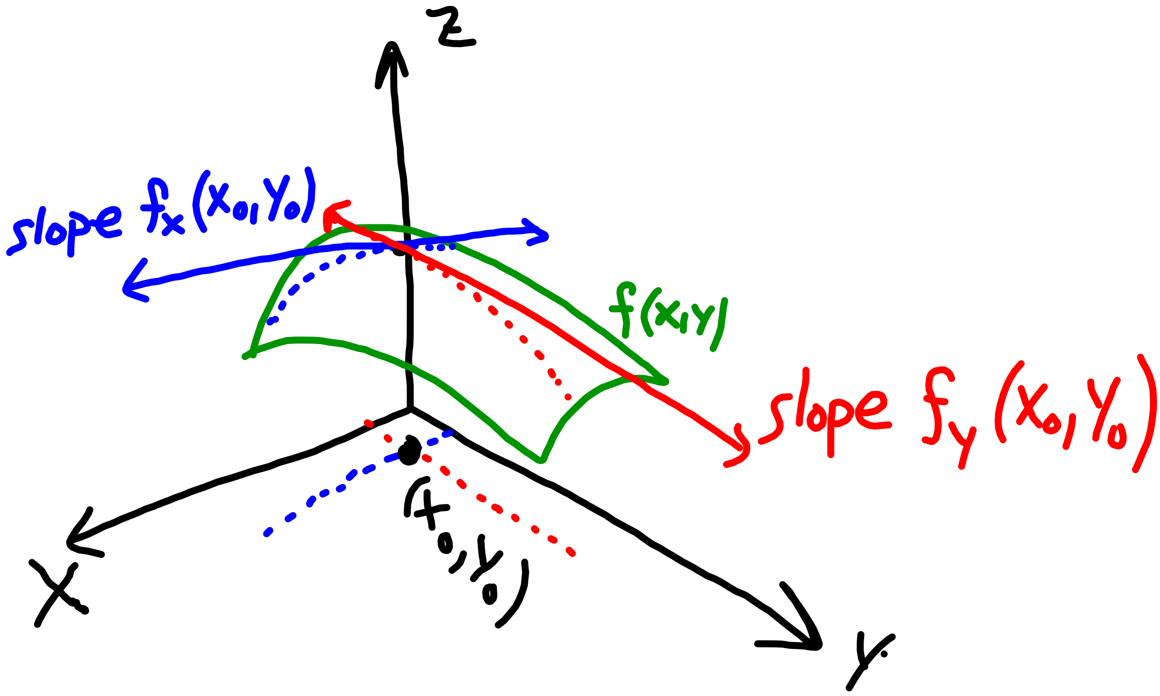
Partial Derivatives

For $f = f(x,y)$ we can discuss derivatives with respect to x, y independently:

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \underbrace{\frac{f(x+h,y) - f(x,y)}{h}}_{y \text{ is constant}}.$$

$$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = \lim_{h \to 0} \frac{\overbrace{f(x,y+h) - f(x,y)}^{x \text{ is constant}}}{h}.$$

Partial derivatives are the slopes of the tangent lines in the coordinate directions.



Ex: Find f_x, f_y if $f(x,y) = x + y + \sin(x)\cos(y)$.

Hold y fixed...

$$f_x = 1 + \cos(x)\cos(y).$$

Hold x fixed...

$$f_y = 1 - \sin(x)\sin(y).$$

EX: Calculate f_x, f_y for

(a) $f(x,y) = \ln(x^2+xy+y^2)$

$$f_x = \frac{1}{x^2+xy+y^2}\frac{\partial}{\partial x}\left(x^2+xy+y^2\right)$$

$$\Rightarrow f_x = \frac{2x+y}{x^2+xy+y^2}$$

$$f_y = \frac{2y+x}{x^2+xy+y^2}$$

(b) $f(x,y) = xy\sin(x+y)$

$f_x = y\sin(x+y) + xy\cos(x+y)$

$f_y = x\sin(x+y) + xy\cos(x+y)$

Implicit differentiation

Let $z = z(x,y)$ satisfy

$$x^3 + y^2 + z^4 + xyz = 1.$$

Then find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ by differentiating through the whole equation...

