

Lecture 5: Vector derivatives, integrals, arc length

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} .$$

Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. This is the same as solving 3 "Calc-I" problems:

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle .$$

Ex. 1: Helix... $\vec{r}(t) = \langle \cos(t), \sin(t), 2t \rangle$.
Find $\vec{r}'(t)$.

$$x'(t) = -\sin(t)$$

$$y'(t) = \cos(t)$$

$$z'(t) = 2$$

$$\Rightarrow \vec{r}'(t) = \langle -\sin(t), \cos(t), 2 \rangle.$$

A closer look at that limit definition:

$$\begin{aligned}\vec{r}'(t) &= \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \\ &= \lim_{h \rightarrow 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h} \right\rangle \\ &= \left\langle \lim_{h \rightarrow 0} \frac{x(t+h) - x(t)}{h}, \lim_{h \rightarrow 0} \frac{y(t+h) - y(t)}{h}, \lim_{h \rightarrow 0} \frac{z(t+h) - z(t)}{h} \right\rangle \\ &= \langle x'(t), y'(t), z'(t) \rangle.\end{aligned}$$

Notation... the unit tangent vector will be $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

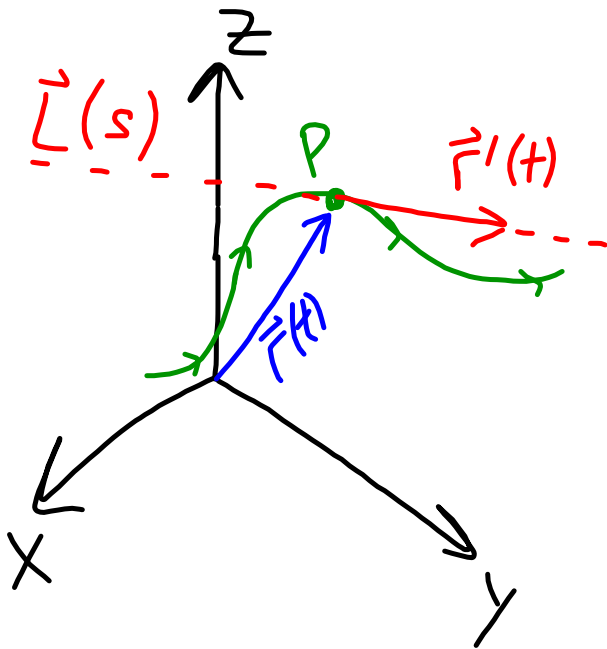
Ex.2: Let $\vec{r}(t) = \langle t, t^2, t^2 \rangle$.

Find the unit tangent vector.

$$\vec{r}'(t) = \langle 1, 2t, 2t \rangle, |\vec{r}'| = \sqrt{1 + 4t^2 + 4t^2}$$

$$\vec{T}(t) = \frac{1}{\sqrt{1 + 8t^2}} \langle 1, 2t, 2t \rangle.$$

Tangent Lines



- $\vec{r}(t)$ points to P
- $\vec{r}'(t)$ gives direction vector for line
- $\vec{L}(s) = \vec{r}(t) + s\vec{r}'(t)$

Ex. 3: Find the tangent line for
 $\vec{r}(t) = \langle 15t, t^2 - 1, \frac{t}{t+1} \rangle$ at
the point $(45, 8, \frac{3}{4})$.

Note $\vec{r}(3) = \langle 45, 8, \frac{3}{4} \rangle$, so $\underline{t=3}$.

$$\vec{r}'(t) = \left\langle 15, 2t, \frac{t+1-t}{(t+1)^2} \right\rangle$$

$$\Rightarrow \vec{r}'(3) = \langle 15, 6, \frac{1}{16} \rangle.$$

$$\vec{L}(s) = \left\langle 45 + 15s, 8 + 6s, \frac{3}{4} + \frac{s}{16} \right\rangle.$$

Ex. 4: Find the tangent line to

$$\vec{r}(t) = \langle t^2 - t + 2, t^2 + 5t, t \rangle$$

when $t = 5$.

