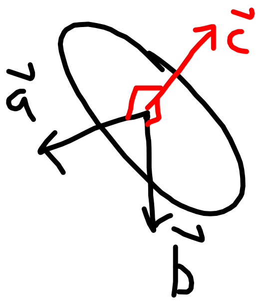
Math 2110: Multivariable Calculus

Lecture 3: Cross-products, lines, planes



Given $\vec{a}$, $\vec{b}$ find $\vec{c}$ such that

$\vec{a} \perp \vec{c}$ & $\vec{b} \perp \vec{c}$?

Could try solving

$$\left.\begin{aligned} \vec{a}\cdot\vec{c} &= 0 \\ \vec{b}\cdot\vec{c} &= 0 \end{aligned}\right\} \begin{aligned} &\text{2 equations} \\ &\text{3 unknowns} \\ &\langle c_1, c_2, c_3 \rangle \end{aligned}$$

$\vec{c}$ is not unique, but one important example is the cross-product

$$\vec{c} = \vec{a} \times \vec{b}$$

$$= \langle a_2 b_3 - b_2 a_3,\ b_1 a_3 - a_1 b_3,\ a_1 b_2 - b_1 a_2 \rangle.$$

You can check using this definition that

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} = 0.$$

Determinants are used to remember the formula...

2x2 determinant: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$

3x3: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

Now we can write

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k}$$

$$= \langle a_2b_3 - b_2a_3 ,\ b_1a_3 - a_1b_3 ,\ a_1b_2 - b_1a_2 \rangle$$

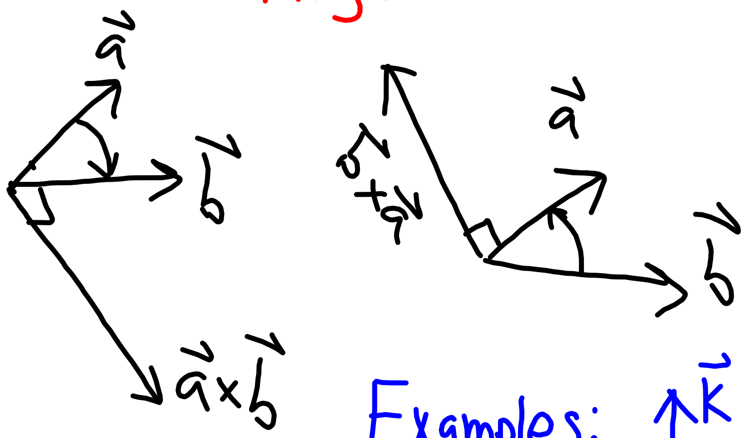
Ex. 1: Find $\vec{a} \times \vec{b}$; $\vec{a} = \langle 0, 1, -1 \rangle$
$\vec{b} = \langle 2, 4, 6 \rangle$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = (6-(-4))\vec{i} - (0\cdot 6-(-2))\vec{j} + (0\cdot 4-2)\vec{k}$$

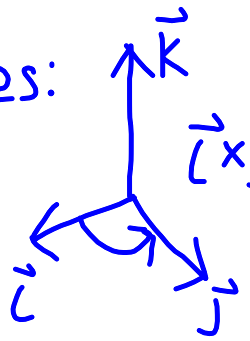
$$= \langle 10, -2, -2 \rangle.$$

What direction is $\vec{a} \times \vec{b}$?

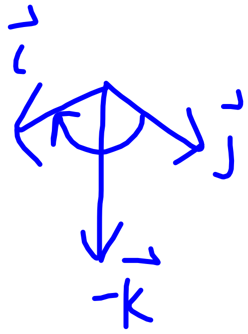
"Right-hand rule"



Examples:

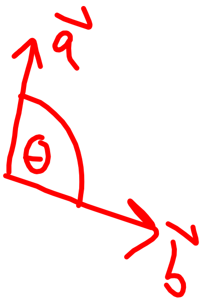


$\vec{i} \times \vec{j} = \vec{k}$

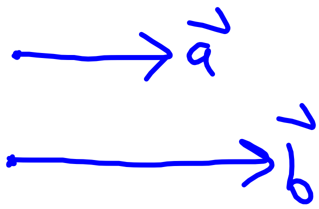


$\vec{j} \times \vec{i} = -\vec{k}$

There is a relationship with the acute angle between vectors:



