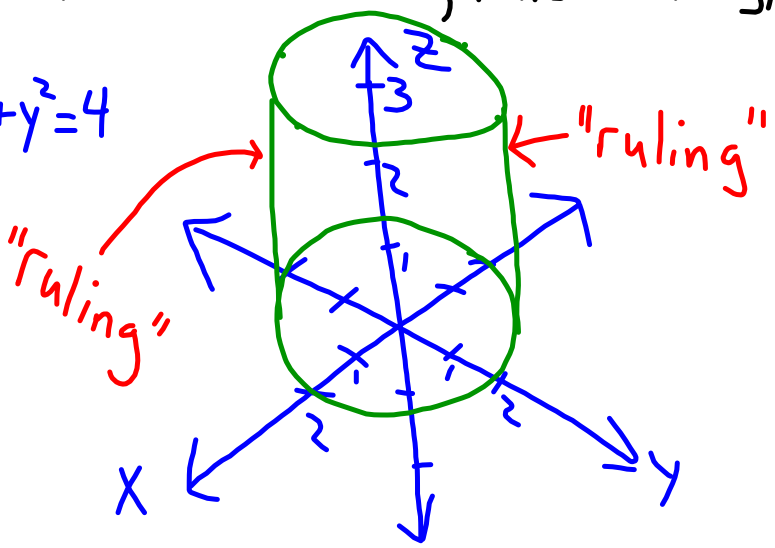
Lecture 4: Cylinders, quadric surfaces and vector functions

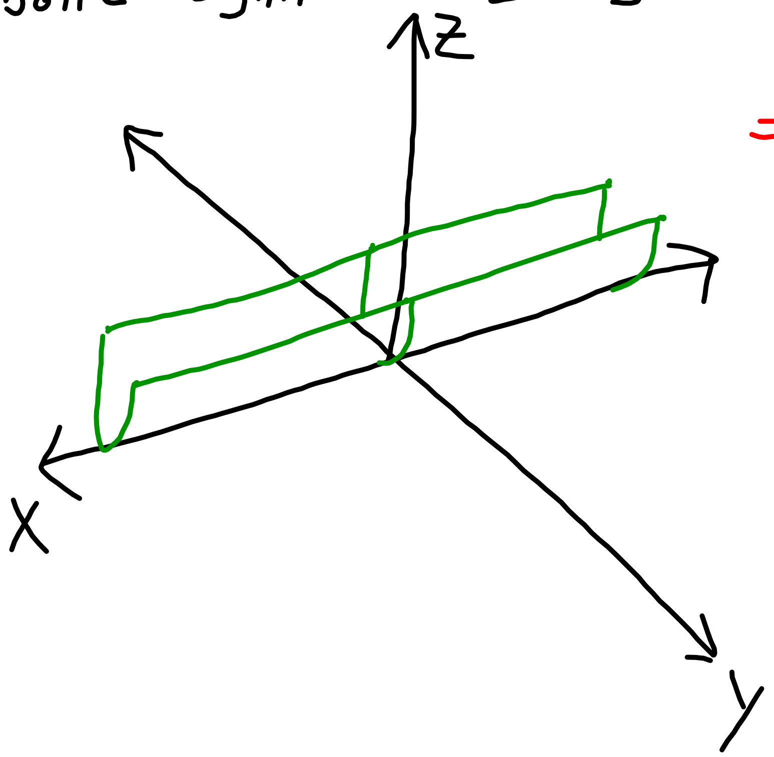
If we translate a curve along a single direction to form a surface, this is a "cylinder".