$$\vec{r}(5) = \langle 25 - 5 + 2, 25 + 25, 5 \rangle \\ = \langle 22, 50, 5 \rangle.$$

$$\vec{r}'(t) = \langle 2t - 1, 2t + 5, 1 \rangle$$

$$\Rightarrow \vec{r}'(5) = \langle 9, 15, 1 \rangle$$

$$\Rightarrow \vec{L}(s) = \langle 22 + 9s, 50 + 15s, 5 + s \rangle.$$

Differentiation Rules

$$(1) \frac{d}{dt} (\vec{u} + \vec{v}) = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt}.$$

$$(2) \frac{d}{dt} (c\vec{u}) = c \frac{d\vec{u}}{dt}.$$

$$(3) \frac{d}{dt} (f(t)\vec{u}(t)) = \left(\frac{df}{dt}\right)\vec{u} + f \frac{d\vec{u}}{dt}.$$

$$(4) \frac{d}{dt} (\vec{u} \cdot \vec{v}) = \frac{d\vec{u}}{dt} \cdot \vec{v} + \vec{u} \cdot \frac{d\vec{v}}{dt}.$$

$$(5) \frac{d}{dt} (\vec{u} \times \vec{v}) = \left(\frac{d\vec{u}}{dt}\right) \times \vec{v} + \vec{u} \times \left(\frac{d\vec{v}}{dt}\right).$$

$$(6) \frac{d}{dt} \vec{u}(f(t)) = \left(\frac{d\vec{u}(f(t))}{dt}\right) \frac{df(t)}{dt}.$$

Ex. 5: (Useful trick.) Find $\frac{d}{dt} |\vec{r}(t)|^2$.

Apply $|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$, so

$$\begin{aligned} \frac{d}{dt} |\vec{r}|^2 &= \frac{d}{dt} (\vec{r} \cdot \vec{r}) = \vec{r}' \cdot \vec{r} + \vec{r} \cdot \vec{r}' \\ &= 2\vec{r}(t) \cdot \vec{r}'(t). \end{aligned}$$

Integration of vectors

$$\text{Definite: } \int_a^b \vec{r}(t) dt = \left\langle \int_a^b x dt, \int_a^b y dt, \int_a^b z dt \right\rangle.$$

Indefinite: Let $X(t) = \int x(t) dt$, $Y(t) = \int y(t) dt$, ...

$$\begin{aligned} \Rightarrow \int \vec{r}(t) dt &= \langle X(t) + C_1, Y(t) + C_2, Z(t) + C_3 \rangle \\ &= \langle X(t), Y(t), Z(t) \rangle + \vec{C}. \end{aligned}$$

Ex. 6: Let $\vec{r}(t) = \langle t, \cos(t), 1-t^2 \rangle$.

Find $\int_0^\pi \vec{r}(t) dt$.

$$\int_0^\pi t dt = \frac{1}{2} \pi^2$$

$$\int_0^\pi \cos(t) dt = 0$$

$$\int_0^\pi 1-t^2 dt = \pi - \frac{1}{3} \pi^3$$

$$\int_0^\pi \vec{r}(t) dt = \left\langle \frac{1}{2} \pi^2, 0, \pi - \frac{1}{3} \pi^3 \right\rangle.$$

Ex. 7: Find $\int \left\langle \frac{1}{t}, e^t, t^{10} \right\rangle dt$.

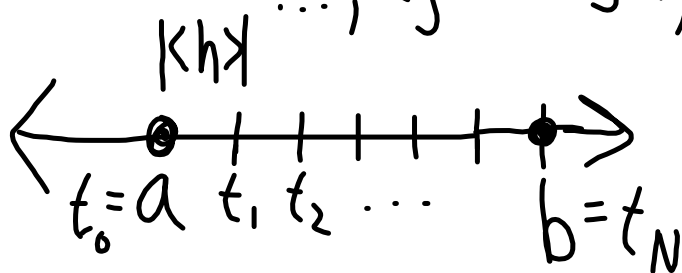
$$= \left\langle \ln|t|, e^t, \frac{t^{11}}{11} \right\rangle + \vec{C}.$$

One comment... what is $\vec{r}''(t)$?

Note that $\vec{r}'(t)$ traces out some curve, so then $\vec{r}''(t)$ is tangent to that curve.

Arc Length - Consider $\vec{r}(t)$ with $a \leq t \leq b$.