$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.$$



<u>parallel</u> $\Leftrightarrow \theta = 0$

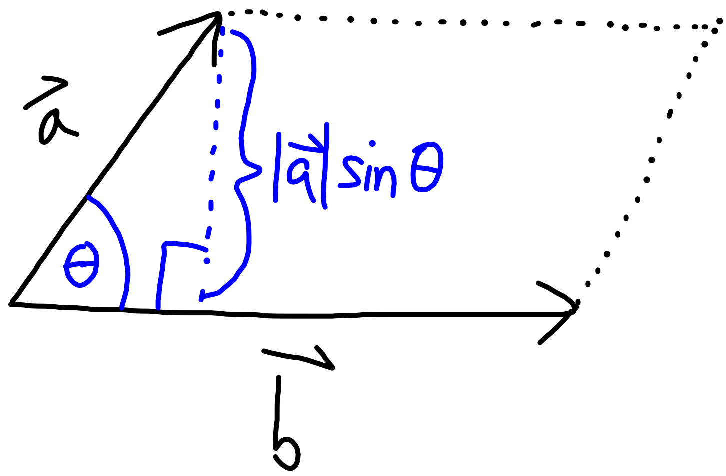
or

$\theta = \pi$

$\Leftrightarrow \sin\theta = 0$

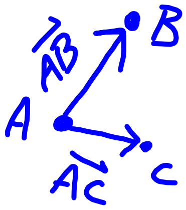
$\Leftrightarrow \underline{\vec{a}\times\vec{b} = 0}.$

A geometric relationship:



$$\text{Area} = |\vec{a}||\vec{b}|\sin\theta = |\vec{a}\times\vec{b}|.$$

Ex. 2: Find $\vec{v}$ orthogonal to the plane containing points $\underbrace{(2,1,1)}_{A}$, $\underbrace{(0,3,-1)}_{B}$ & $\underbrace{(2,-2,3)}_{C}$.



Find $\vec{AB}\times\vec{AC}$... $\vec{AB}=\langle -2,2,-2\rangle$

$\vec{AC}=\langle 0,-3,2\rangle$

$$\Rightarrow \vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-2&2&-2\\0&-3&2\end{vmatrix}=(4-6)\vec{i}-(-4)\vec{j}+(6)\vec{k}$$

$$=\langle -2,4,6\rangle.$$

Ex. 3: What is the area of the triangle with the points in the previous example?

Take $\frac{1}{2}$ area of parallelogram...

$$A = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = \frac{1}{2}\left|\langle -2, 4, 6\rangle\right|$$

$$= \frac{1}{2}\sqrt{4+16+36} = \frac{1}{2}\sqrt{56}.$$

PROPERTIES

(1) $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$ (not associative)

(2) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (not commutative)

(3) $(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times(c\vec{b})$.

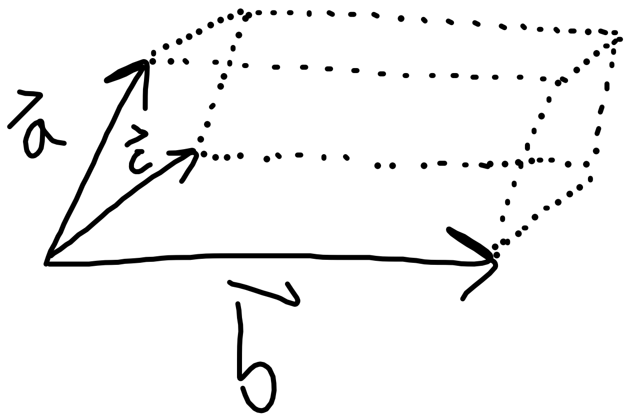
(4) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

(5) $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.

(6) $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ } $\vec{a}\cdot(\vec{b}\times\vec{c})$ "Triple product"

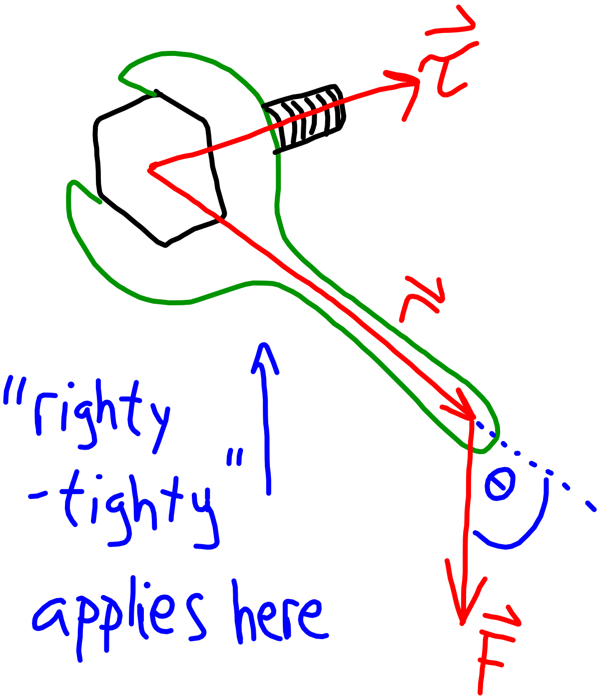
(7) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

Triple product gives the volume of a parallelipiped



$$V = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|.$$

Application: Torque $\vec{\tau}$.



