

The Chain Rule revisited

Recall  $y=y(x)$  &  $x=x(t)$

$$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad (\text{Chain Rule})$$

But now we can have  $f=f(x,y)$

with  $x=x(t)$  &  $y=y(t)$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} .$$

↑ "note  
partials" ↑

Similarly, for  $f = f(x, y, z)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

} • Clear  
• Precise

"Compact" notation:

$$f'(t) = f_x \cdot X' + f_y \cdot Y' + f_z \cdot Z'$$

} • Fast, easy  
• Need to recall meaning

$$\text{Let } x = x(s, t), \quad y = y(s, t) \\ f = f(x, y)$$

Now there are two cases; s and t

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

EX:  $f(x, y) = e^{x+y}$ ,  $x = st$ ,  $y = s^2 - t^2$ .

Find  $f_s, f_t$ .

$$\frac{\partial x}{\partial s} = t \quad \frac{\partial y}{\partial s} = 2s$$
$$\frac{\partial x}{\partial t} = s \quad \frac{\partial y}{\partial t} = -2t$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = e^{x+y} (t + 2s)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = e^{x+y} (s - 2t).$$

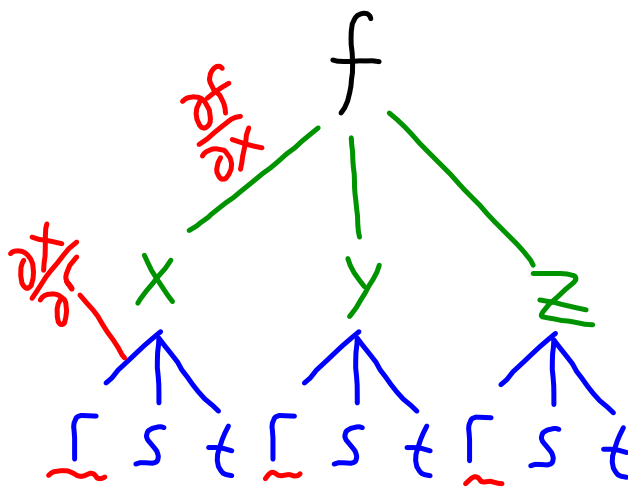
## Tree diagrams

$$f=f(x,y,z)$$

$$x=x(r,s,t)$$

$$y=y(r,s,t)$$

$$z=z(r,s,t)$$

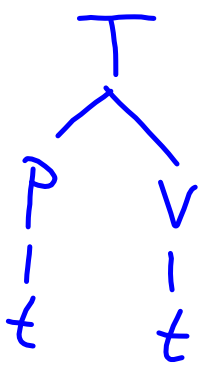


$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}$$

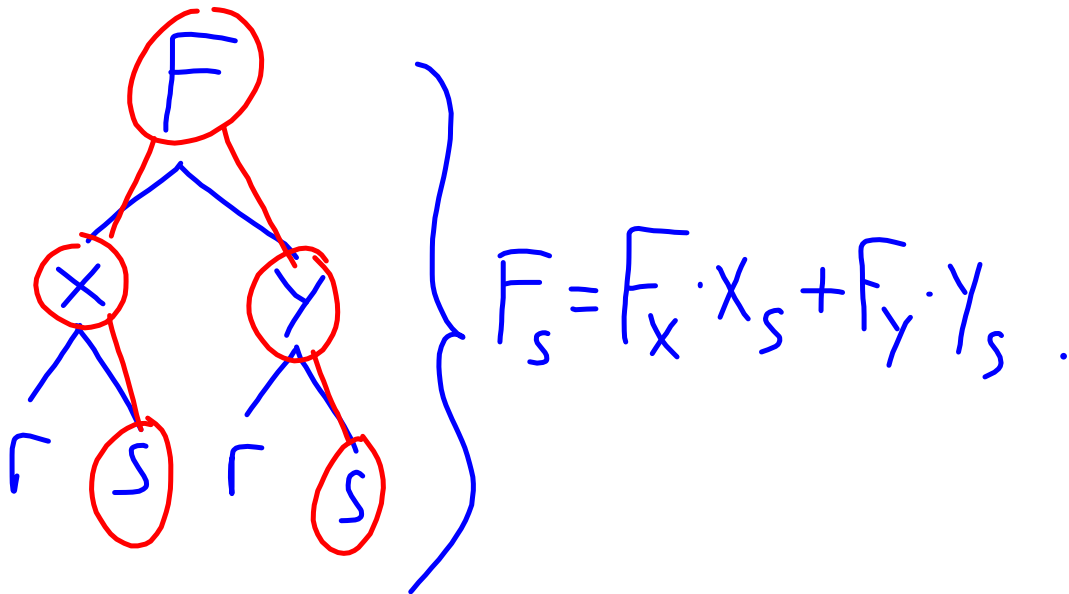
EX: Temperature  $T = T(P, v)$

$P$ : pressure } depend on time "t".  
 $V$ : volume }

Find  $\frac{dT}{dt}$ .

