Functions of multiple variables

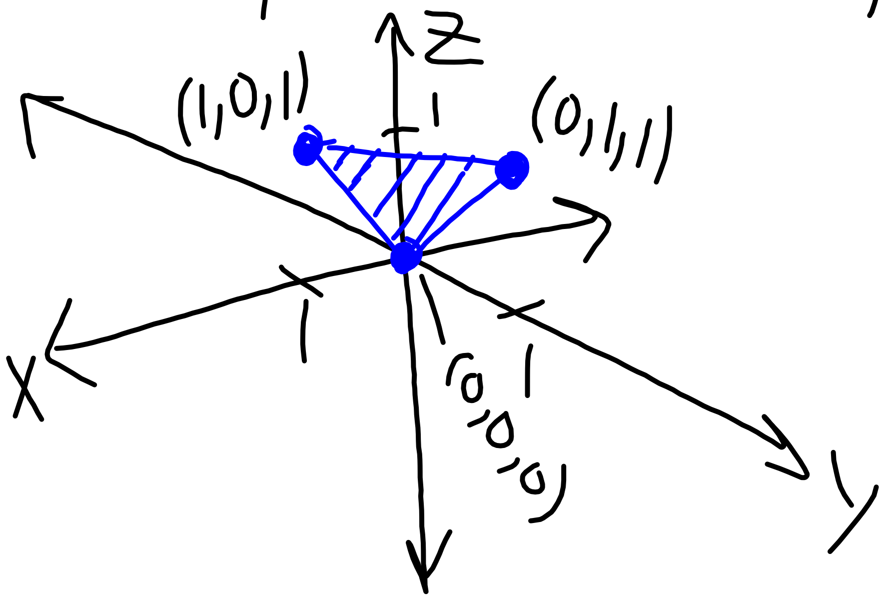
$$f = f(x_1, x_2, x_3, \ldots, x_n)$$

with $f(\cdot)$ a real number

We say that f is a real, scalar-valued function of n variables;

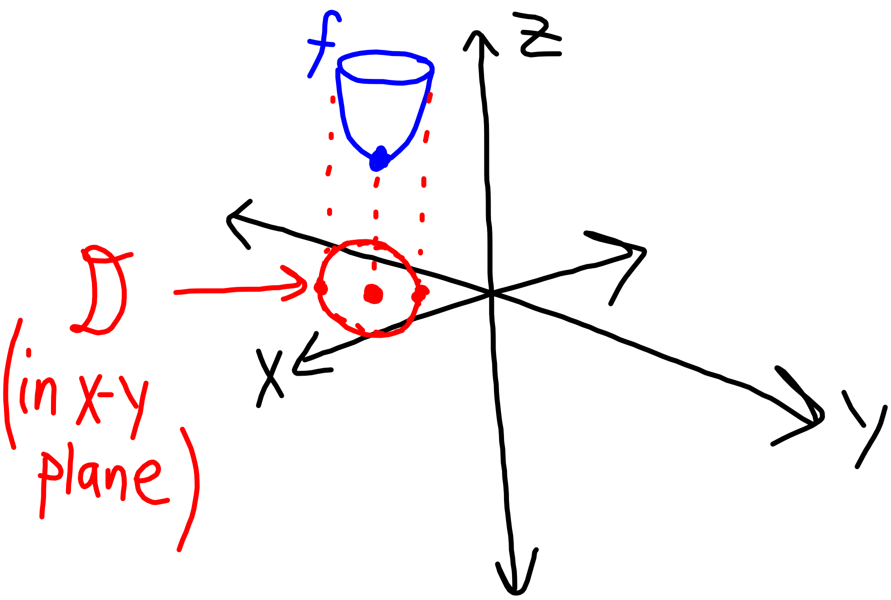
$$f : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

Consider $f(x,y) = x + y$; $f: \mathbb{R}^2 \to \mathbb{R}$.
We visualize sometimes via a "surface plot"... set $z = f(x,y)$.



$$f : D \subset \mathbb{R}^n \to \mathbb{R}$$

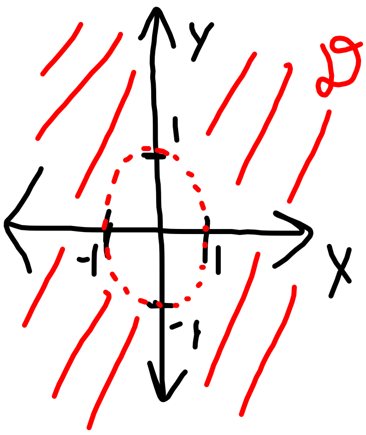
we often need to restrict the domain



Usually, the domain may be determined by inspection.