EX: $x^2+y^2=4$

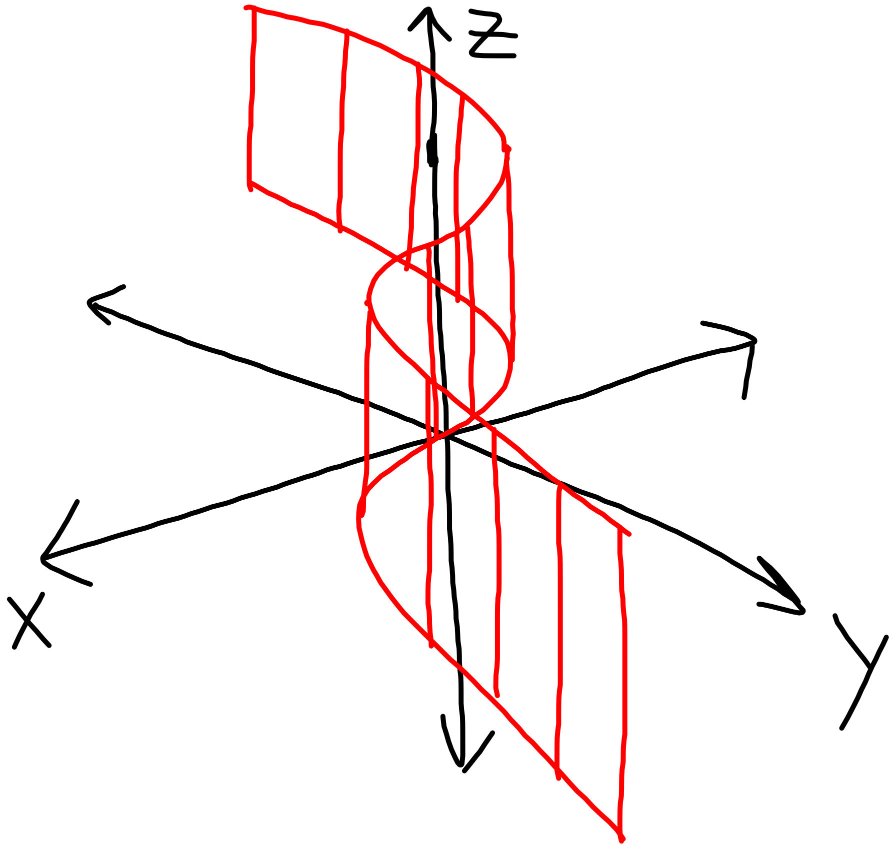


Parabolic cylinder $z = y^2$ } x-independent

$\Rightarrow$ runs along x-axis



Cubic cylinder $y = x^3$



Quadric surfaces

The term $x^i y^j z^k$ is of "order $i+j+k$", e.g. xy^2z^3 is "order 6". Quadric surfaces have equations with terms up to order 2:

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz$$
$$+Gx+Hy+Iz+J=0.$$

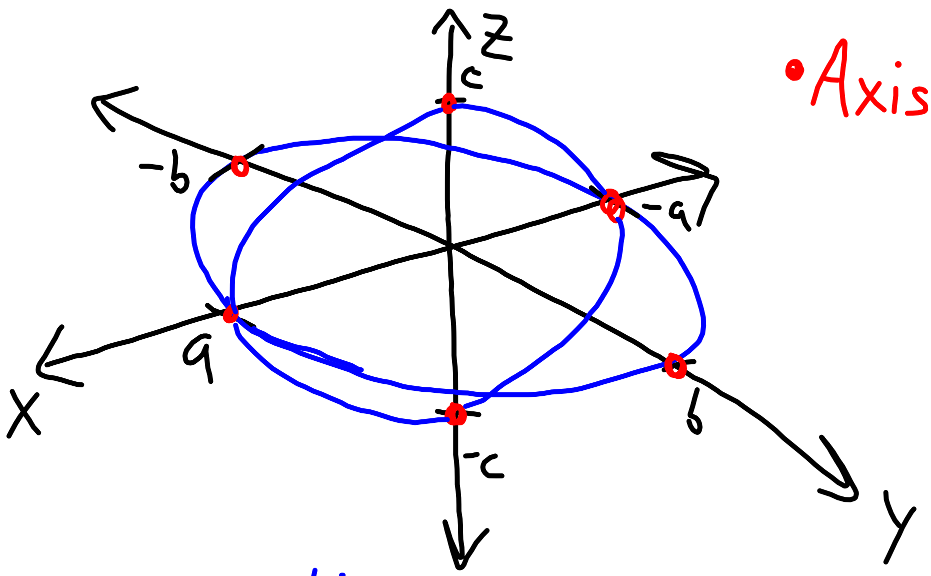
We can use translations and rotations to reduce to two forms:

(1) $Ax^2+By^2+Cz^2+J=0$

(2) $Ax^2+By^2+Iz=0$

There are 4 surfaces of type (1) and 2 of type (2).