$$\frac{\partial}{\partial x}\left(x^3 + y^2 + z^4 + xyz\right) = \frac{\partial}{\partial x}(1) = 0$$

$$\Rightarrow 3x^2 + 4z^3\frac{\partial z}{\partial x} + yz + xy\frac{\partial z}{\partial x} = 0.$$

Now group $\frac{\partial z}{\partial x}$ terms and factor it out:

$$3x^2+yz+\frac{\partial z}{\partial x}\left\{4z^3+xy\right\}=0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x}=\frac{-3x^2-yz}{4z^3+xy}}.$$

$$\frac{\partial}{\partial y}\left(x^3+y^2+z^4+xyz\right)=\frac{\partial}{\partial y}(1)=0$$

$$\Rightarrow 2y+4z^3\frac{\partial z}{\partial y}+xz+xy\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{\partial z}{\partial y}(4z^3+xy) = -xz-2y$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-xz-2y}{4z^3+xy}}.$$

* If $(x, y, z(x,y))$ solve the original equation then you can plug them in to get the partial derivatives.

Higher-order derivatives

2^{nd}-order of $f(x,y)$: $\begin{matrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{matrix}$

THEOREM: If f_{xy} & f_{yx} are both continuous then $f_{xy} = f_{yx}$.

$\underline{EX}$: Find all 2^{nd}-order derivatives of

$$f(x,y) = x^3 + y^3 + x^2y^2 + xy + 1.$$

$$f_x = 3x^2 + 2xy^2 + y \qquad f_y = 3y^2 + 2x^2y + x$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$f_{xy} = 4xy + 1 \xleftrightarrow{\text{equal}} f_{yx} = 4xy + 1$$

$$f_{xx} = 6x + 2y^2 \qquad f_{yy} = 6y + 2x^2$$

Some other notations

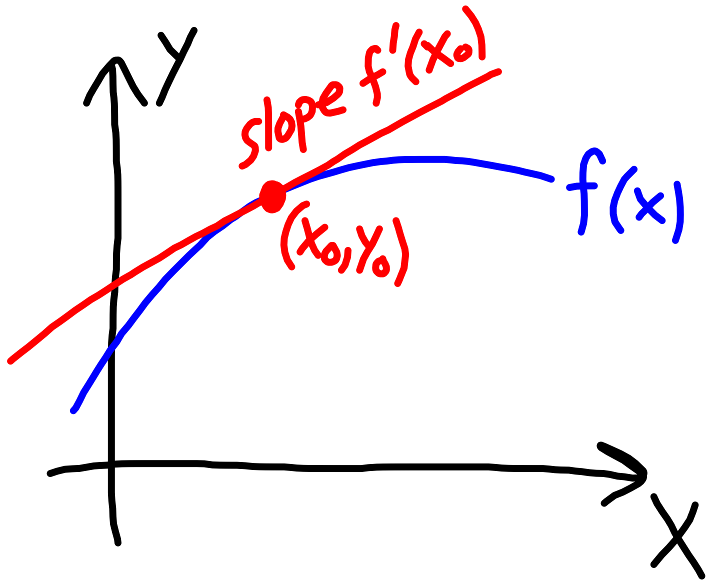
$$\frac{\partial}{\partial x} f = \frac{\partial f}{\partial x} = D_x f = f_x$$

$$\frac{\partial^2}{\partial y \partial x} f = \frac{\partial^2 f}{\partial y \partial x} = D_{xy} f = f_{xy}$$

These have various uses depending on the circumstances and are worth being familiar with for convenience in writing.