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$\vec{F}$: force

$\vec{r}$: lever

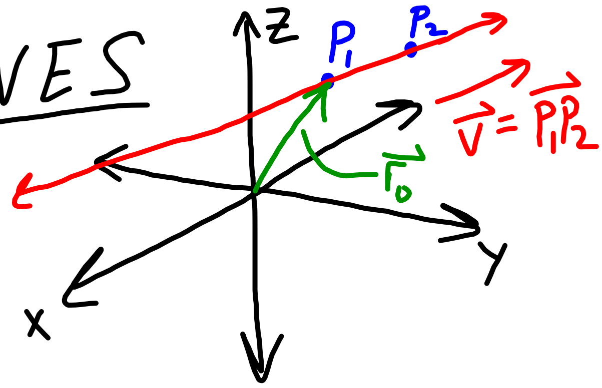
Ex. 4: How much torque is generated if a force of 50N is applied at an angle of 85° to the end of a 0.1m-long wrench?

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}| \sin\theta$$

$$= (0.1)(50) \sin 85^\circ$$

$$= 5 \sin 85^\circ \quad (N\text{-}m).$$

LINES



Red line can be created by adding multiples of $\vec{v}$ to $\vec{r}_0$ ...

$$\langle x, y, z \rangle = \vec{r}_0 + t\vec{v}$$ vector equation for the line

$$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

Then

$$\langle x, y, z \rangle = \langle x_0 + t v_1 , y_0 + t v_2 , z_0 + t v_3 \rangle$$

"Parametric equations"

$$\boxed{\begin{aligned} x &= x_0 + t v_1 \\ y &= y_0 + t v_2 \\ z &= z_0 + t v_3 . \end{aligned}}$$

"Symmetric" equations...

Solve for t: $x = x_0 + t v_1 \Rightarrow t = \dfrac{x - x_0}{v_1}$

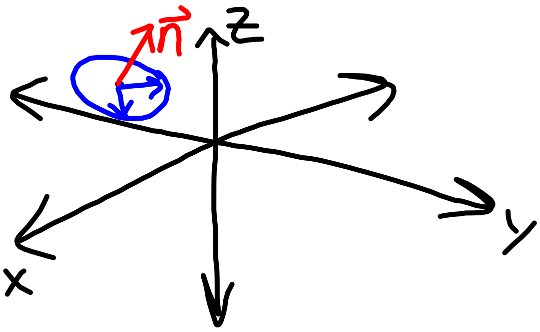
Also, $t = \boxed{\dfrac{y - y_0}{v_2} = \dfrac{z - z_0}{v_3} = \dfrac{x - x_0}{v_1}}$

symmetric equations

If, e.g. $v_3 = 0$, then $z = z_0 + \cancel{t v_3}^{\,0} = z_0$

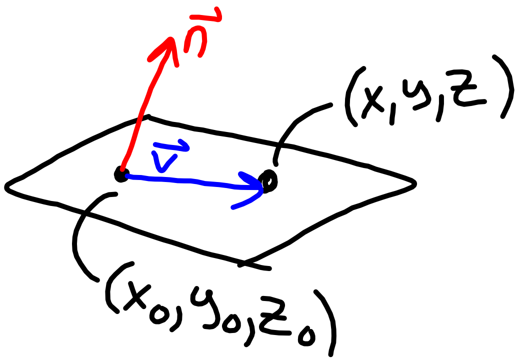
$\Rightarrow \boxed{\dfrac{x - x_0}{v_1} = \dfrac{y - y_0}{v_2} \ \& \ z = z_0.}$

PLANES



Cross two non-parallel vectors in a plane and a normal vector $\vec{n}$ results.

⊛ If $\vec{v}$ is in the desired plane then $\boxed{\vec{n} \cdot \vec{v} = 0.}$



$$\vec{n} = \langle n_1, n_2, n_3 \rangle$$

$$\vec{v} = \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$\vec{n} \cdot \vec{v} = \underbrace{n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0}_{\text{Scalar equation for plane}}$$

Note how we can algebraically manipulate the equation of the plane...

$$n_1x+n_2y+n_3z-n_1x_0-n_2y_0-n_3z_0=0.$$

looks like $\underbrace{ax+by+cz+d=0}$.