$$f(x,y) = \ln\underbrace{(x^2+y^2-1)}_{>0}$$

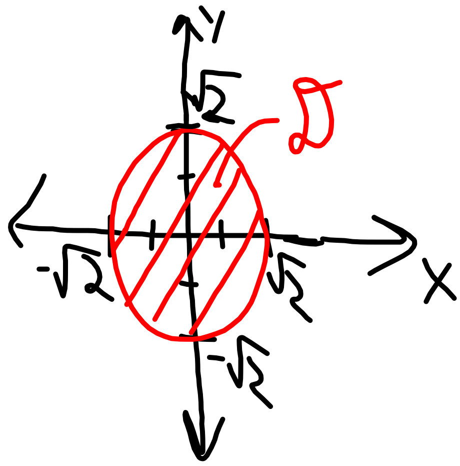
$$D = \{(x,y) \mid x^2+y^2 > 1\}$$



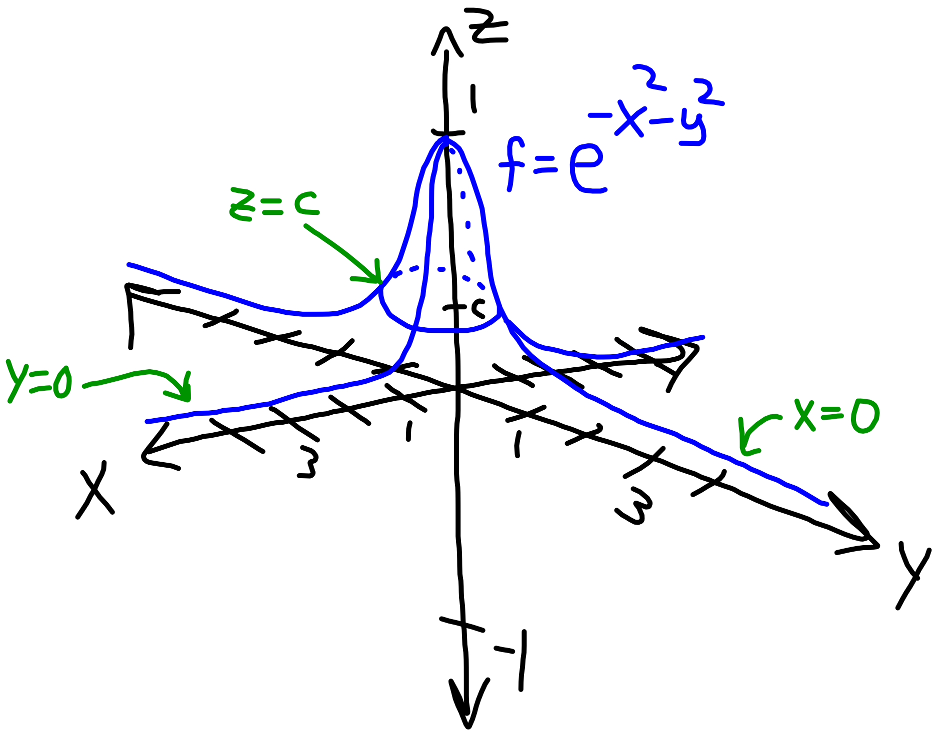
EX: Find the domain of

$$f(x,y)=\sqrt{2-x^2-y^2}.$$

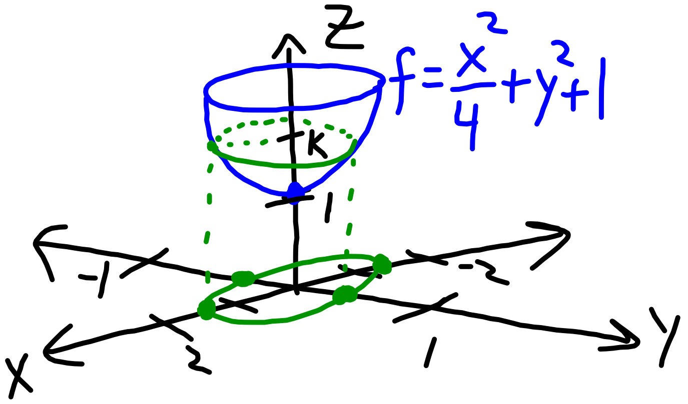
Then $2-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 2$



