

# Math 5520 Lecture 11

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Non-symmetric problems require a generalization of our previous analysis

$$\begin{aligned} -\nabla \cdot A \nabla u + \vec{b} \cdot \nabla u + a_0 u &= f \text{ on } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Here,  $\vec{b} \in \mathbb{R}^d$  and  $\|\vec{b}\|_{L^\infty(\Omega)} \leq b_{max}$ . We take the function space  $V = H_0^1(\Omega)$  and the weak problem is  
*Find  $u \in V$  such that*

$$\int_{\Omega} \left( A \nabla u \cdot \nabla v + (\vec{b} \cdot \nabla u) v + a_0 u v \right) dx = \int_{\Omega} f v dx, \quad \forall v \in V.$$

Denote the left-hand side by  $a(u, v)$  in the usual manner. Then

$$a(u, v) \neq a(v, u).$$

We cannot use this bilinear form to derive an inner product on  $V$ .

The Lax-Milgram Theorem solves this dilemma;  
symmetricity is not important

### Theorem (Lax-Milgram)

Let  $V$  be a Hilbert space. Let  $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$  be a bilinear form. Assume that  $a(\cdot, \cdot)$  is continuous and coercive:

1.  $|a(u, v)| \leq C_1 \|u\|_V \|v\|_V$ .
2.  $a(u, u) \geq \alpha_0 \|u\|_V^2$ .

Let  $L(\cdot) : V \rightarrow \mathbb{R}$  with  $L \in V^*$ . Then there exists a unique solution  $u \in V$  such that

$$a(u, v) = L(v), \quad \forall v \in V.$$

Moreover,  $\|u\|_V \leq \frac{1}{\alpha_0} \|L\|_{V^*}$ .

## Proof of Lax-Milgram

Fix  $\phi \in V$ . Define  $L_\phi(v) = a(\phi, v)$ , for all  $v \in V$ . Then  $L_\phi$  is a continuous linear functional. There exists a unique  $\tilde{\phi} \in V$  such that

$$L_\phi(v) = \langle \tilde{\phi}, v \rangle, \quad \forall v \in V.$$

Thus  $\langle \tilde{\phi}, v \rangle = a(\phi, v)$ , for all  $v \in V$ . We define  $A : V \rightarrow V$  by  $A\phi = \tilde{\phi}$ , so that

$$a(\phi, v) = \langle A\phi, v \rangle, \quad \forall \phi, v \in V.$$

Next, one shows that (homework):

1.  $A$  is linear,
2.  $\mathcal{R}(A) \subset V$  is a closed subspace of  $V$ ,
3.  $A$  is onto  $V$  (use Projection Theorem), and subsequently
4. the desired results follow (use the RRT).

## We can now apply Lax-Milgram to our non-symmetric model problem

Let us assume:

- ▶  $\alpha_0 z^T z \leq z^T A z \leq \alpha_1 z^T z$
- ▶  $0 \leq a_{min} \leq a_0(x) \leq a_{max} < \infty$
- ▶  $\vec{b} \in (L^\infty(\Omega))^d$

First, we show continuity:

$$\begin{aligned} |a(u, v)| &\leq \int_{\Omega} |A^{1/2} \nabla u \cdot A^{1/2} \nabla v| + |\vec{b} \cdot \nabla u| |v| + a_0 |u| |v| \, dx \\ &\leq \left( \int_{\Omega} |A^{1/2} \nabla u|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |A^{1/2} \nabla v|^2 \, dx \right)^{1/2} \\ &\quad + \sum_{k=1}^d \|b_k\|_{L^\infty(\Omega)} \int_{\Omega} \left| \frac{\partial}{\partial x_k} u \right| |v| \, dx + a_{max} \int_{\Omega} |u| |v| \, dx \end{aligned}$$