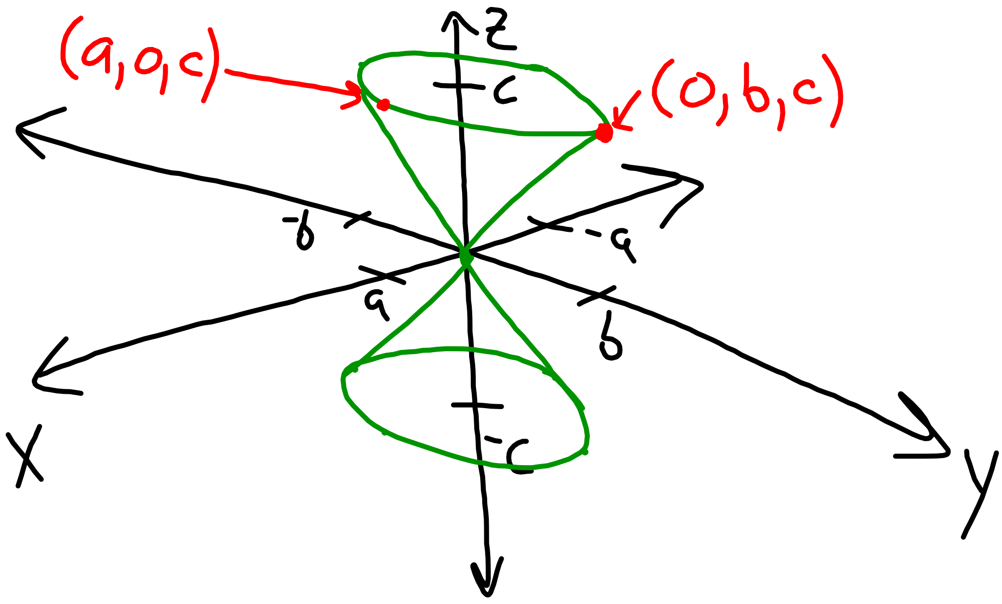
Ellipsoids: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$ } STANDARD FORM



• Axis intercepts

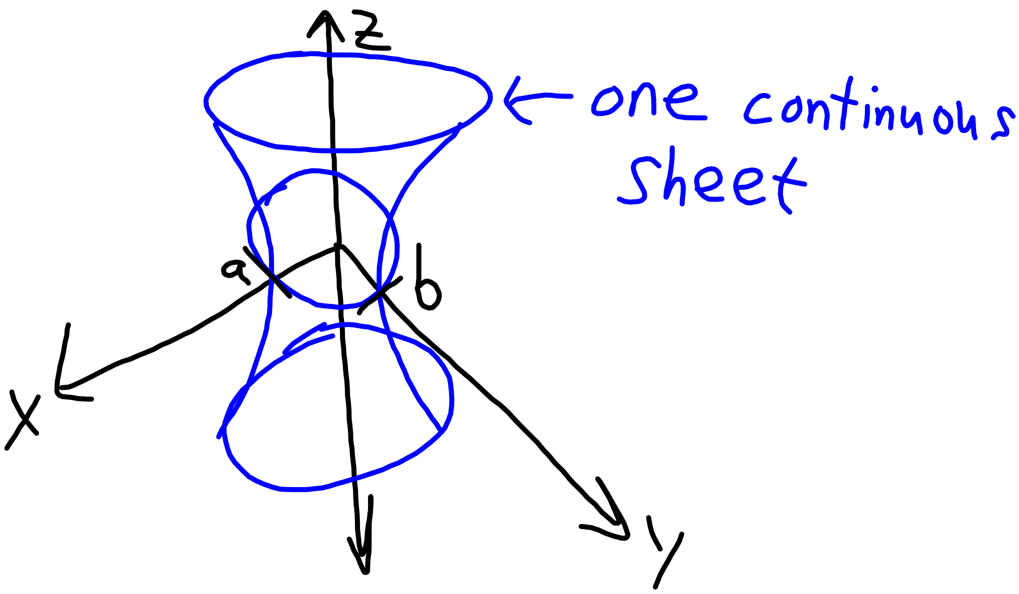
Cross-sections are ellipses.