So if you see this, then you know $\vec{n}=\langle a,b,c\rangle$. ☺

Ex. 5: Find the equation of the plane passing through the points $(1,0,0) \leftarrow A$, $(0,2,0) \leftarrow B$, $(0,0,-3) \leftarrow C$

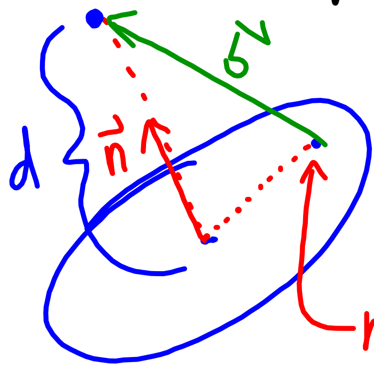
$\overrightarrow{AB} = \langle -1, 2, 0 \rangle$; $\overrightarrow{AC} = \langle -1, 0, -3 \rangle$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & -3 \end{vmatrix} = \langle -6, -3, 2 \rangle.$$

choose $(x_0, y_0, z_0) = (1, 0, 0)$ ... $\vec{n} \cdot \langle x-1, y-0, z-0 \rangle = 0$

$$\Rightarrow -6(x-1) - 3y + 2z = 0.$$

Ex. 6: Find the distance from $(3,1,1)$ to the plane $\underbrace{2x-2y+3z}= 1$.



So take $\vec{n} = \langle 2,-2,3\rangle$.

$$d = \left|\text{comp}_{\vec{n}}\, \vec{b}\right| = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{n}|} \quad (*)$$

need a point in the plane... $\left(\frac{1}{2},0,0\right)$

$$\Rightarrow \vec{b} = \left\langle 3-\frac{1}{2},\, 1-0,\, 1-0\right\rangle = \left\langle \frac{5}{2}, 1, 1\right\rangle$$

$$\Rightarrow d = \frac{\left|\frac{5}{2}\cdot 2-2+3\right|}{\sqrt{4+4+9}} = \frac{6}{\sqrt{17}}.$$

Practice...

(#1) Calculate the triple product

$\vec{u}\cdot(\vec{v}\times\vec{w})$, if $\vec{u}=\langle 1,-1,2\rangle$

$\vec{v}=\langle -1,-1,0\rangle$

$\vec{w}=\langle 0,3,-5\rangle$.

$\vec{v}\times\vec{w}=\langle 5,-5,-3\rangle$

$\vec{u}\cdot(\vec{v}\times\vec{w})=5(1)+5-6=4$

(#2) Consider two lines:

$$\begin{cases} x=1+2t \\ y=1+3t \\ z=1+4t \end{cases} \quad \& \quad \begin{cases} x=-1-4t \\ y=5-6t \\ z=7-8t \end{cases}.$$

Are the lines parallel? Derive the symmetric equations for the line on the left.

(Yes, parallel.)

$$t=\frac{x-1}{2}=\frac{y-1}{3}=\frac{z-1}{4}.$$

#3 Consider 3 planes:

(1) $x-2y+5z-9=0$

(2) $2x-3y+10z+7=0$

(3) $-3x+6y-15z+1=0$.

Which of these planes (if any) are parallel?

Yes, (1) & (3) ... multiply (1) by -3 to see this.

(#4) Derive equations for the line passing through $(10, 4, -3)$ in the direction $\langle -5, 0, 6 \rangle$.

$$\langle x, y, z \rangle = \langle 10-5t, 4, -3+6t \rangle$$

$$x = 10-5t$$

$$y = 4$$

$$z = -3+6t$$

#5 Derive an equation for the plane passing through $(1,-1,4)$ with normal $\langle -3,-3,1\rangle$.

$$\vec{n}\cdot\vec{r}=0=\langle -3,-3,1\rangle\cdot\langle x-1,y+1,z-4\rangle=0$$

$$-3(x-1)-3(y+1)+z-4=0.$$

#6 Find the distance between the point $(-4, 5, 10)$ and the plane $x+y+z=1$.

$(0,0,1)$ in the plane

$\vec{n} = \langle 1, 1, 1 \rangle$

"$\vec{b}$" $= \langle -4, 5, 9 \rangle$

$$d = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{n}|} = \frac{|-4+5+9|}{\sqrt{3}} = \frac{10}{\sqrt{3}}.$$