Tree:   $\left. \begin{array}{l} \frac{dT}{dt} = \frac{\partial T}{\partial P} \cdot \frac{dP}{dt} + \frac{\partial T}{\partial V} \cdot \frac{dV}{dt} \end{array} \right\}$

EX: Find  $\frac{\partial F}{\partial s}$ ;  $F = F(x, y)$  and  $x, y$  depend on  $r, s$ .



Implicit differentiation revisited

$F = F(x, y) = 0$  is often encountered.

E.g.  $x^3 + y^3 = -xy$  } so  $F(x, y) = x^3 + xy + y^3 = 0$ .

Sometimes\* we may think of  $y = y(x)$  but lack an EXPLICIT formula... so  $y'(x) = ?$

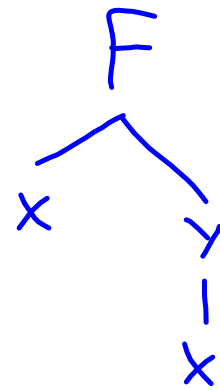
\* "Implicit function theorem" says when  $y = y(x)$  is valid... depends on  $x, y$  choice.



Differentiate  $F$  to figure out  $dy/dx$

$$F(x, y) = 0$$
$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = - \frac{F_x}{F_y}$$



EX:  $F = x^3 + xy + y^3 = 0 \dots \frac{dy}{dx} = ?$

$$F_x = 3x^2 + y \quad F_y = x + 3y^2$$

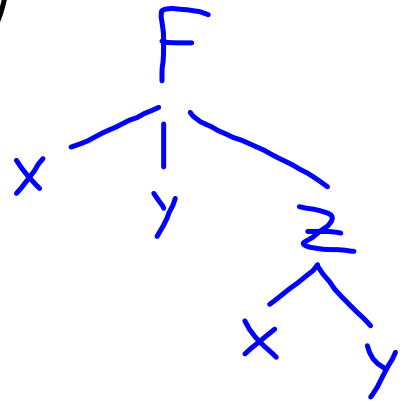
$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 + y}{x + 3y^2}.$$

Similarly, if  $F = F(x, y, z) = 0$ ,  
 $z = z(x, y)$ ,

$$\text{then } F_x + F_z \cdot z_x = 0$$

$$F_y + F_z \cdot z_y = 0$$

$$\Rightarrow z_x = -\frac{F_x}{F_z}$$
$$z_y = -\frac{F_y}{F_z}.$$



## Directional derivatives

Recall  $f_x$  : rate of change,  $x$ -direction  
 $f_y$  : rate of change,  $y$ -direction

$\hat{u} = \langle a, b \rangle$  : unit vector

Rate of change in direction  $\hat{u}$  is

$$D_{\hat{u}} f = \lim_{h \rightarrow 0} \frac{f(x+ah, y+bh) - f(x, y)}{h}$$

So how do we calculate this?

$$\begin{aligned} &= \lim_{h \rightarrow 0} \left[ \frac{f(x+ah, y+bh) - f(x, y+bh)}{h} + \frac{f(x, y+bh) - f(x, y)}{h} \right] \\ &= a \lim_{h \rightarrow 0} \frac{f(x+ah, \cdot) - f(x, \cdot)}{ah} + b \lim_{h \rightarrow 0} \frac{f(\cdot, y+bh) - f(\cdot, y)}{bh} \\ &= a \lim_{z \rightarrow 0} \frac{f(x+z, \cdot) - f(x, \cdot)}{z} + b \lim_{z \rightarrow 0} \frac{f(\cdot, y+z) - f(\cdot, y)}{z} \\ &= a \frac{\partial f}{\partial x}(x, y) + b \frac{\partial f}{\partial y}(x, y). \end{aligned}$$

