apply the fact that all norms are equivalent on $\mathbb{P}_k(\hat{E})$.
  - Map back to $E$.

- Prove that (1) implies (2) by using the mesh regularity assumption.
- Show that (3)-(4) follow from (1)-(2).

A calculus reminder:

$$|DF_E| = 2|E|.$$

3. Prove there exists $C \in \mathbb{R}^+$ independent of $E \in \tau_h$, such that for any edge $e \subset \partial E$ and any $v \in H^1(E)$, the following trace inequality holds:

$$\|v\|_{L^2(e)} \leq C|e|^{1/2}|E|^{-1/2} \left(\|v\|_{L^2(E)} + h_E \, |v|_{H^1(E)}\right).$$

It could help to note that for any $a \geq 0$ and $b \geq 0$,

$$a^2 + b^2 \leq (a+b)^2.$$