

Curvature: how quickly does a curve change direction?

$$K = \left| \frac{d\vec{T}}{ds} \right| \text{ is curvature.}$$

$$\vec{T} = \vec{T}(s) \left. \begin{array}{l} \uparrow \text{arc length} \\ \end{array} \right\} \text{ independent of choice of parameterization}$$

$$\left| \frac{d\vec{T}}{ds} \right| = ??? \dots \text{ apply Chain Rule}$$

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} \Rightarrow \left| \frac{d\vec{T}}{ds} \right| = \frac{|d\vec{T}/dt|}{|ds/dt|} \left. \begin{array}{l} \text{Easier} \\ \text{to} \\ \text{find.} \end{array} \right\}$$

Recall $\frac{ds}{dt} = |\vec{r}'(t)|$

$$\Rightarrow \left| \frac{d\vec{T}}{ds} \right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

OK, but we still need $\frac{d}{dt} \left(\frac{|\vec{r}'(t)|}{|\vec{r}'(t)|} \right)$

can be tough

It turns out there is another formula:

$$K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

Ex. 1: Find curvature of $\vec{r}(t) = \langle 1+t, 2t, t^2 \rangle$.

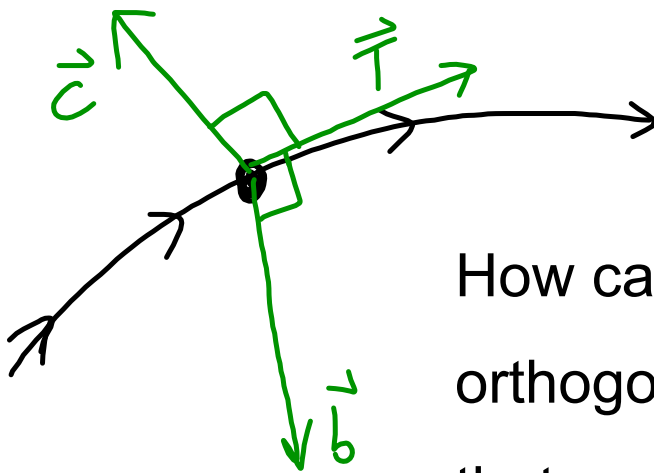
You could try finding $\vec{T}'(t)$... it will get ugly fast. Quicker to use the other formula: $K = |\vec{r}' \times \vec{r}''| / |\vec{r}'|^3$.

$$\vec{r}' = \langle 1, 2, 2t \rangle \dots |\vec{r}'| = \sqrt{5+4t^2}$$

$$\vec{r}'' = \langle 0, 0, 2 \rangle$$

$$\vec{r}' \times \vec{r}'' = \langle 4, -2, 0 \rangle \Rightarrow |\vec{r}' \times \vec{r}''| = \sqrt{16+4} = 2\sqrt{5}$$
$$\Rightarrow K = \frac{2\sqrt{5}}{(5+4t^2)^{3/2}}$$

The normal and binormal vectors



How can we find three orthogonal unit vectors that move with a particle through space?

\vec{T} points in the direction of motion, so that will be the first vector. Note

$$\text{that } |\vec{T}| = 1 \Rightarrow |\vec{T}|^2 = \vec{T} \cdot \vec{T} = 1$$

$$\Rightarrow \frac{d}{dt}(\vec{T} \cdot \vec{T}) = \frac{d}{dt}(1) = 0$$

$$\Rightarrow 2\vec{T} \cdot \vec{T}' = 0$$

$$\Rightarrow \vec{T} \perp \vec{T}'$$

so normalize
this...
 $\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$

$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$: NORMAL vector
for a curve (not a plane).

Bi-normal : $\vec{B} = \vec{T} \times \vec{N}$

$$|\vec{B}| = |\vec{T}| |\vec{N}| \sin(90^\circ) = 1 \cdot 1 \cdot 1 = 1.$$

$\{\vec{T}, \vec{N}, \vec{B}\}$: the "TNB frame".