DEFINE :  $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$   
"gradient"

$$\Rightarrow D_{\hat{u}} f = \nabla f \cdot \hat{u}$$

$$\langle f_x, f_y \rangle \cdot \langle a, b \rangle = a f_x + b f_y$$

EX:  $f(x,y) = x^2 + 4y^2 + 10$ . Find the derivative of  $f$  in the direction  $\langle 1, 2 \rangle$  at  $(x=3, y=-1)$ .

$$\nabla f = \langle 2x, 8y \rangle$$

$$\nabla f(3, -1) = \langle 6, -8 \rangle$$

$$\Rightarrow \hat{u} = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$$

$$D_{\hat{u}} f = \langle 6, -8 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{6-16}{\sqrt{5}} = \frac{-10}{\sqrt{5}}$$

Maximum rate of change: It turns out that

(1) the fastest rate of change is  $|\nabla f|$

(2) this occurs in the direction of  $\nabla f$

Proof:  $\nabla f \cdot \hat{u} = |\nabla f| \underbrace{|\hat{u}|}_{1} \cos \theta$

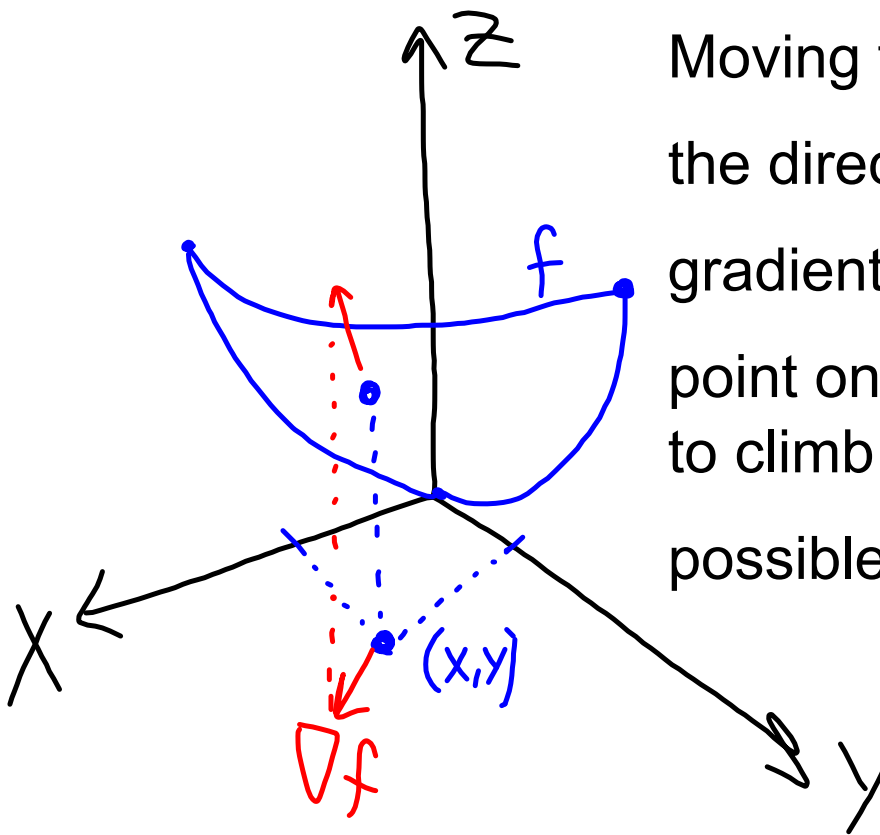
$$\Rightarrow -|\nabla f| \leq \nabla f \cdot \hat{u} \leq |\nabla f|$$

$\theta = \pi$   
 $\Rightarrow \hat{u}, \nabla f$   
opposite direction

$\theta = 0$   
 $\Rightarrow \hat{u}, \nabla f$  same  
direction



# Picture of the gradient



Moving from  $(x, y)$  in the direction of the gradient causes the point on the surface to climb as fast as possible.