Tangent planes

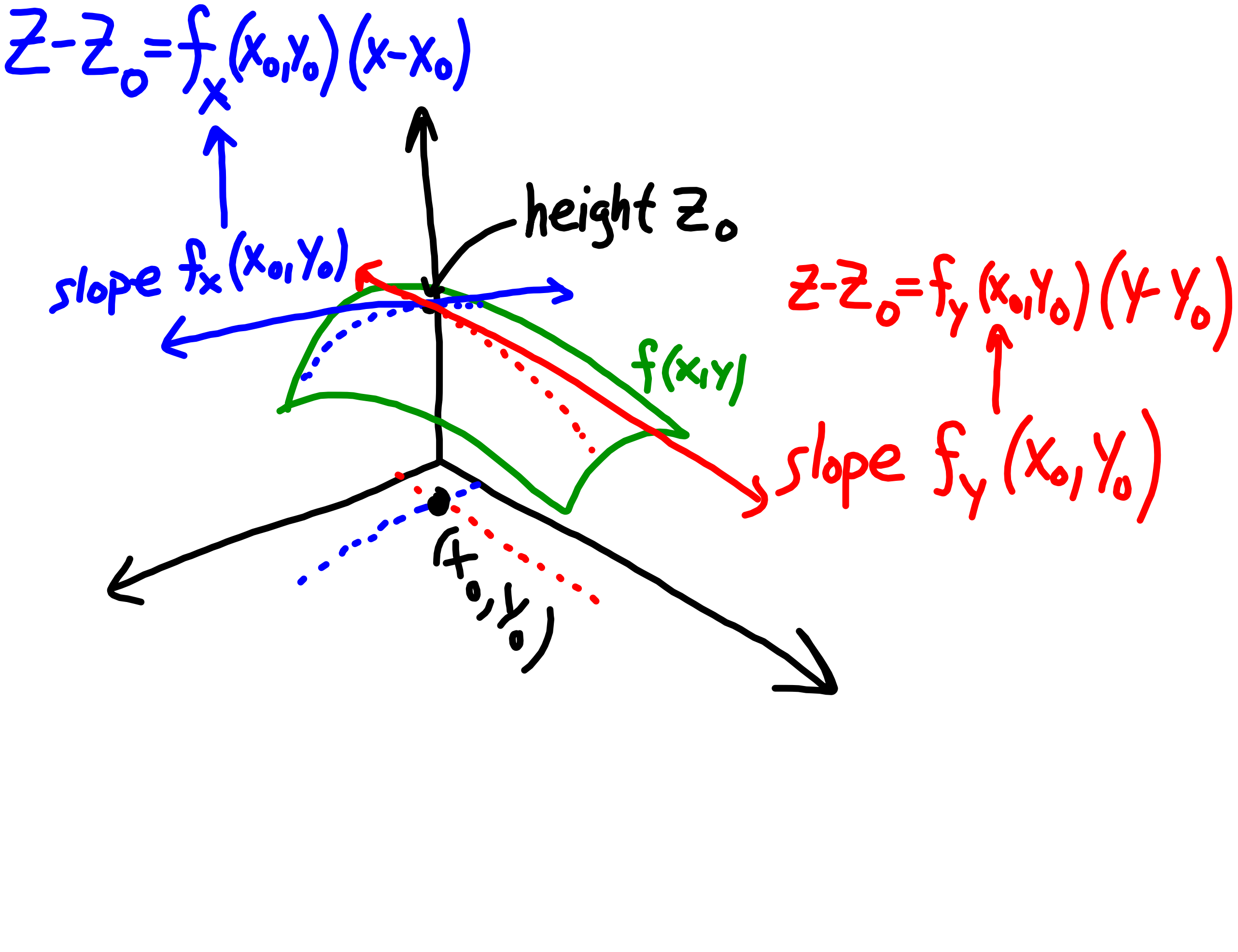
Let's recall tangent lines in 1D:



$$y - y_0 = m(x - x_0)$$

$$= f'(x_0)(x - x_0).$$

Now recall the tangent lines on a surface. The plane containing these is the tangent plane.