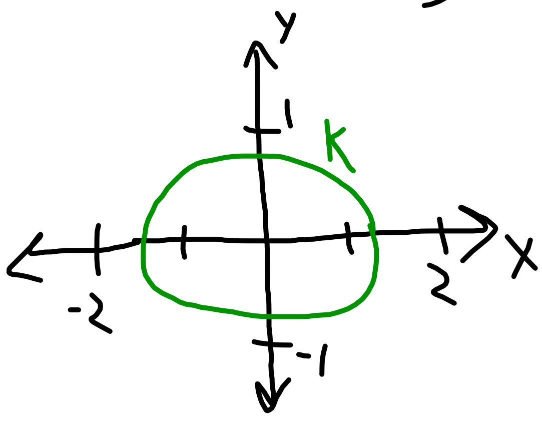
We have seen before that traces may be used to help sketch surface plots.



Level curves are another visualization tool.

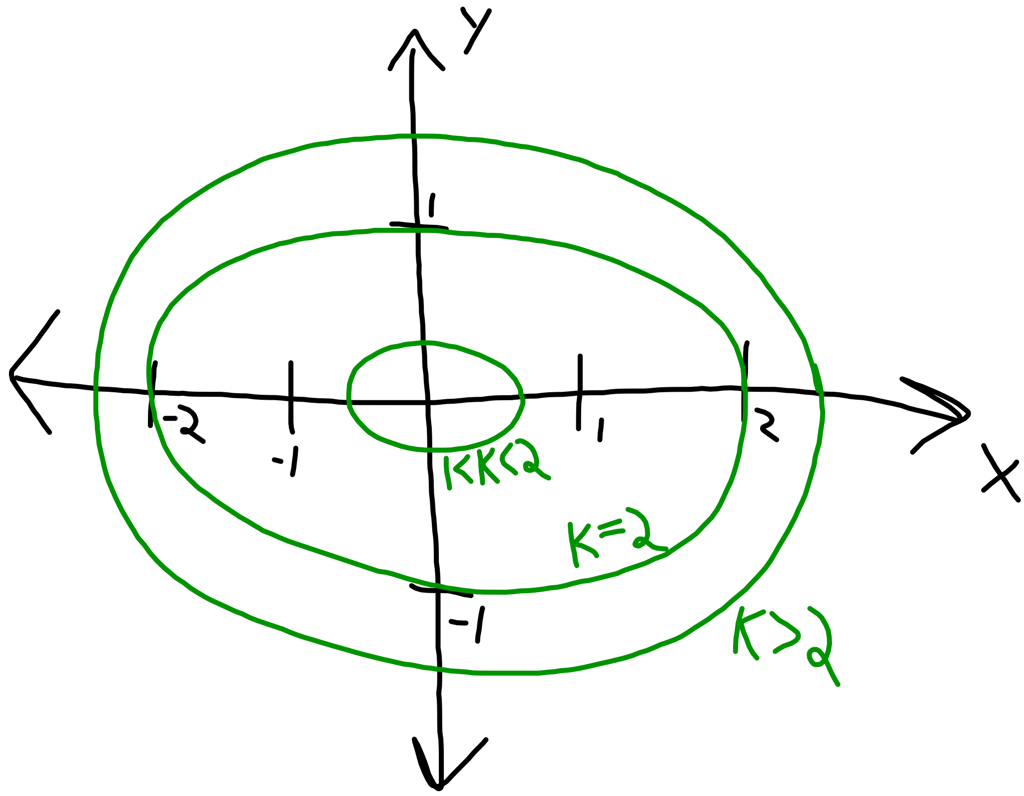


Project traces $z = k$ down into x-y plane.