EX: Find the max rate of change  
for  $f(x, y) = \cos(x - 2y)$  at  $(1, \frac{1}{2})$   
and at  $(\frac{\pi}{2}, 0)$ .

$$\nabla f = \langle -\sin(x - 2y), 2 \sin(x - 2y) \rangle$$

$$|\nabla f(1, \frac{1}{2})| = |\langle -\sin(0), 2 \sin(0) \rangle| = 0$$

$$|\nabla f(\frac{\pi}{2}, 0)| = |\langle -1, 2 \rangle| = \sqrt{5}.$$

Tangent plane to a surface

We will think of something like

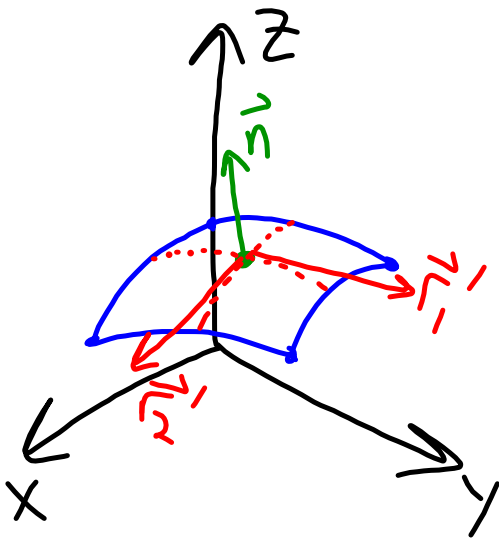
$$z = x^2 + y^2 \text{ as } F(x, y, z) = 0,$$

$$F = x^2 + y^2 - z \text{ or } F = z - x^2 - y^2.$$

Now consider a point  $(x_0, y_0, z_0)$  and **ANY** curve on the surface, passing through this point;  $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ .

Since  $(x(t), y(t), z(t))$  is on the surface,  $F(x(t), y(t), z(t)) = 0$   
 $\Rightarrow \frac{dF}{dt} = F_x \cdot x' + F_y \cdot y' + F_z \cdot z' = 0$

$$\Rightarrow \underbrace{\langle F_x, F_y, F_z \rangle}_{\nabla F} \cdot \underbrace{\langle x', y', z' \rangle}_{\frac{d\vec{r}}{dt}} = 0$$



Given two surface curves  
through  $P = (x_0, y_0, z_0)$ , say  
 $\vec{r}_1(t), \vec{r}_2(t)$  it follows

$$\nabla F \cdot \vec{r}_1' = 0$$

$$\nabla F \cdot \vec{r}_2' = 0$$

$$\Rightarrow \nabla F \parallel \vec{n} \leftarrow \text{normal to tangent plane}$$

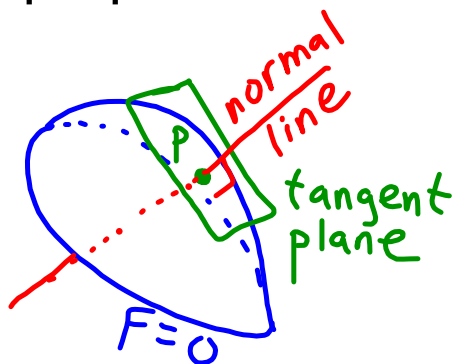
\*  $\nabla F$  is a normal for the tangent plane

\*  $P = (x_0, y_0, z_0)$  is in the plane

$$\Rightarrow \nabla F \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

TANGENT PLANE

Normal line: The line through the point P that is perpendicular to the tangent plane at P.



Since  $\nabla F$  is the direction vector for the normal line,

$$\frac{x-x_0}{F_x} = \frac{y-y_0}{F_y} = \frac{z-z_0}{F_z} = t.$$

EX: Find the tangent plane for  
 $\frac{x^2}{9} + y^2 + z^2 = 1$  (ellipsoid) at  $\underbrace{\left(\sqrt{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)}_P$ .

$$F = \frac{x^2}{9} + y^2 + z^2 - 1 = 0$$

$$\nabla F(P) = \left\langle \frac{2x}{9}, 2y, 2z \right\rangle \Big|_P = \left\langle \frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} \right\rangle$$

$$\Rightarrow \left\langle \frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} \right\rangle \cdot \left\langle x - \sqrt{3}, y - \frac{\sqrt{3}}{3}, z - \frac{\sqrt{3}}{3} \right\rangle = 0.$$



EX: Find the normal line through  $P$  in the previous example.