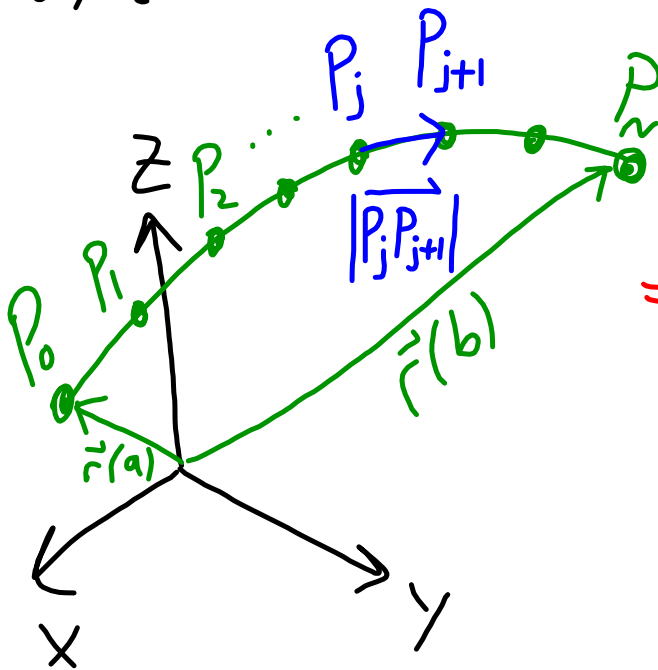
- DIVIDE the arc by choosing
 $t_0 = a$, $t_1 = a+h$, $t_2 = a+2h$,
 \dots , $t_j = a+jh$, \dots , $t_N = b$.



- Let P_j be the point for $\vec{r}(t_j)$.

Let L be the length of the curve.

$$L \approx \sum_{j=0}^{N-1} |\vec{P}_j \vec{P}_{j+1}|.$$



But $\vec{P}_j \vec{P}_{j+1} = \vec{r}(t_{j+1}) - \vec{r}(t_j)$

$$\Rightarrow L \approx \sum_{j=0}^{N-1} |\vec{r}(t_{j+1}) - \vec{r}(t_j)|$$

$$= h \sum_{j=0}^{N-1} \left| \frac{\vec{r}(t_{j+1}) - \vec{r}(t_j)}{h} \right|$$

Riemann sum!

$$L = \lim_{N \rightarrow \infty} h \sum_{j=0}^{N-1} \left| \frac{\vec{r}(t_{j+h}) - \vec{r}(t_j)}{h} \right|$$
$$= \lim_{N \rightarrow \infty} h \sum_{j=0}^{N-1} \left| \vec{r}'(t_j^*) \right| = \int_a^b |\vec{r}'(t)| dt.$$

$t_j \leq t_j^* \leq t_{j+1}$

$$L = \int_a^b |\vec{r}'(t)| dt$$

Ex. 8: Find the length of the helical segment of $\langle \cos(t), \sin(t), t \rangle$ for $0 \leq t \leq 2\pi$.

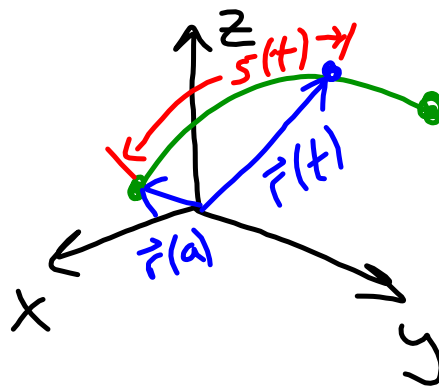
$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$|\vec{r}'(t)| = \sqrt{\underbrace{\sin^2(t) + \cos^2(t)}_{=1} + 1} = \sqrt{2}$$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{2} dt = 2\pi\sqrt{2}.$$

The arc-length function

$$S(t) = \int_a^t |\vec{r}'(\tau)| d\tau.$$



Note that $S'(t) = |\vec{r}'(t)|$
so $|\vec{r}'(t)|$ is the rate of increase
of the arc length at t .

