Math 5521 Homework 2

Due Spring 2018

In all problems, let Ω be an open, bounded, simply-connected subset of $\mathbb{R}^d$ with Lipschitz-continuous boundary, for $d = 2$ or $d = 3$, and let K be a constant, SPD matrix.

1. Let $V = H_{DIV}(\Omega)$ and $W = L^2(\Omega)$. Given $f \in W$, use the results from class to prove the equivalence of problems $(P8.1)$ and $(P8.2)$:

$$(P8.1)\left\{\begin{array}{c} \text{Find } \vec{u} \in V \text{ and } p \in W \text{ such that} \\ L^*(\vec{u}, p) = \inf_{\vec{v}\in V} \sup_{w\in W} L^*(\vec{v}, w), \\ \text{where } L^*(\vec{v}, w) \equiv \frac{1}{2}\left(K^{-1}\vec{v}, \vec{v}\right) - (\nabla\cdot\vec{v}, w) + (f, w) \end{array}\right.$$

$$(P8.2)\left\{\begin{array}{c} \text{Find } \vec{u} \in V_f \equiv \{\vec{v} \in V \,|\, \nabla\cdot\vec{v} = f\} \text{ such that} \\ J^*(\vec{u}) = \inf_{\vec{v}\in V_f} J^*(\vec{v}), \\ \text{where } J^*(\vec{v}) \equiv \frac{1}{2}\left(K^{-1}\vec{v}, \vec{v}\right) \end{array}\right.$$

2. Consider the problem to solve for $\vec{u} = -K\nabla p$ such that $\nabla \cdot \vec{u} = f \in L^2(\Omega)$, and subject to boundary conditions $\vec{u}\cdot\hat{n} = g_N \in H^{-1/2}(\partial\Omega)$. Assume that the required compatability condition holds:

$$\int_\Omega f\,dx = \int_{\partial\Omega} g_N\,d\sigma,$$

and for uniqueness of the pressure,

$$\overline{p} \equiv \frac{1}{|\Omega|}\int_\Omega p\,dx = 0.$$

Very that the necessary assumptions hold to apply Theorem 11.1 from class, and thereby prove well-posedness of the dual mixed formulation for this "pure Neumann" problem. Follow along as in the notes for the Dirichlet case, but consider the auxiliary problem

$$\left.\begin{array}{r} -\Delta\phi = w \in L_0^2(\Omega), \\ \nabla\phi\cdot\hat{n} = 0, \text{ on } \partial\Omega, \\ \overline{\phi} = 0. \end{array}\right\} \qquad (1)$$

You may use the fact that there exists a fixed, positive constant C_P such that

$$\|q - \overline{q}\|_{L^2(\Omega)} \leq C_P \left\|\nabla q\right\|_{L^2(\Omega)}, \ \forall q \in H^1(\Omega).$$

It may help to show that the space

$$X \equiv \left\{q \in H^1(\Omega) \,|\, \overline{q} = 0\right\}$$

is a Hilbert space when equipped with the inner-product $(\nabla q_1, \nabla q_2)$.

3. Consider now that $\Omega \subset \mathbb{R}^2$ is convex and has a polygonal boundary. Consider applying the mixed FEM with RT_k elements for homework problem 2 above. Show that the (discrete) LBB^h condition will hold. You can use the fact that under the given assumptions on Ω here, the conclusions of the elliptic regularity result (Lemma 21.1 in class) still hold. That is, problem (1) above has a solution that satisfies

$$\|\phi\|_{H^2(\Omega)} \leq C\|w\|_{L^2(\Omega)},$$

where C is independent of w.