Equation for the tangent plane

Recall the plane equation of the form $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$$\Rightarrow c(z-z_0)=-a(x-x_0)-b(y-y_0)$$

$$\Rightarrow z-z_0=\frac{-a}{c}(x-x_0)-\frac{b}{c}(y-y_0)$$

$$\Rightarrow z-z_0=A(x-x_0)+B(y-y_0).$$

$y = y_0$ fixed $\Rightarrow z - z_0 = A(x - x_0)$

$$= \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0)$$

$x = x_0$ fixed $\Rightarrow z - z_0 = B(y - y_0)$

$$= \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$\Rightarrow \boxed{z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).}$$

TANGENT PLANE

<u>EX</u>: Let $f(x,y)=\ln(x^2+y^2)$... find the tangent plane equation at $(1,1,\ln(2))$.

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} = \left.\left(\frac{2x}{x^2+y^2}\right)\right|_{(1,1)} = \frac{2}{2} = 1.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \left.\left(\frac{2y}{x^2+y^2}\right)\right|_{(1,1)} = 1$$

$$z-\ln(2)=(x-1)+(y-1)=x+y-2.$$

EX: Let $f(x,y)=\sqrt{x^2-y^2+1}$. Find the equation of the tangent plane at the point $(\sqrt{8},0,3)$.

$$f_x(\sqrt{8},0)=\frac{2x}{2\sqrt{x^2-y^2+1}}\Bigg|_{(\sqrt{8},0)}=\frac{\sqrt{8}}{\sqrt{9}}=\frac{1}{3}\sqrt{8}.$$

$$f_y(\sqrt{8},0)=\frac{-2y}{2\sqrt{x^2-y^2+1}}\Bigg|_{(\sqrt{8},0)}=0.$$

$$\Rightarrow \boxed{z-3=\frac{\sqrt{8}}{3}\left(x-\sqrt{8}\right)}$$

Differentials (particularly useful in engineering)

$$\underbrace{z - z_0}_{\Delta z} = (f_x)\underbrace{(x - x_0)}_{\Delta x} + (f_y)\underbrace{(y - y_0)}_{\Delta y}$$

$$\Delta z = \frac{\partial f}{\partial x} \cdot \Delta x + \frac{\partial f}{\partial y} \cdot \Delta y \approx \Delta f$$

For small perturbations of design variables we may estimate the effect on the output in this way.

EX: A cylinder has volume $V = \pi r^2 h$

r: radius h: height

Consider machining a cylinder (solid metal) and can control r to $\pm 0.1\%$ and $h \pm 0.01\%$. Put limits on how big/small the cylinder could turn out.

$$\Delta V \cong V_r \Delta r + V_h \Delta h = 2\pi r h \Delta r + \pi r^2 \Delta h.$$

$$\Delta r = 0.001 r \quad \Delta h = 0.0001 \cdot h$$

MAX V:

$$V_{MAX} \approx \pi r^2 h + \left(0.001 \cdot 2\pi r^2 h + 0.0001 \cdot \pi r^2 h\right)$$

$$\approx V + \Delta V$$

$$V_{MIN} \approx \pi r^2 h - \left(0.001 \cdot 2\pi r^2 h + 0.0001 \pi r^2 h\right).$$

$$\approx V - \Delta V$$

Practice!