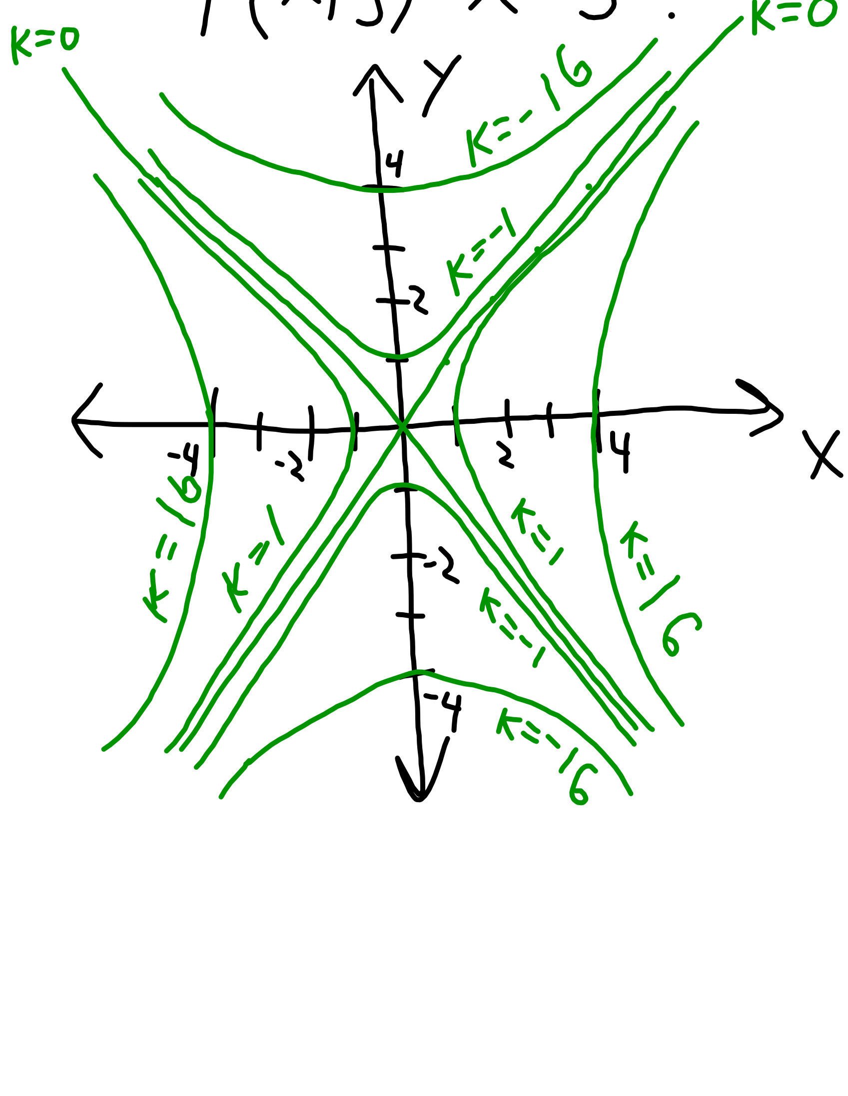
Visualize just in the x-y plane

Now add in more k-values...



Ex: Sketch a contour plot for

$$f(x,y) = x^2 - y^2.$$



Level surfaces

Consider $f(x,y,z)=k$

e.g. $f(x,y,z)=\underbrace{x^2+y^2+z^2=k}_{\text{spheres of radius } \sqrt{k}} > 0$

One could try to visualize these together,

but this is difficult (how do you see one sphere

within the next?) so we will not focus on level surfaces.

LIMITS If $f(x,y) \to L$ as $(x,y) \to (a,b)$ the we say

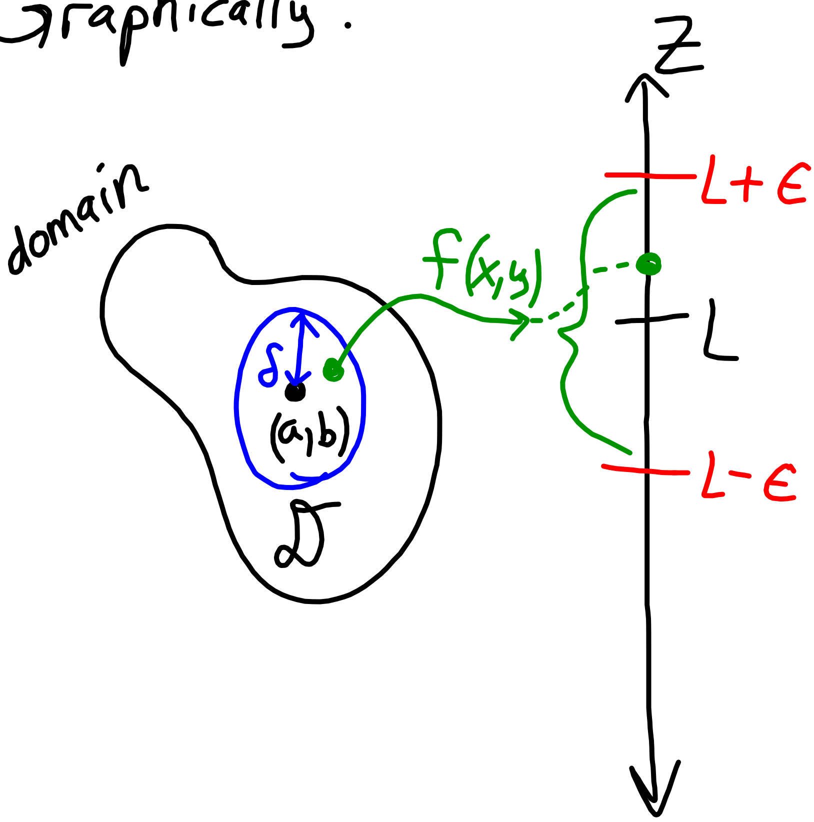
$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