CONES : $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{z^2}{c^2}$



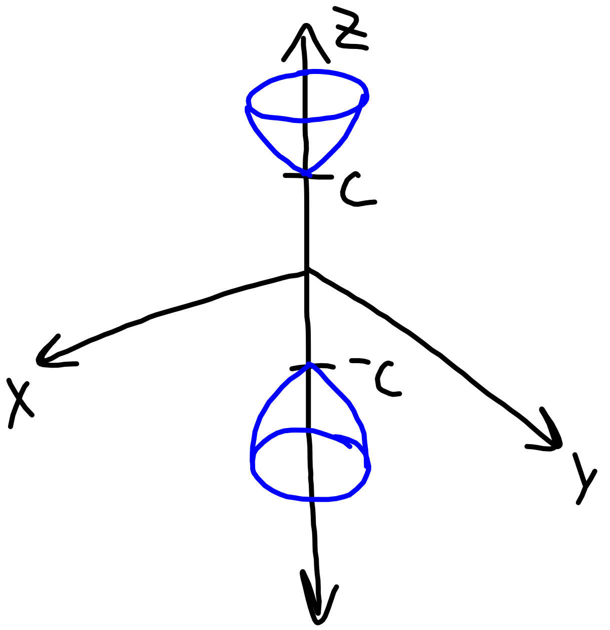
Hyperboloid of one sheet:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$



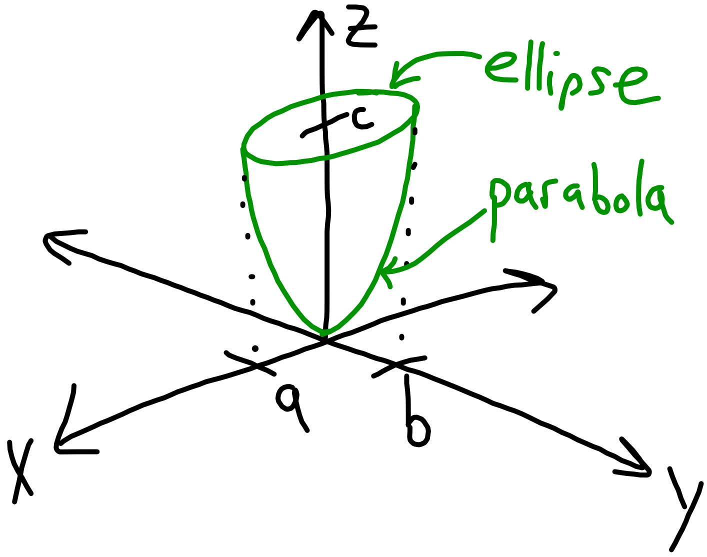
Hyperboloid of two sheets:

$$\frac{-x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$



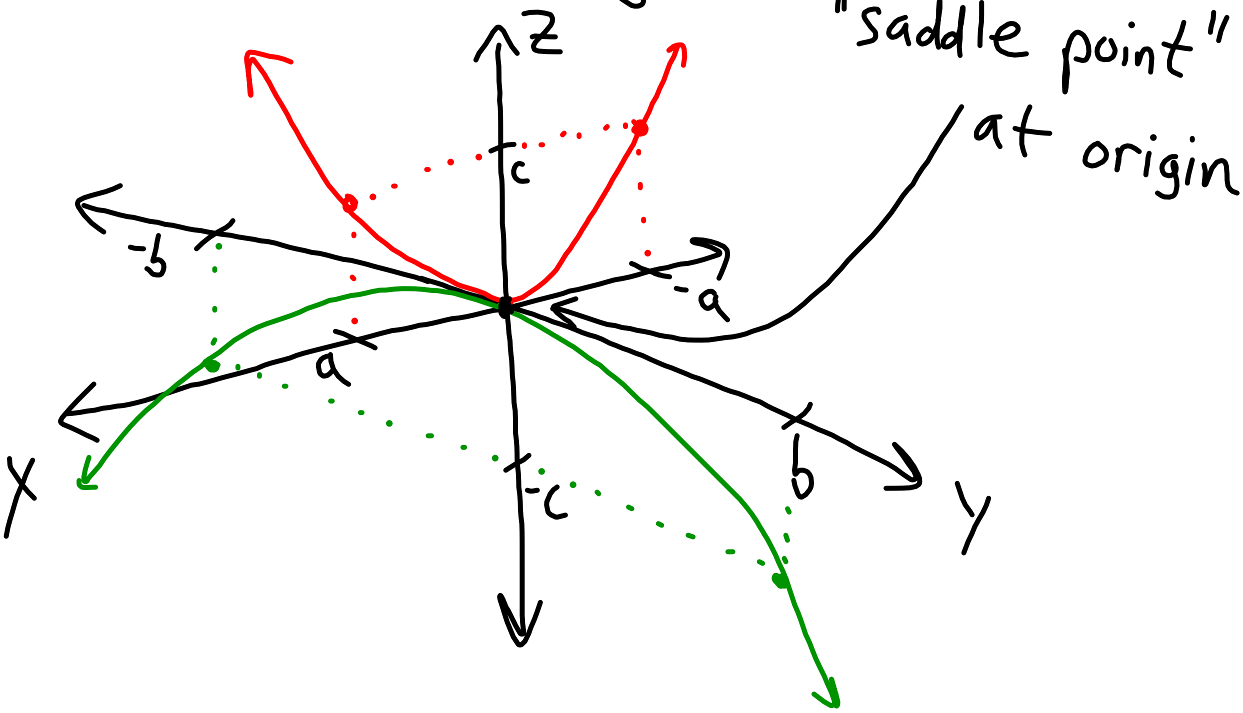
ELLIPTIC PARABOLOID

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 0$$



Hyperbolic Paraboloid:

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$



Ex. 1: Classify the surfaces:

(a) $x = y^2 - z^2$ hyperbolic paraboloid

(b) $x^2 + y^2 + 2z^2 = 1$ ellipsoid

(c) $2y = 3x^2 + 4z^2$ elliptic paraboloid

(d) $\frac{1}{4}z^2 = x^2 + y^2$ cone

(e) $2x^2 + z^2 - 3y^2 = -6$ two-sheet hyperboloid

(f) $z^2 + \frac{1}{4}y^2 - \frac{1}{9}x^2 = 1$ one-sheet hyperboloid

(g) $x^2 + z^2 = 4$ cylinder

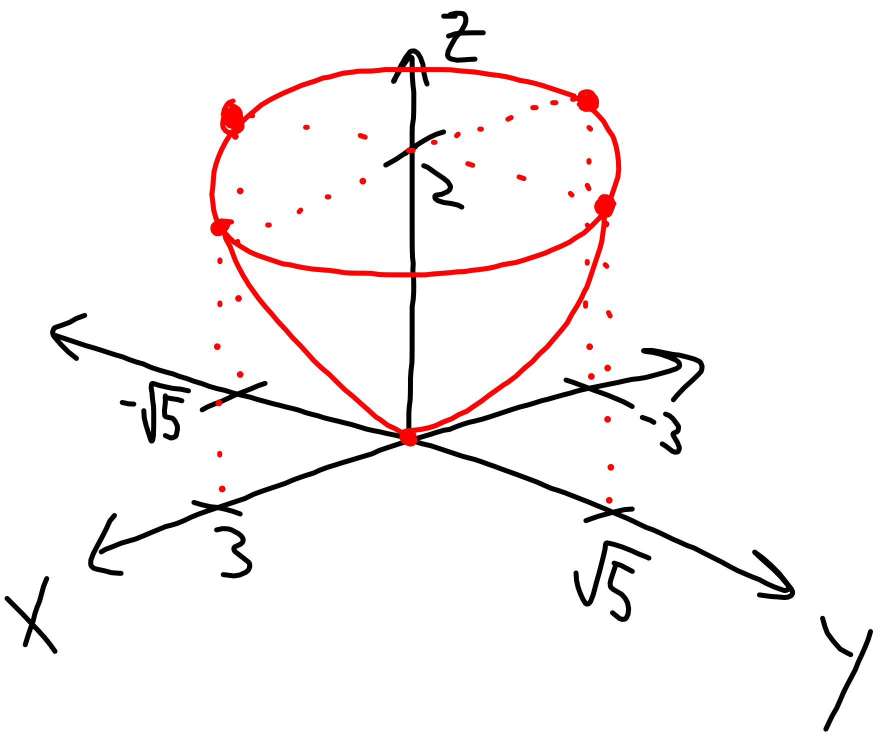
Ex. 2: Put $45z - 10x^2 - 18y^2 = 0$ into standard form and sketch the graph.

Elliptic paraboloid.

$$\frac{45z}{90} - \frac{10x^2}{90} - \frac{18y^2}{90} = 0$$

$$\Rightarrow \frac{z}{\underbrace{2}_{c}} = \frac{x^2}{9} + \frac{y^2}{5} = \frac{x^2}{\underbrace{3}_{a}{}^2} + \frac{y^2}{(\underbrace{\sqrt{5}}_{b})^2}.$$

Sketch:



Ex. 3: What sort of surface is

$$-\frac{(x-1)^2}{4} - \frac{(y+1)^2}{16} + z^2 = 1 \; ?$$

Two-sheet hyperboloid, translated so it is centered about $x=1$ & $y=-1$.

