Math 5511 Homework 1

Due May 10, 2019

Solve 3 problems total from the 7 below.

Theoretical

(1) Let $f(x) = \frac{x}{1+x^2}$. Find the largest real value $a > 0$ such that for any initial guess $x_0 \in (-a, a)$, the Newton's method iterates x_k, $k = 0, 1, \ldots$ converge. Prove that your a is maximal and that $x_k \to 0$ with a quadratic convergence rate.

(2) Let $p(x) : \mathbb{R} \to \mathbb{R}$ be a polynomial of order $n > 1$ with real coefficients and n distinct, real-valued roots. Let $x = \eta$ be a root of $p(x)$. Let x_k, $k = 0, 1, \ldots$, be approximations to η calculated using the method

$$x_{k+1} = x_k - \frac{p(x_k)}{p[x_{k-2}, x_k]}.$$

This is a modification to the secant method that is less accurate in general. Prove that if x_0, x_1 and x_2 are sufficiently close to η then this method will converge with a rate q, where q is the largest root of the equation $q^3 = q^2 + 1$.

(3) Let A be an $n \times n$ real, SPD matrix and let b be a real, $n \times 1$ array. Prove that the following problems are equivalent: Find $x \in \mathbb{R}^{n \times 1}$ such that

$$\begin{aligned} &(\text{P1}): \; Ax = b \\ &(\text{P2}): \; \frac{1}{2}x^T Ax - b^T x \leq \frac{1}{2}y^T Ay - b^T y, \text{ for all } y \in \mathbb{R}^{n \times 1}. \end{aligned}$$

Computational

Submit your MATLAB code (e-mail if possible) along with your solutions. Explain how to run the code where appropriate.

(4) Write two codes: the standard Newton's method and Broyden's method. Use the initial guesses $x_0 = 1$ and $x_0 = 6$ with both methods to try to find the root of $\arctan(x) = 0$ with an accuracy of 10^{-8} or better. In case of convergence, provide tables with each iterate x_k in one column and the ratios

$$\frac{|x_{k+1} - x_k|}{|x_k - x_{k-1}|^2}$$

in the next column. Explain your results.

(5) Write a code to implement Maehly's method to find the roots of the following polynomials:

(a) $p(x) = 2x^2 + 19x - 10$

(b) $q(x) = 40x^6 - 243x^5 + 137x^4 + 1037x^3 - 1041x^2 + 46x + 24$

(c) $r(x) = x^3 - x^2 - x + 1$

In each case, ensure that the accuracy of each root is 10^{-8} or better and list the computed roots.

(6) Write a code to implement Muller's method. Use it to find all roots (real and complex) of

$$p(x) = x^3 - 7x^2 + 12x - 10.$$

In case of a conjugate pair, only one of the pair needs to be found. Ensure that the accuracy of each root is 10^{-8} or better. You will need to choose initial values x_0, x_1 and x_2. Provide the calculated values of each root as well as the initial guesses used in each case. Provide the convergence rate in each case based upon the last three successive differences $e_k = |x_k - x_{k-1}|$, using the formula

$$\text{rate} = \frac{\log\left(e_{k+1}/e_k\right)}{\log\left(e_k/e_{k-1}\right)}.$$

This rate calculation is related to "Richardson extrapolation".

(7) Code up the BFGS method with exact line search to minimize the quadratic functional $h : \mathbb{R}^n \to \mathbb{R}$, where

$$h(x) = \frac{1}{2}x^T Ax + b^T x,$$

with A an $n \times n$ matrix and b an $n \times 1$ vector. In particular, if the entries of A and b are $A_{i,j}$ and b_i, respectively, then $b_i = 1$ and $A_{i,i} = 3$ for $1 \le i \le n$, $A_{i,i-1} = A_{i-1,i} = -1$ for $2 \le i \le n$ and A has zeros elsewhere. (This problem is related to solving a certain boundary-value ODE problem.) Have your code calculate the 2-norm of the error, which should be round-off sized, and output the final number of iterations, say m. Your code should work for any $n > 1$; run it for various cases. What relationship do you observe between m and n?