If we can invert $s=s(t)$ to get $t=t(s)$ then we can parameterize \vec{r} in terms of s (arc length):

$$\vec{r}(t) = \vec{r}(t(s)) = \vec{r}(s).$$

Ex. 9: Reparameterize the curve

$\vec{r}(t) = \langle t^2, t^2, 2t^2 \rangle$ starting
from the origin in terms of arc length.

$$\vec{r}'(t) = \langle 2t, 2t, 4t \rangle$$

$$\Rightarrow s(t) = \int_0^t \sqrt{4\tau^2 + 4\tau^2 + 16\tau^2} d\tau$$

$$= \sqrt{24} \int_0^t \tau d\tau = 2\sqrt{6} \int_0^t \tau d\tau = t^2 \sqrt{6}$$

$$\dots \text{ so } s = t^2 \sqrt{6} \Rightarrow t^2 = \frac{s}{\sqrt{6}}$$

$$\vec{r}(t) = \langle t^2, t^2, 2t^2 \rangle$$

$$\Rightarrow \vec{r}(s) = \left\langle \frac{s}{\sqrt{6}}, \frac{s}{\sqrt{6}}, \frac{2s}{\sqrt{6}} \right\rangle.$$

* So, for example, when
arc length = $\frac{1}{2} = s$ you now know

$$\vec{r} = \frac{1}{\sqrt{6}} \left\langle \frac{1}{2}, \frac{1}{2}, 1 \right\rangle.$$

#1 Find a vector tangent to
 $\vec{r}(t) = \langle t^3, 1+t^2, 4t \rangle$
at the point $(8, 5, 8)$.

$$\vec{r}' = \langle 3t^2, 2t, 4 \rangle$$

$$\vec{r}'(2) = \langle 12, 4, 4 \rangle$$

#2 Find $\vec{T}(t)$ if $\vec{r}(t) = \langle \cos(t), \sin(t), 4t \rangle$.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{\sqrt{17}} \langle -\sin(t), \cos(t), 4 \rangle$$

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 4 \rangle$$

$$|\vec{r}'(t)| = \sqrt{\cos^2(t) + \sin^2(t) + 16} = \sqrt{17}$$

#3 Find $\int_0^1 \vec{r}(t) dt$; $\vec{r}(t) = \langle e^{2t}, 3t, 1 \rangle$.

$$\int_0^1 e^{2t} dt = \left(\frac{1}{2} e^{2t} \right) \Big|_{t=0}^{t=1}$$
$$\int_0^1 3t dt = \frac{3}{2} t^2 \Big|_0^1 = \frac{3}{2} (e^2 - 1) - \frac{1}{2} \Big|_{t=0}$$

$$\int_0^1 \vec{r} dt = \left\langle \frac{1}{2}(e^2 - 1), \frac{3}{2}, 1 \right\rangle.$$

#4 Let $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 3t \rangle$.
Find the length of the curve traced
out between $(2, 0, 0)$ & $(0, 2, \frac{3\pi}{2})$.
 $t=0$ $t=\pi/2$

$$\vec{r}' = \langle -2\sin(t), 2\cos(t), 3 \rangle$$

$$|\vec{r}'| = \sqrt{4(\sin^2(t) + \cos^2(t)) + 9} = \sqrt{13}$$

$$\int_0^{\pi/2} |\vec{r}'| dt = \int_0^{\pi/2} \sqrt{13} dt = \frac{\pi\sqrt{13}}{2}$$

#5 Reparameterize $\vec{r}(t) = \frac{1}{4} \langle \cos(t), \sin(t), t\sqrt{2} \rangle$
in terms of arc length from $(\frac{1}{4}, 0, 0)$.

$$s = \int_{t=0}^t |\vec{r}'(\tau)| d\tau = \frac{t\sqrt{3}}{4} \Rightarrow t = \frac{4s}{\sqrt{3}} \quad t=0$$

$$\vec{r}' = \frac{1}{4} \langle -\sin(t), \cos(t), \sqrt{2} \rangle$$

$$|\vec{r}'| = \frac{1}{4} \sqrt{1+2} = \frac{\sqrt{3}}{4}$$

$$\vec{r}(s) = \frac{1}{4} \left\langle \cos\left(\frac{4s}{\sqrt{3}}\right), \sin\left(\frac{4s}{\sqrt{3}}\right), 4s\sqrt{\frac{2}{3}} \right\rangle$$