Math 5511 Homework 3

Due May 10, 2019

Instructions: submit solutions for any two problems below.

Theoretical:
(1) You are given the model problem to find $y : [0,T] \to \mathbb{R}$ for some $T > 0$ such that

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad t \in [0,T], \\ y(0) &= y_0 \in \mathbb{R}. \end{aligned}$$

(1a) Show that Crank-Nicolson is an A-stable method for IVPs.

(1b) Show that Crank-Nicolson is consistent of order 2 for any $y \in C^3[0,T]$ that solves this problem.

(1c) Given that f satisfies a uniform Lipschitz condition

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|, \quad \forall t \in [0,T], \quad \text{and any } y_1,\ y_2 \in \mathbb{R},$$

prove that for a unique solution $y \in C^3[0,T]$, the Crank-Nicolson method is convergent of order 2.

(2) You are given the boundary value problem to find $u : [a,b] \to \mathbb{R}$ such that

$$\begin{aligned} u(x) - u''(x) &= f(x), \quad x \in (a,b), \\ [u(a), u(b)] &= [\alpha, \beta] \in \mathbb{R}^2, \end{aligned}$$

with $f(x)$ a smooth function. Derive a finite-difference method (FDM) to approximate the solution to this problem that is consistent of order 2 for smooth solutions $u(x)$. (You must show the second-order truncation error). Write the FDM using matrix notation as $M\vec{u} = \vec{b}$, where $\vec{u}$, $\vec{b}$ are column vectors and M a square matrix. Here, $\vec{u}$ represents the nodal values of the discrete solution and should not contain the boundary values. Provide formulas for the entries of M and $\vec{b}$.

Computational:
Submit your MATLAB code (e-mail if possible) along with your solutions. Explain how to run the code where appropriate.

(3) Solve $y'(t) = \lambda y(t)$, $0 \le t \le 1$, with $y(0) = 10$ for $\lambda = -1, -50$ using forward-Euler, backward-Euler, the midpoint method and Crank-Nicolson. You can easily calculate the update formulas by hand and put them all in a single loop. You do not need to store the solution at all time steps; you will only use the last time step. Use time steps sizes $\Delta t = 1/2^k$ for $k = 2, 3, \ldots, 9$. Since we know the true solution, use it to calculate the errors $e = e(\Delta t)$ at time $t = 1$. Plot $\log|e(\Delta t)|$ versus $\log \Delta t$ for each of the four methods using only two plots; one for each λ-value. Explain the results.

(4) Implement your finite difference method for problem (2) above, in the case $\alpha = 1$, $\beta = 2$, $f(x) = (1 - \gamma^2)e^{\gamma(x+1)}$, $\gamma = \ln(2)/2$, on the domain $[a,b] = [-1,1]$. The true solution is given by

$u(x) = e^{\gamma(x+1)}$. Use grid sizes $h = 1/2^k$ for $k = 2, 3, \ldots, 7$. Presumably, you should be able to apply the error *anzatz* $e_h = \max_i |u(x_i) - u_i| = Ch^2$, from which it follows that

$$\log e_h = \log C + 2 \log h.$$

Plot $\log e_h$ versus $\log h$ and explain your results.