#1 Find all 2^{nd}-order partial derivatives of $f(x,y) = \sin(x^2y^2)$.

$$f_x = \cos(x^2y^2)\frac{\partial}{\partial x}(x^2y^2)$$

$$= 2xy^2\cos(x^2y^2)$$

$$f_y = 2yx^2\cos(x^2y^2)$$

$$f_{xx} = \underbrace{2y^2\cos(x^2y^2) - 2xy^2 \cdot \underbrace{2xy^2\sin(x^2y^2)}_{\text{Chain Rule}}}_{\text{Product rule}}$$

So $f_{xx} = 2y^2\cos(x^2y^2) - 4x^2y^4\sin(x^2y^2)$

$$f_{xy} = f_{yx} = \frac{\partial}{\partial y}\left\{2xy^2\cos(x^2y^2)\right\}$$

$$= 4xy\cos(x^2y^2) - 2xy^2 \cdot 2x^2y\sin(x^2y^2)$$

$$= 4xy\cos(x^2y^2) - 4x^3y^3\sin(x^2y^2).$$

$f_{yy} = 2x^2\cos(x^2y^2) - 4y^2x^4\sin(x^2y^2)$

just switch x & y in f_{xx} formula because for this example $f = \sin(x^2y^2)$ is symmetric with respect to x & y.

#2 Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(1,1,-1)$ if

$z=z(x,y)$ satisfies

check: these should solve the equation

$$xy+xz^2+y^2z=1.$$

$$\frac{\partial}{\partial x}\left(xy+xz^2+y^2z\right)=\frac{\partial}{\partial x}(1)=0$$

$$y+z^2+2xz\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial x}=0$$

Easiest to plug in $x=y=1$ & $z=-1$ now...

$$1+1-2\frac{\partial z}{\partial x}+\frac{\partial z}{\partial x}=0$$

solve now for $\frac{\partial z}{\partial x}$...

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = 1+1 = 2.}$$

Now for $\partial z/\partial y$,

$$\frac{\partial}{\partial y}\left(xy + xz^2 + y^2 z\right) = \frac{\partial}{\partial y}(1) = 0$$

$$x + 2xz\frac{\partial z}{\partial y} + 2yz + y^2\frac{\partial z}{\partial y} = 0$$

$$\left.\begin{matrix} x=y=1 \\ z=-1 \end{matrix}\right\} \Rightarrow 1 - 2\frac{\partial z}{\partial y} - 2 + \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow -1 - \frac{\partial z}{\partial y} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial y} = -1.}$$

#3 Find the tangent plane to $f(x,y) = x^2 + xy + y^3$ when $x=2$, $y=-1$.

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$x_0 = 2$ $\qquad$ $y_0 = -1$

$z_0 = f(x_0,y_0) = 4 - 2 - 1 = 1$

$f_x = 2x + y \Rightarrow f_x(2,-1) = 4 - 1 = 3$

$f_y = x + 3y^2 \Rightarrow f_y(2,-1) = 2 + 3 = 5$

$$\boxed{z - 1 = 3(x-2) + 5(y+1)}$$

#4 Consider a box of length L, width W that is 4 ft. high. You want to construct the box with L=2 ft and W=1 ft, but are only able to measure these to within 1/8 inch precision. Estimate the largest volume the box could potentially have by applying differentials.

So $V = 4 \cdot L \cdot W \Rightarrow \Delta V \approx \frac{\partial V}{\partial L}\Delta L + \frac{\partial V}{\partial W}\Delta W$

$\Rightarrow \Delta V \approx 4W\Delta L + 4L\Delta W$ $\left\{ \begin{array}{l} W = 1 \text{ ft.} \\ L = 2 \text{ ft.} \\ \text{use } \Delta L = \Delta W = \frac{1}{96} \text{ ft.} \end{array} \right.$

Then $\Delta V_{MAX} \approx 4 \cdot 1 \cdot \frac{1}{96} + 4 \cdot 2 \cdot \frac{1}{96}$

$$= \frac{12}{96} = \frac{1}{8} \quad (ft.^3)$$

$$\Rightarrow V_{MAX} = V + \Delta V_{MAX} = 4 \cdot 2 \cdot 1 + \frac{1}{8}$$

$$= 8 + \frac{1}{8} = \frac{65}{8} \ ft^3.$$