## Continuity and coercivity

$$\begin{aligned} \Rightarrow |a(u, v)| &\leq \alpha_1 |u|_{H^1(\Omega)} |v|_{H^1(\Omega)} \\ &+ \left\{ \sum_{k=1}^d \|b_k\|_{L^\infty(\Omega)}^2 \right\}^{1/2} |u|_{H^1(\Omega)} \|v\|_{L^2(\Omega)} + a_{max} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\left( \text{set } B = \left\{ \sum_{k=1}^d \|b_k\|_{L^\infty(\Omega)}^2 \right\}^{1/2} \right) \\ &\leq (\alpha_1 + B + a_{max}) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned}$$

Coercivity:

$$\begin{aligned} a(v, v) &= \int_{\Omega} \left( A \nabla v \cdot \nabla v + (\vec{b} \cdot \nabla v) v + a_0 |v|^2 \right) dx \\ &\geq \min\{\alpha_0, a_{min}\} \int_{\Omega} |\nabla v|^2 + |v|^2 dx + \int_{\Omega} (\vec{b} \cdot \nabla v) v dx \end{aligned}$$

## Coercivity continued

In fact, we allow for  $a_{\min} = 0$ , in which case

$$\begin{aligned} a(v, v) &\geq \min\{1, C_P^{-2}\} \frac{\alpha_0}{2} \|v\|_{H^1(\Omega)}^2 + \int_{\Omega} (\vec{b} \cdot \nabla v) v \, dx \\ &\geq \min\{1, C_P^{-2}\} \frac{\alpha_0}{2} \|v\|_{H^1(\Omega)}^2 - \frac{B}{2} \|v\|_{H^1(\Omega)}^2. \end{aligned}$$

Coercivity follows under the condition that

$$\min\{1, C_P^{-2}\} \alpha_0 - B \geq \alpha > 0$$

for some fixed  $\alpha \in \mathbb{R}$ . In this case,

$$a(v, v) \geq \alpha \|v\|_{H^1(\Omega)}^2, \quad \forall v \in V.$$

Lax-Milgram may now be applied, so long as  $L \in V^*$ .

## Some other conditions under which the problem is well-posed

Coercivity is always the trick.

### Theorem (Gårding Inequality)

For  $a(\cdot, \cdot)$  defined as above, there exists some  $K < \infty$  such that

$$a(v, v) + K\|v\|_{L^2(\Omega)}^2 \geq \frac{\alpha_0}{2}\|v\|_{H^1(\Omega)}^2, \quad \forall v \in H^1(\Omega).$$

*Proof.* We have that

$$a(v, v) + K\|v\|_{L^2(\Omega)}^2 \geq \alpha_0|v|_{H^1(\Omega)}^2 + \int_{\Omega} (\vec{b} \cdot \nabla v)v \, dx + (K + a_{min})\|v\|_{L^2(\Omega)}^2.$$

Recall that

$$\left| \int_{\Omega} (\vec{b} \cdot \nabla v)v \, dx \right| \leq B|v|_{H^1(\Omega)}\|v\|_{L^2(\Omega)}.$$

## Some other conditions under which the problem is well-posed

Apply Young's inequality to get

$$\left| \int_{\Omega} (\vec{b} \cdot \nabla v) v \, dx \right| \leq B \left( \frac{\alpha_0}{2B} |v|_{H^1(\Omega)}^2 + \frac{B}{2\alpha_0} \|v\|_{L^2(\Omega)}^2 \right).$$

It follows that

$$a(v, v) + K \|v\|_{L^2(\Omega)}^2 \geq \frac{\alpha_0}{2} |v|_{H^1(\Omega)}^2 + \left( K + a_{\min} - \frac{B^2}{2\alpha_0} \right) \|v\|_{L^2(\Omega)}^2.$$

Then take any  $K \geq \frac{B^2}{2\alpha_0} + \frac{\alpha_0}{2} - a_{\min}$ .

Note that if  $a_{\min} \geq \frac{B^2}{2\alpha_0} + \frac{\alpha_0}{2}$ , then we can take  $K = 0$  and get coercivity. Yet another condition for coercivity is that  $a_0(x) - \frac{1}{2} \nabla \cdot \vec{b}(x) \geq 0$ , for all  $x \in \Omega$  (recall  $V = H_0^1(\Omega)$ ).