Math 5521 Homework 1

Due Spring 2018

In all problems, let Ω be an open, bounded, simply-connected subset of $\mathbb{R}^d$ with Lipschitz-continuous boundary, for $d = 2$ or $d = 3$.

1. Let $f \in H^k(\Omega)$, where k may be *any* (fixed) integer. Show that

$$\frac{\partial f}{\partial x_i} \in H^{k-1}(\Omega),\ 1 \le i \le d.$$

2. Let Ω_j, $1 \le j \le N < \infty$, be bounded, simply-connected subsets of $\mathbb{R}^d$ with Lipschitz-continuous boundaries, such that $\overline{\Omega} = \cup_{1 \le j \le N} \Omega_j$. Here each set Ω_j is closed with non-empty interior, and for $i \ne j$, $\mathring{\Omega}_i \cap \mathring{\Omega}_j = \varnothing$. Let $\hat{n}_j$ be the outward-pointing unit normal vectors for $\partial \Omega_j$, and let $\vec{u}_j = \vec{u}|_{\Omega_j}$, $1 \le j \le N$, where $\vec{u} \in (L^1(\Omega))^d$. Suppose that

$$\vec{u}_j \in H_{DIV}(\Omega_j),\ 1 \le j \le N$$

and that

$$\vec{u}_i \cdot \hat{n}_i + \vec{u}_j \cdot \hat{n}_j = 0,\ 1 \le i, j \le N.$$

Prove that $\vec{u} \in H_{DIV}(\Omega)$.

3. The purpose of this exercise is to prove that the trace operator $\gamma_0 : H^1(\Omega) \to Range(\gamma_0) = H^{1/2}(\partial\Omega)$ admits a continuous, linear right inverse, say E. Use the norm

$$\|g\|_{H^{1/2}(\partial\Omega)} \equiv \inf_{\substack{v \in H^1(\Omega) \\ \gamma_0(v) = g}} \|v\|_{H^1(\Omega)}.$$

You may use the fact that $H^{1/2}(\partial\Omega)$ is Banach with this norm, and that the trace operator is continuous;

$$\|\gamma_0(v)\|_{H^{1/2}(\partial\Omega)} \le \|v\|_{H^1(\Omega)}.$$

To construct the inverse, given $g \in H^{1/2}(\partial\Omega)$, consider solving for $\phi = E(g) \in H^1(\Omega)$ with $\gamma_0(\phi) = g$ such that

$$(\phi, \psi) + (\nabla\phi, \nabla\psi) = 0,\ \forall \psi \in H_0^1(\Omega).$$

Show that $Range(E)$ is closed. Apply the Open Mapping Theorem to conclude that E is continuous.