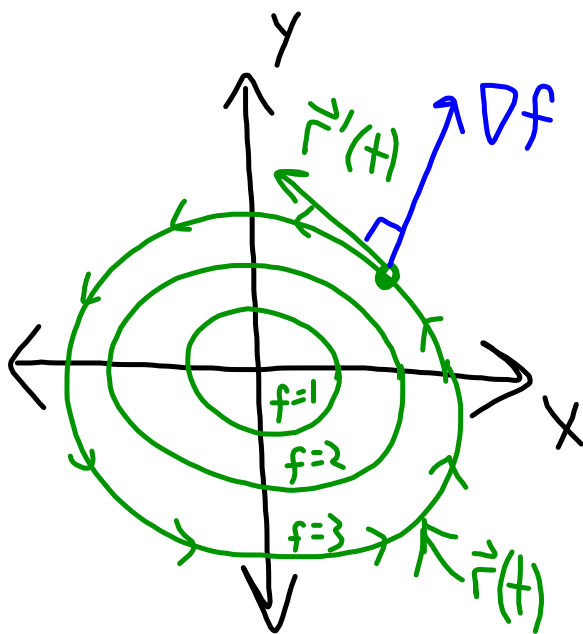
$$\nabla F = \left\langle \frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} \right\rangle$$

$$\Rightarrow \frac{9(x-\sqrt{3})}{2\sqrt{3}} = \frac{3(y-\sqrt{3}/3)}{2\sqrt{3}} = \frac{3(z-\sqrt{3}/3)}{2\sqrt{3}}$$

$$\Rightarrow 3(x-\sqrt{3}) = y - \frac{\sqrt{3}}{3} = z - \frac{\sqrt{3}}{3} = t$$

$$x = \frac{1}{3}t + \sqrt{3}, \quad y = t + \frac{\sqrt{3}}{3}, \quad z = t + \frac{\sqrt{3}}{3}.$$

Similarly,  $\nabla f(x,y)$  is perpendicular to level curves  $f(x,y)=K$ .



$$\left. \begin{aligned} f &= K, \text{ constant} \\ \Rightarrow \frac{df}{dt} &= f_x \cdot x' + f_y \cdot y' = 0 \\ \Rightarrow \nabla f \cdot \vec{r}' &= 0 \end{aligned} \right\}$$

Practice!

① Let  $f(x, y) = x^2 + xy + y^2$ ,  $x = \cos(t)$ ,  $y = \sin(t)$ .  
Find  $\frac{df}{dt}$  using the Chain Rule.

$$f_x = 2x + y \quad f_y = x + 2y$$

$$x' = -\sin(t) \quad y' = \cos(t)$$

$$\frac{df}{dt} = f_x \cdot x' + f_y \cdot y'$$

$$= (2x + y)(-\sin(t)) + (x + 2y)\cos(t).$$

(#2) Let  $f = f(x, y, z)$  and  $x, y, z$  are all functions of  $r, s, t, u, v$ . Find  $\frac{\partial f}{\partial u}$ .

$$\frac{\partial f}{\partial u} = f_x \cdot x_u + f_y \cdot y_u + f_z \cdot z_u$$

#3 Find the derivative of  $f(x,y)=2x+\sqrt{y}$  at  $(3,4)$  in the direction of  $\langle -1,1 \rangle$ .

$$\begin{aligned} \hat{u} &= \frac{1}{\sqrt{2}} \langle -1, 1 \rangle \\ D_{\hat{u}} f &= \nabla f \cdot \hat{u} \end{aligned} \left. \begin{array}{l} f_x = 2 \\ f_y = \frac{1}{2\sqrt{y}} \end{array} \right\}$$
$$= \left\langle 2, \frac{1}{4} \right\rangle \cdot \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{-2}{\sqrt{2}} + \frac{1}{4\sqrt{2}}$$
$$= \frac{-7}{4\sqrt{2}}$$

#4 Find the tangent plane and normal line through  $z = \sin(x+y)$  at  $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ .

$$F = \sin(x+y) - z = 0$$

$$\left. \begin{array}{l} F_x = \cos(x+y) \\ F_y = \cos(x+y) \\ F_z = -1 \end{array} \right\} \begin{array}{l} x_0 = \frac{\pi}{2} \\ y_0 = \frac{\pi}{2} \\ z_0 = 0 \end{array} \Rightarrow \begin{array}{l} F_x = -1 \\ F_y = -1 \\ F_z = -1 \end{array}$$

$$-\left(x - \frac{\pi}{2}\right) - \left(y - \frac{\pi}{2}\right) - z = 0. \quad \left( \begin{array}{l} \text{TANGENT} \\ \text{PLANE} \end{array} \right)$$

Normal line:

$$\frac{x-x_0}{F_x} = \frac{y-y_0}{F_y} = \frac{z-z_0}{F_z}$$

$$\Rightarrow \frac{x - \frac{F_x}{2}}{-1} = \frac{y - \frac{F_y}{2}}{-1} = \frac{z - 0}{-1}$$