Vector-valued functions

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

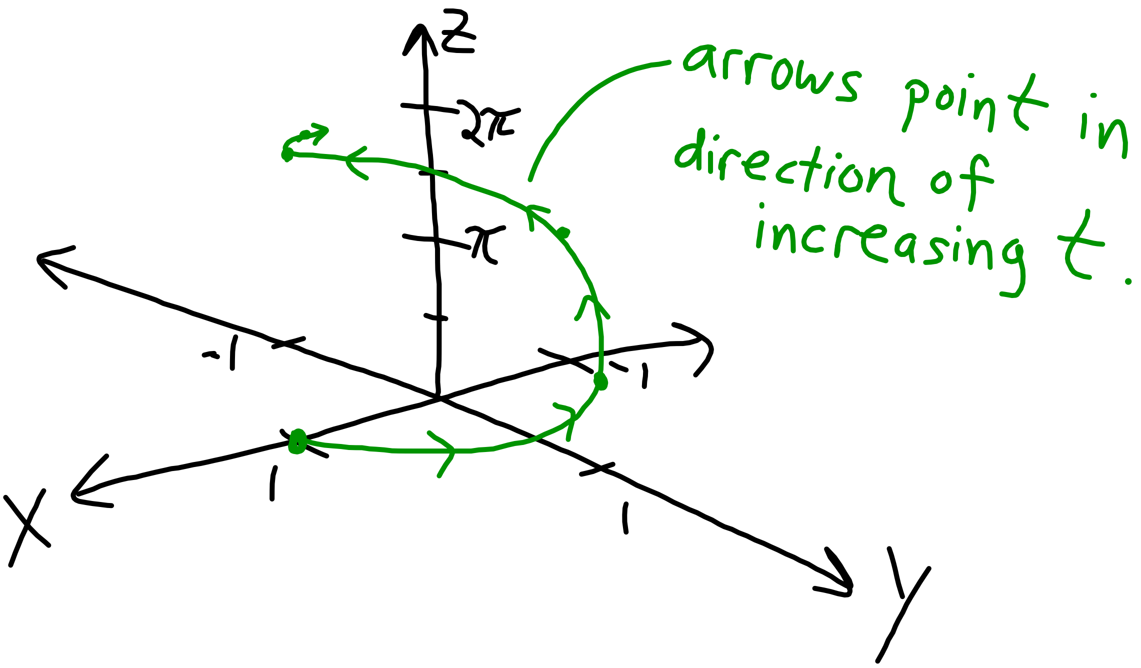
$\Rightarrow \vec{r}(t)$ is a "vector-valued" function of t.

$\vec{r}(t)$ traces out a curve in 3-D.

Ex: $\vec{r}(t) = \langle t, 2t, 3t+1 \rangle$ is a line.

EX: "Helix"

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle.$$



<u>Limits</u> are handled component-wise.

$$\lim_{t \to a} \vec{r}(t)$$

$$\left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle.$$

Each of the three limits needs to exist.

Ex. 4: Find $\lim_{t\to 0} \vec{r}(t)$; $\vec{r}(t) = \langle t, t^2+1, \frac{\sin(t)}{t} \rangle$.

$\lim_{t\to 0} t = 0$, $\lim_{t\to 0} t^2+1 = 1$, $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$

$\Rightarrow \lim_{t\to 0} \vec{r}(t) = \langle 0, 1, 1 \rangle$.

Ex: Find $\lim_{t\to 0} \langle t, 1+t, \sin(1/t) \rangle$.

Limit D.N.E. since $\lim_{t\to 0} \sin\left(\frac{1}{t}\right)$ D.N.E.

Ex. 5: Parameterize the curve formed by the intersection of

$$z = x^2 \text{ \& } -2y + z = 0.$$

Note $z = z(x)$ & $y = \frac{1}{2}z = y(z(x))$.