Ex. 2: Find the TNB frame for
 $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ @ $t = \pi/2$.

$$\left. \begin{array}{l} \vec{r}' = \langle -\sin(t), \cos(t), 1 \rangle \\ |\vec{r}'| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2} \end{array} \right\} \vec{T} = \frac{1}{\sqrt{2}} \vec{r}'$$

$$\vec{T}'(t) = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle \quad \left. \right\} |\vec{T}'| = \frac{1}{\sqrt{2}}$$

$$\text{So } \vec{N}(t) = \langle -\cos(t), -\sin(t), 0 \rangle.$$

Plug in $t = \pi/2 \dots$

$$\vec{T} = \frac{1}{\sqrt{2}} \langle -\sin(\frac{\pi}{2}), \cos(\frac{\pi}{2}), 1 \rangle = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle.$$

$$\vec{N} = \langle -\cos(\frac{\pi}{2}), -\sin(\frac{\pi}{2}), 0 \rangle = \langle 0, -1, 0 \rangle$$

$$\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle.$$

SUMMARY

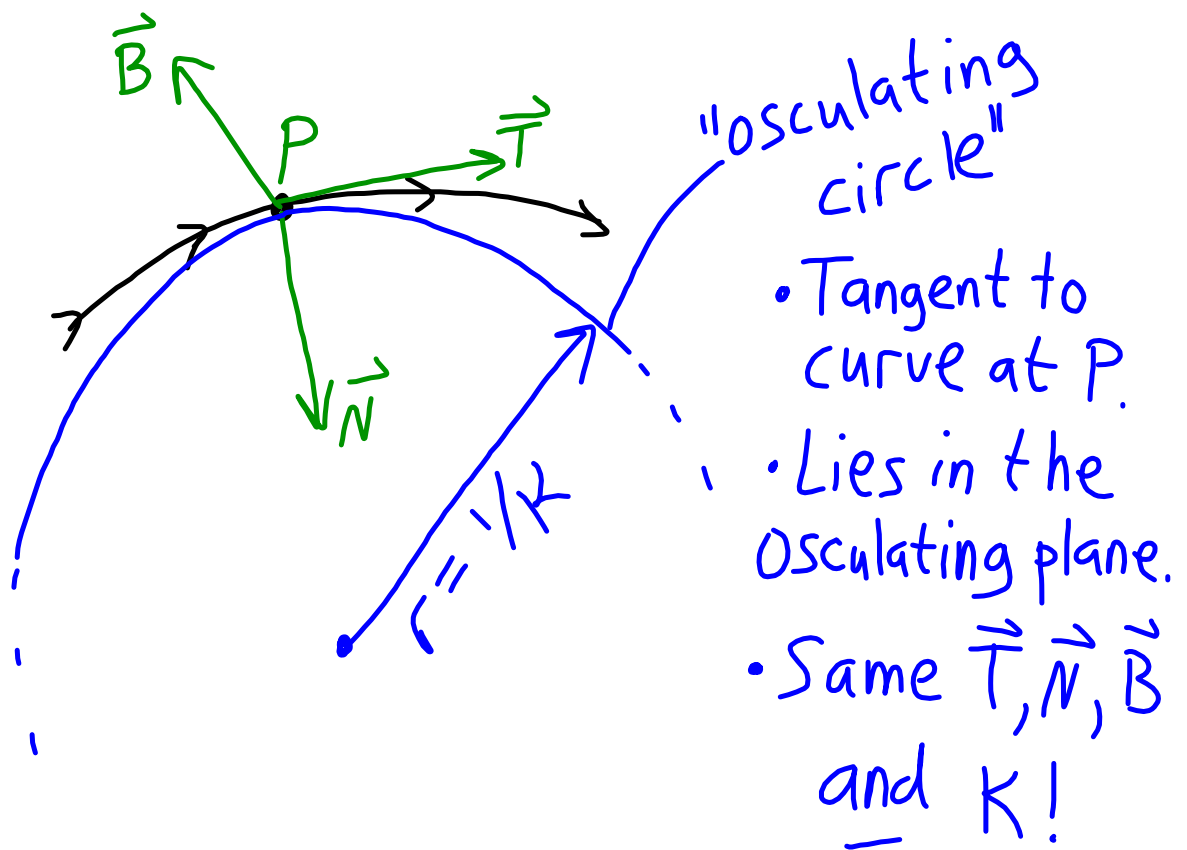
$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|}$$

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$$

$$K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}.$$

$$\vec{B} = \vec{T} \times \vec{N}$$

The osculating plane - contains \vec{T} & \vec{N}



Velocity and acceleration

$\vec{r}(t)$: position at time t .