More precisely, if for ANY $\epsilon > 0$ there is some $\delta = \delta(\epsilon) > 0$ such that (x,y) in $\mathcal{D}$,

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$$

then the limit exists.

Graphically:

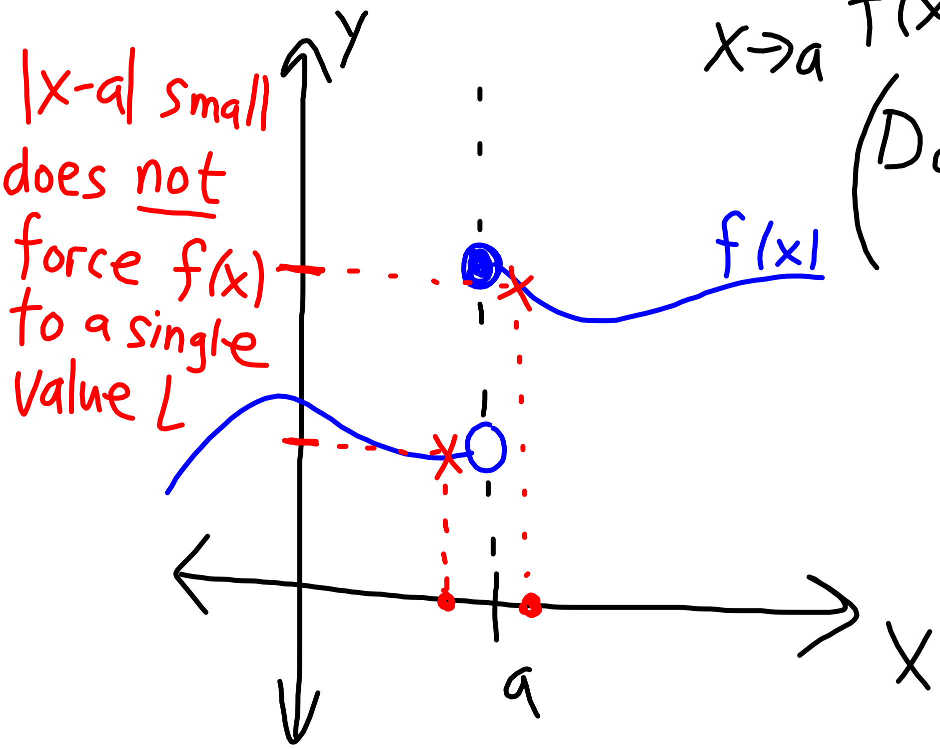


- If we choose $\epsilon > 0$ very small there must be a neighborhood of (a,b) that f maps into $(L-\epsilon, L+\epsilon)$.