So choose $x = t \Rightarrow z = t^2$

$\Rightarrow y = \frac{1}{2}t^2$

$$\vec{r}(t) = \langle t, \tfrac{1}{2}t^2, t^2 \rangle.$$

#1 Classify each surface type:

(a) $5x^2+6y^2-\frac{1}{2}z^2=9$

one-sheet hyperboloid

(b) $x^2+y^2=z^2$

cone

(c) $z=x^2+5$

parabolic cylinder

(d) $x^2+4y^2+6z^2=12$

ellipsoid

(e) $\frac{x^2}{2}+\frac{y^2}{3}-z^2=-1$

2-sheet hyperboloid

(f) $\frac{z}{7}=x^2-y^2$

hyperbolic paraboloid

(g) $x^2+y^2-z=0$

paraboloid (elliptic)

(h) $x^2/4 + y^2/9 = 1.$

elliptic cylinder

#2 Classify...

(a) $x^2+6z^2-2y^2=1$

one-sheet hyperboloid

(b) $y/5 = x^2/2 + z^2/3$

elliptic paraboloid

(c) $-x^2+y^2+z^2=-2$

2-sheet hyperboloid

(d) $z=x^2-2y^2$

hyperbolic paraboloid

(e) $x^2+\frac{z^2}{2}=7$

elliptic cylinder

(f) $2x^2+3z^2=4y^2$

elliptic cone

(g) $9x^2+16y^2+25z^2=1$

ellipsoid

(h) $x^2-y^2-z^2=1.$

2-sheet hyperboloid

#3 List any/all axis intercepts:

(a) $6x^2 - y^2 + z^2 = -6$

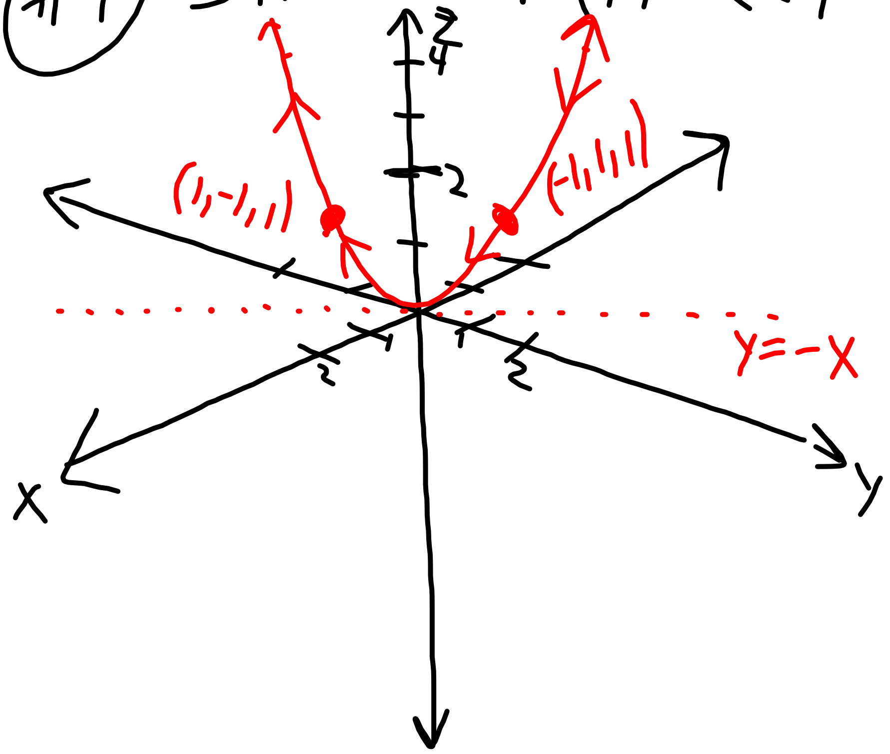
Standard form: $-x^2 + \frac{y^2}{6} - \frac{z^2}{6} = 1$

2-sheet hyperboloid intersects y-axis: $(0, \pm\sqrt{6}, 0)$.

(b) $y = x^2 + 2z^2$ - paraboloid through origin

$(0, 0, 0)$

#4 Sketch $\vec{r}(t) = \langle t, -t, t^2 \rangle$.



#5 Parameterize the curve formed by the intersection of $y=4x^2$ & $z=1-x^2$.

$y=y(x)$ & $z=z(x)$

$$\left.\begin{aligned} x &= t \\ y &= 4t^2 \\ z &= 1-t^2 \end{aligned}\right\} \vec{r}(t) = \langle t, 4t^2, 1-t^2 \rangle.$$