$\vec{v}(t) = \vec{r}'(t)$: velocity at time t .

$$\vec{v} = |\vec{v}| \hat{v}$$

speed \uparrow \uparrow direction

$\vec{a}(t) = \vec{v}'(t)$
 $= \vec{r}''(t)$: acceleration

Given acceleration, we can also find velocity and position by integrating...

$$\vec{v} = \int \vec{a}(t) dt$$

$$\vec{r} = \int \vec{v}(t) dt$$

We require unknown constants to be added at each step; their values are often determined later from "initial conditions".

Ex. 3: Let $\vec{a} = \langle 100, 100, -9.8 \rangle$ (m/s^2)

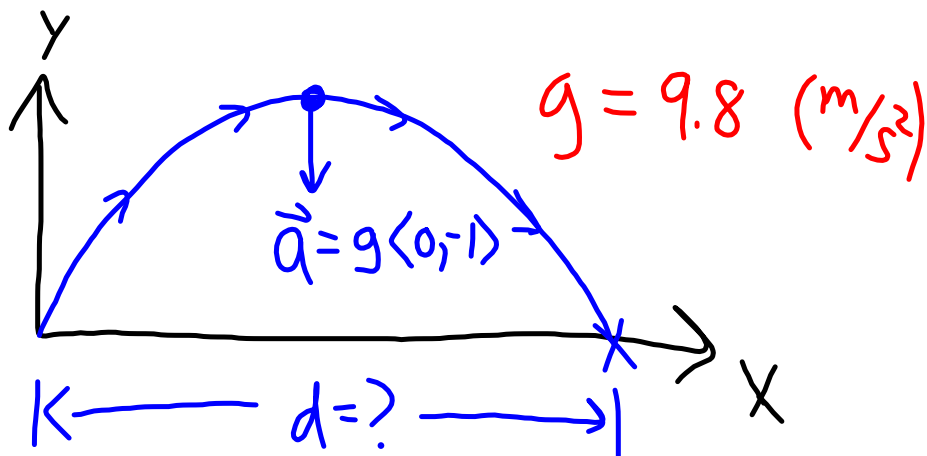
Find position at time t .

$$\vec{v}(t) = \int \vec{a} dt = t \langle 100, 100, -9.8 \rangle + \vec{C}_1$$

$$\vec{r}(t) = \int \vec{v}(t) dt = \frac{1}{2} t^2 \langle 100, 100, -9.8 \rangle + \vec{C}_1 t$$

*Note $\vec{r}(0) = \vec{C}_2$ & $\vec{v}(0) = \vec{C}_1$. $+ \vec{C}_2$.

Ex. 4: A projectile is fired from a cannon at a 45 degree angle across flat terrain. The muzzle velocity is 300 m/s. Neglecting air friction, find the distance away that the projectile lands.



$$\begin{aligned}\vec{v}(t) &= t \langle 0, -9.8 \rangle + \vec{v}(0) \\ &= t \langle 0, -9.8 \rangle + \frac{300}{\sqrt{2}} \langle 1, 1 \rangle.\end{aligned}$$

$$\vec{r}(t) = \int \vec{v} dt = \frac{1}{2} t^2 \langle 0, -9.8 \rangle + 150\sqrt{2} t \langle 1, 1 \rangle$$