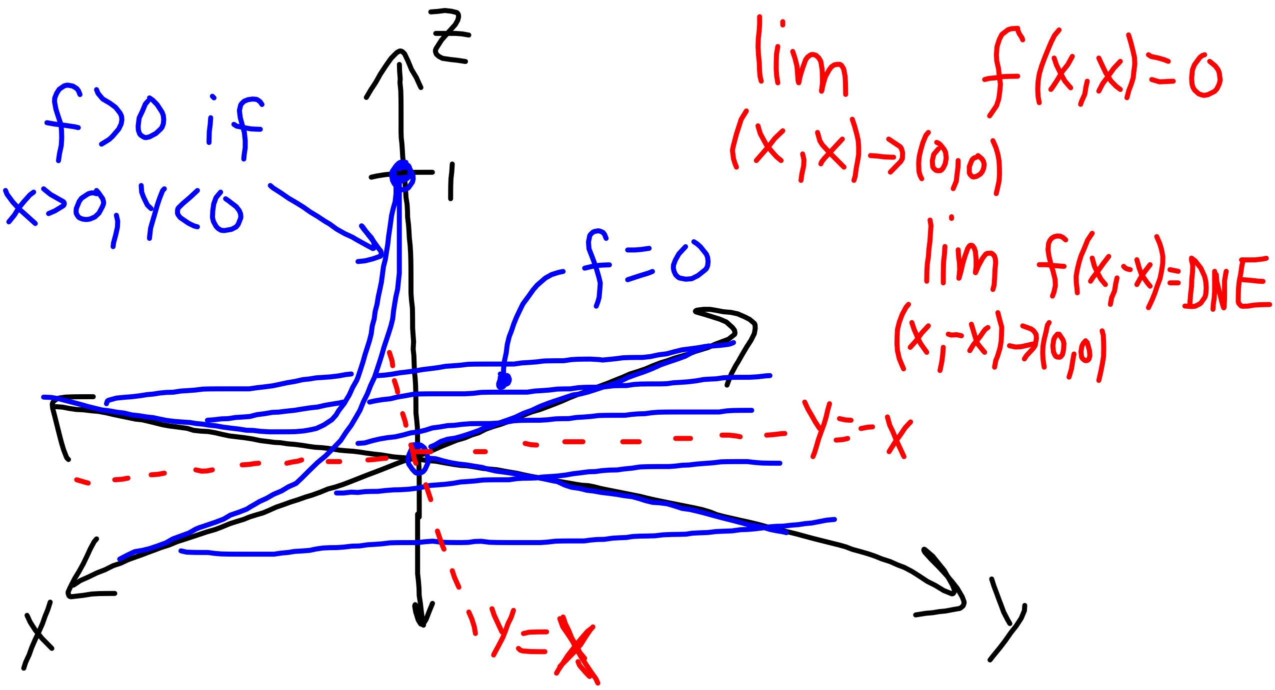
1D reminder:

$\lim_{x \to a} f(x) = ?$

(Does not exist)



In 2D (or higher) the limit may only exist along specific paths.



EX: Consider $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$.

Note $(0,0)$ is NOT in the domain, but we can still talk about $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$.

(Same as $\lim\limits_{r\to 0^+} \dfrac{\sin(r)}{r} = 1$)

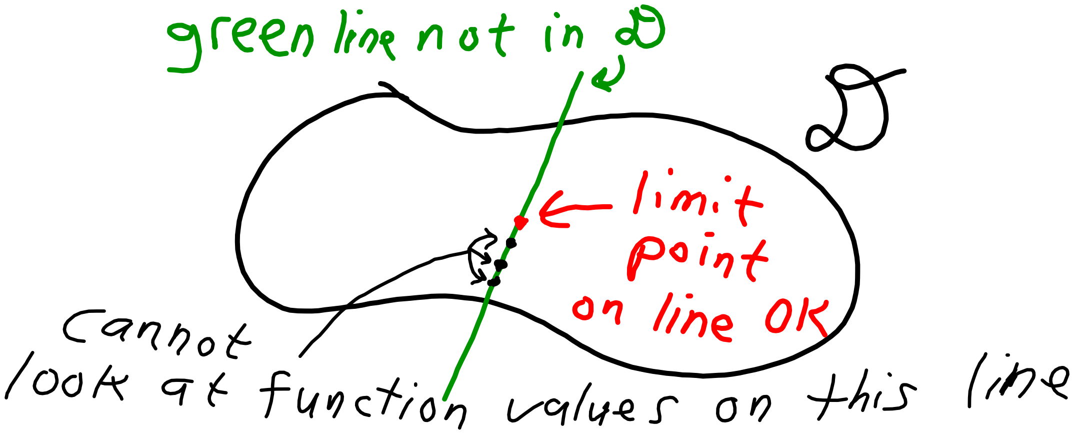
Ex: Find $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ or write DNE if the limit does not exist.

The way you usually show DNE is to find two paths that yield different limits.

