Math 2110Q Worksheet 15 Solutions

1. Let $\mathscr{D} = \{(x,y) \,|\, 0 \le x \le 1,\ -x \le y \le 2x\}$. Use the transformation $u = x+y$, $v = 2x-y$ to calculate

$$\int\int_{\mathscr{D}} 2(x+y)(2x-y-3)\,dA.$$

Solution: We find the corresponding bounds in uv-space by transforming each boundary edge in turn. Along the edge $y=-x$, $x+y=0$ so that $u=0$. Along the edge $y=2x$, $2x-y=0$ so that $v=0$. Then there is the edge where $x=1$. Then we have both $u=1+y$ and $v=2-y$. Then $u-1=y$ and we substitute for y in the v-equation to get $v=2-(u-1)=3-u$, which is a line in uv-space. So the transformation corresponds to a triangle $0 \le u \le 3$, $0 \le v \le 3-u$ in uv-space. Now, we find the magnitude of the Jacobian; first we invert our transformation. We could substitute $x=u-y$ into the v-equation to get $v=2(u-y)-y=2u-3y$. Then $y=(2u-v)/3$. Substitute this back into $x=u-y=u-(2u-v)/3=(u+v)/3$. Therefore, the Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}.$$

Recall we take the absolute value of the Jacobian when we apply the change of variables, so that

$$\begin{aligned}\int\int_{\mathscr{D}} 2(x+y)(2x-y-3)\,dA &= \int_0^3\int_0^{3-u} 2u(v-3)\frac{1}{3}\,dv\,du = \frac{1}{3}\int_0^3 u(v-3)^2|_{v=0}^{v=3-u}\,du \\ &= \frac{1}{3}\int_0^3 u^3-9u\,du = \frac{1}{3}\left[\frac{1}{4}u^4-\frac{9}{2}u^2\right]|_0^3 \\ &= \frac{1}{3}\left(\frac{81}{4}-\frac{81}{2}\right) = -\frac{27}{4}.\end{aligned}$$

Some comments:

1. There are many ways to explain the domain transformation. Since the change of variables is linear, we could note that the new region in uv-space must also be triangular, and simply map the vertices of the triangle in xy-space to those in uv-space, then connect the dots.
2. One could also find the inverse transformation first, then insert the new equations for $x=x(u,v)$ and $y=y(u,v)$ into the equations for the boundary edges of the triangle in xy-space to convert it to uv-space. In this case, this is probably the most work.

2. Describe the circle of intersection of the spheres $x^2+y^2+z^2=16$ and $x^2+y^2+(z-3)^2=4$ using spherical coordinates. (4 pts.)

Solution: convert these equations to spherical coordinates. The first sphere is just $\rho=4$. The second is

$$\begin{aligned}\rho^2\sin^2(\phi)+(\rho\cos(\phi)-3)^2=4 &\Rightarrow \rho^2\sin^2(\phi)+\rho^2\cos^2(\phi)-6\rho\cos(\phi)+9=4 \\ &\Rightarrow \rho^2-6\rho\cos(\phi)+5=0.\end{aligned}$$

The intersection occurs when $\rho=4$, so that

$$16-24\cos(\phi)+5=21-24\cos(\phi)=0 \Rightarrow \cos(\phi)=\frac{21}{24}=\frac{7}{8} \Rightarrow \phi=\cos^{-1}\left(\frac{7}{8}\right).$$

The circle of intersection wraps around the z-axis. In spherical coordinates, the curve is completely described by

$$\rho=4,\ \phi=\cos^{-1}\left(\frac{7}{8}\right),\ 0 \le \theta \le 2\pi.$$