Look for y-component of $\vec{r}(t)$ ~~+ $\vec{r}(0)$~~ $\rightarrow \vec{0}$

$\vec{r}(t)$ to be zero (height of projectile):

$$-4.9t^2 + 150\sqrt{2}t = 0$$

$$t(150\sqrt{2} - 4.9t) = 0$$

$$t = \frac{150\sqrt{2}}{4.9} \text{ (seconds)}$$

$$\text{Distance} = x \left(\frac{150\sqrt{2}}{4.9} \right)$$

$$= \frac{(150\sqrt{2})^2}{4.9} \text{ (meters)}$$

$$\approx 9.18 \text{ km}$$

Newton's Second Law

$$\vec{F} = m \vec{a}$$

force *mass*

Often we just want the acceleration:

$$\vec{a} = \frac{1}{m} \vec{F}$$

Ex. 5: Consider a particle of mass $m=11,110$ kg orbiting the origin according to

$$\vec{r}(t) = |\vec{r}| \langle \cos(\omega t), \sin(\omega t) \rangle$$

$$|\vec{r}| = 6.9113 \cdot 10^6 \text{ m}$$

$$\omega = \frac{2\pi}{5790} \text{ s}^{-1}$$

Find the force acting on the particle.

(represents the Hubble Space Telescope)

$\vec{F} = m\vec{a} \dots$ need \vec{a} .

$$\vec{r} = |\vec{r}| \langle \cos(\omega t), \sin(\omega t) \rangle$$

$$\vec{v} = \vec{r}' = |\vec{r}| \omega \langle -\sin(\omega t), \cos(\omega t) \rangle.$$

$$\vec{a} = \vec{v}' = -|\vec{r}| \omega^2 \langle \cos(\omega t), \sin(\omega t) \rangle.$$

$$\vec{F} = m\vec{a} \Rightarrow |\vec{F}| = |\vec{r}| m \omega^2 \quad \text{unit vector}$$

$$\approx 90.4 \text{ kN.}$$

Practice!

(#1) Find K for $\vec{r} = \langle t, 2t, e^t \rangle$.

$$\begin{aligned}\vec{r}' &= \langle 1, 2, e^t \rangle & |\vec{r}'| &= \sqrt{5 + e^{2t}} \\ \vec{r}' \times \vec{r}' &= \langle 0, 0, e^t \rangle \\ \|\vec{r}' \times \vec{r}'\| &= \|\langle 2e^t, -e^t, 0 \rangle\| = \sqrt{4e^{2t} + e^{2t}} \\ K &= \frac{e^t \sqrt{5}}{(5 + e^{2t})^{3/2}}.\end{aligned}$$

(#2) Find the equation of the osculating plane through $(-\sqrt{3}, 0, 2\pi+4)$ for the curve $\vec{r}(t) = \langle \sqrt{3} \cos(t), \sqrt{3} \sin(t), 2t+4 \rangle$.

$$\vec{r}' = \langle -\sqrt{3} \sin(t), \sqrt{3} \cos(t), 2 \rangle$$

$$|\vec{r}'| = \sqrt{3(\sin^2(t) + \cos^2(t)) + 4} = \sqrt{7}$$

$$\vec{T} = \frac{1}{\sqrt{7}} \langle -\sqrt{3} \sin(t), \sqrt{3} \cos(t), 2 \rangle.$$

$$\vec{T}' = \frac{1}{\sqrt{7}} \langle -\sqrt{3} \cos(t), -\sqrt{3} \sin(t), 0 \rangle.$$

$$\vec{N} = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$\Rightarrow \vec{T} = \left\langle 0, -\frac{\sqrt{3}}{7}, \frac{2}{\sqrt{7}} \right\rangle$$

$$\vec{N} = \langle 1, 0, 0 \rangle$$

$$\vec{B} = \left\langle 0, \frac{2}{\sqrt{7}}, \frac{\sqrt{3}}{7} \right\rangle$$

$$\vec{B} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

$$\Rightarrow \frac{2y}{\sqrt{7}} + \sqrt{\frac{3}{7}} (z - (2\pi + 4)) = 0$$

#3 Let a particle of mass 10 kg be acted upon by force

$$\vec{F} = \langle -10, 10, 20 \rangle \text{ (newtons).}$$

If initial velocity is $\langle 1, 0, 0 \rangle$ and it starts at the origin, find $\vec{r}(t)$.

$$\vec{a} = \frac{1}{m} \vec{F} = \langle -1, 1, 2 \rangle$$

$$\vec{v} = \langle -1, 1, 2 \rangle t + \langle 1, 0, 0 \rangle$$

$$\vec{r} = \frac{1}{2} t^2 \langle -1, 1, 2 \rangle + t \langle 1, 0, 0 \rangle + \vec{r}(0)$$