1) Set $y=0\ldots$ $\lim_{\substack{(x,0)\to(0,0) \\ (x\neq 0)}} \frac{x\cdot 0}{x^2} = 0$

2) Set $y=-x\ldots$ $\lim_{\substack{(x,-x)\to(0,0) \\ (x\neq 0)}} \frac{x(-x)}{2x^2} = -\frac{1}{2} \neq 0$

Subtle but important point: ONLY look on the domain to evaluate limiting behavior!



Ex: $f(x,y) = \frac{1}{x+y}$, $\lim_{(x,y)\to(0,0)} f(x,y) = ?$

Note $\mathcal{D} = \{(x,y) \mid x+y \neq 0\}$ so we can't look at f on line $y=-x$.

But $x=0, y\neq 0$ is OK...

$f(x=0,y) = \frac{1}{y}$, $\lim_{y\to 0} \frac{1}{y} = \underline{\underline{DNE}}$.

To show that limits exist we will usually cheat a bit and cite CONTINUITY.

When $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ $\Big\}$ • limit exists • equals $f(a,b)$

we say f is continuous (CTS) at (a,b).

* If f is CTS. at every point of a set A

we say "f is CTS on A".

Polynomials are CTS everywhere and rational functions are CTS on their domains.

Ex: Find $\lim\limits_{(x,y)\to(1,1)} \frac{x^2-y^2}{x^2+y^2}$ or write DNE if the limit does not exist.

Since $(1,1)$ is in the domain, we note $\frac{x^2-y^2}{x^2+y^2}$ is rational, hence $\lim\limits_{(x,y)\to(1,1)} \frac{x^2-y^2}{x^2+y^2} = \frac{1^2-1^2}{1^2+1^2} = 0.$

<u>EX</u>: Define $g(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

Show $\lim\limits_{(x,y)\to(0,0)} g(x,y)=$ DNE.

Useful short-cut: try $y=kx$

$$\lim_{(x,kx)\to(0,0)} g(x,kx)=\lim_{x\to 0}\frac{x^2-k^2x^2}{x^2+k^2x^2}=\lim_{x\to 0}\frac{1-k^2}{1+k^2}=\frac{1-k^2}{1+k^2}.$$

The limit is k-dependent $\Rightarrow$ DNE.

EX: $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = \infty$; we can conceivably discuss such limits like in Calc I.

EX: Is $f(x,y) = \begin{cases} 1, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ CTS at $(0,0)$?

No; $\lim_{(x,y)\to(0,0)} f(x,y) = 1 \neq f(0,0) = 0$.

Guidelines to identify continuous functions so you know when certain limits automatically exist

(1) Sums, differences and products of

CTS functions are CTS (check domain)

(2) Ratios of CTS functions (check domain)

(3) Compositions of CTS functions are CTS**

** If $g(a) = b = \lim_{x \to a} g(x)$ & $\lim_{x \to b} f(x) = f(b)$

then $\lim_{x \to a} f(g(x)) = f(b) = f(g(a))$.

EX: Find the limit or write DNE...

$$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = ?$$

$\ln(z)$ is CTS at $z=1$ and $\frac{1+y^2}{x^2+xy}$ is rational / CTS at $(1,0)$ so just plug in $(x,y)=(1,0)$ ... $\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = \ln(1) = 0.$

Ex: $\lim_{(x,y)\to(1,-2)} \underbrace{e^{-x^2y}\,\underbrace{\sin(\underline{x+y^2})}_{\text{CTS}}}_{\text{CTS}} = ?$

CTS

$$= e^{-(1)^2(-2)} \sin(1+(-2)^2) = \boxed{e^2 \sin(5).}$$

Ex: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = ?$

division by zero = red flag...

The limit turns out to be zero, but it takes thought. Note the numerator is effectively "higher-order" than the denominator, so it vanishes faster as $(x,y)\to(0,0)$.

This requires an "ϵ-δ" proof... we will skip.

Ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} = ?$

(0,0) is NOT in the domain... need to investigate.

Along paths like $y=kx$ the limit $=0$.

However, y^8 is a red flag... numerator is 5^{th} order. Take $y=x^{1/4}$...

$$\lim_{(x,x^{1/4})\to(0,0)} \frac{x\cdot x}{x^2+x^2} = \frac{1}{2} \neq 0.$$

$\boxed{\text{DNE}}$

Practice!

#1

(a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{y^2\sin^2 x}{x^4+y^4} = ?$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4} = ?$

$$(a) = \frac{(1)^2\sin^2(1)}{1^4+1^4} = \frac{1}{2}\sin^2(1). \quad (CTS)$$

$$(b) = \lim_{(x,y)\to(0,0)} \frac{y^2x^2}{x^4+y^4}$$

$y=kx$? $\dfrac{k^2\cancel{x^4}}{\cancel{x^4}(1+k^4)}$

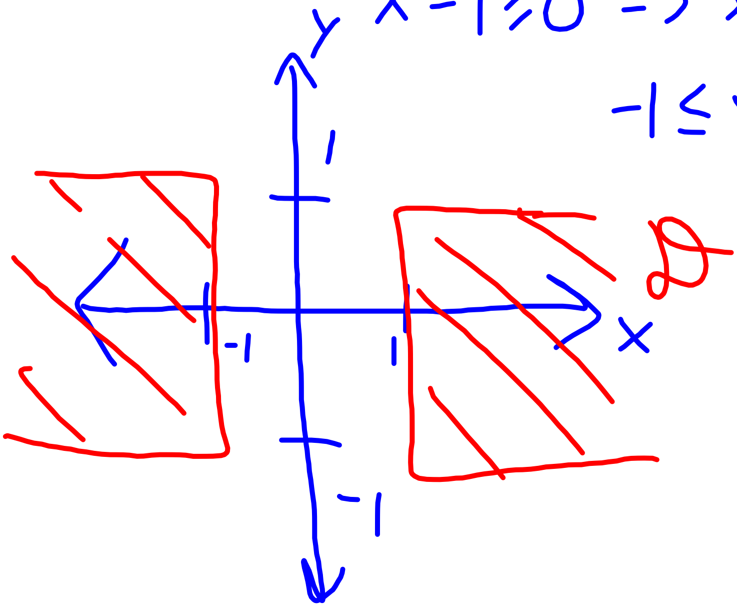
DNE

#2 Sketch the domain of

$$f(x,y)=\sqrt{x^2-1}-\sqrt{1-y^2}$$

$x^2-1\geq 0 \Rightarrow x\leq -1$ or $x\geq 1$

$-1\leq y\leq 1$

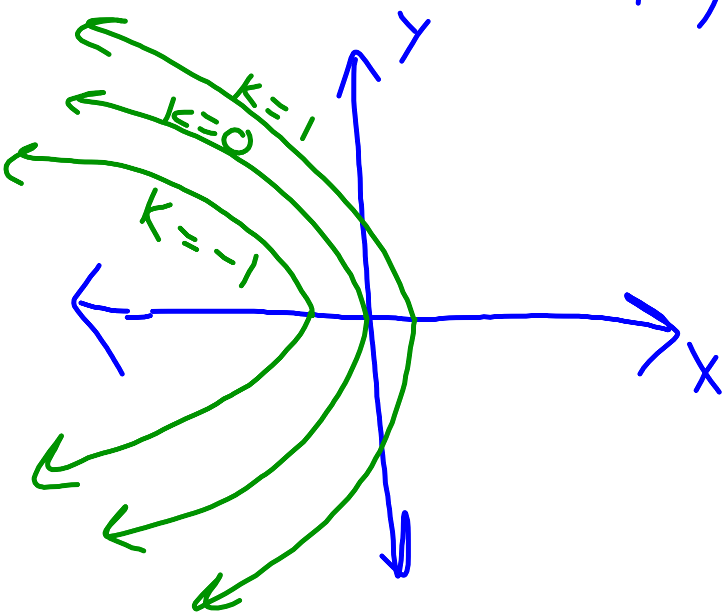


(#3) Sketch level sets of $f(x,y)=x+y^2$.

$k=0 \Rightarrow x=-y^2$

$k=1 \Rightarrow x=1-y^2$

$k=-1 \Rightarrow x=-1-y^2$



#4 $\lim_{(x,y)\to(0,0)} \frac{x+\sin(y)}{x+y+1} = \frac{0+0}{0+0+1} = \frac{0}{1} = 0$ (CTS)

#5 $\lim_{(x,y)\to(1,0)} \ln\left(\frac{2x^2+y^2}{x^2-2y^2}\right) = \ln\left(\frac{2}{1}\right) = \ln(2)$

(CTS)

#6 $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{2x^2+y^2} = $ DNE

$$\frac{x^2+k^2x^2}{2x^2+k^2x^2} = \frac{1+k^2}{2+k^2}$$