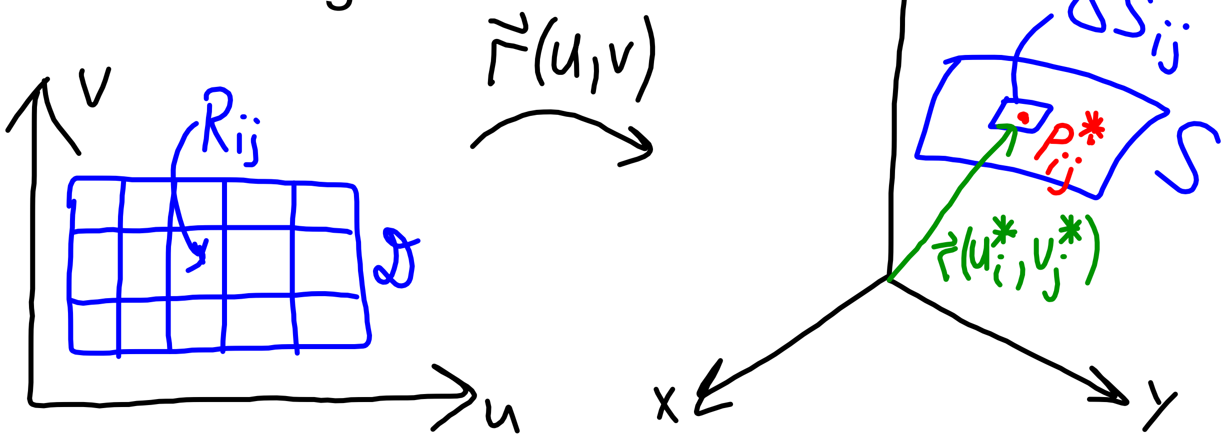
Surface integrals



- $f(x,y,z)$ is defined on S.
- S is parameterized by $\vec{r}(u,v)$.

Recall $\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta u \cdot \Delta v$

$$\Rightarrow \iint_S f(x,y,z)\,dS = \lim_{N\to\infty} \sum_{i,j=1}^{N} f(P_{ij}^{*})\Delta S_{ij}$$

$$= \lim_{N\to\infty} \sum_{i,j=1}^{N} f(\vec{r}(u_i^{*}, v_j^{*}))|\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

$$= \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv.$$

Surface integral for scalar-valued f:

$$\boxed{\iint_S f\, dS = \iint_{\mathcal{D}} f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv.}$$

In case $f \equiv 1$ we recover surface area:

$$|S| = \iint_S dS = \iint_{\mathcal{D}} |\vec{r}_u \times \vec{r}_v|\,du\,dv.$$

Recall for $\vec{r}(x,y) = \langle x, y, z(x,y) \rangle$

we get

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}, \quad \text{so}$$

$$\boxed{\iint_S f \, dS = \iint_D f(x,y,z(x,y)) \sqrt{1 + (z_x)^2 + (z_y)^2} \, dx\,dy.}$$

Special case we use often.

EX: Calculate $\iint_S x^2+y^2\,dS$, where S is the sphere of radius 3 centered at the origin.

Recall $x = 3\cos\theta\sin\phi$
$y = 3\sin\theta\sin\phi$
$z = 3\cos\phi$

Also, $|\vec{r}_\theta \times \vec{r}_\phi| = 3^2\sin\phi = 9\sin\phi$, and

$x^2+y^2 = r^2 = (3\sin\phi)^2 = 9\sin^2\phi$.

$$\iint_S x^2+y^2\,dS = \int_0^{2\pi}\int_0^{\pi} 9\sin^2\phi \cdot 9\sin\phi\, d\phi\, d\theta$$

$$= 81 \cdot 2\pi \cdot \int_0^{\pi} \sin^3\phi\, d\phi = 162\pi \int_0^{\pi} \sin\phi\left(1-\cos^2\phi\right) d\phi$$

$$= 162\pi\left[\int_0^{\pi} \sin\phi\, d\phi - \int_0^{\pi} \sin\phi \cos^2\phi\, d\phi\right]$$

$$= 162\pi\left[-\cos\phi\Big|_0^{\pi} + \frac{1}{3}\cos^3\phi\Big|_0^{\pi}\right] = \boxed{216\pi}.$$

Ex: Calculate $\iint\limits_S z^2\,dS$, where S is given by $f(x,y) = 1+x+y$, $0 \le x \le 1$, $0 \le y \le 2$.

$$f_x = 1 = f_y \Rightarrow \sqrt{1+f_x^2+f_y^2} = \sqrt{3}.$$

$$\iint\limits_S z^2\,dS = \int_0^1\int_0^2 (1+x+y)^2\sqrt{3}\,dy\,dx = \sqrt{3}\int_0^1 \frac{1}{3}(1+x+y)^3\Big|_0^2\,dx$$

$$= \frac{\sqrt{3}}{3}\int_0^1 (3+x)^3-(1+x)^3\,dx = \frac{\sqrt{3}}{3}\left[\frac{1}{4}(3+x)^4-\frac{1}{4}(1+x)^4\right]_0^1$$

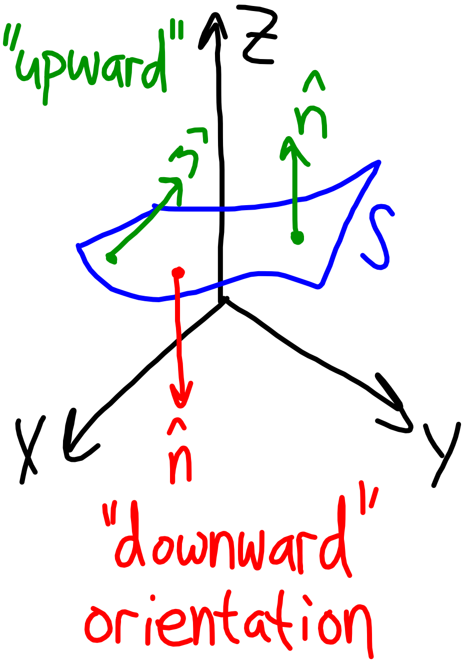
$$= \frac{\sqrt{3}}{12}\left[4^4 - 3^4 - 2^4 + 1\right]$$

$$= \frac{\sqrt{3}}{12}\left[256 - 81 - 16 + 1\right]$$

$$= \frac{\sqrt{3} \cdot 160}{12} = \boxed{\frac{40\sqrt{3}}{3}}$$

Surface orientation

We will assign a unit normal vector to a surface, depending on where you are on that surface.



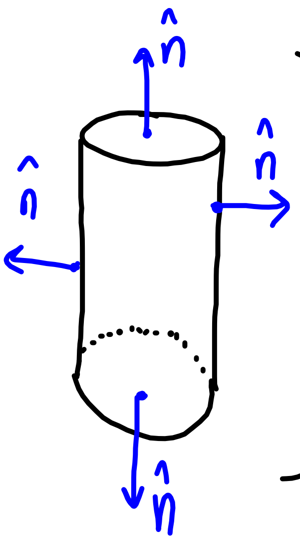
$\hat{n} = \hat{n}(x, y, z)$

We only consider surfaces with 2 distinct sides (no Möbius strips, etc.)

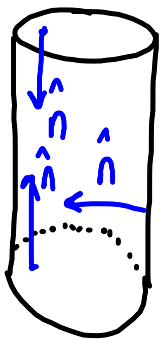
Thus, there are 2 possible "orientations".

More precisely, "upward" means the $\vec{k}$-component of $\hat{n}$ is positive.

Closed surfaces (enclose a volume):

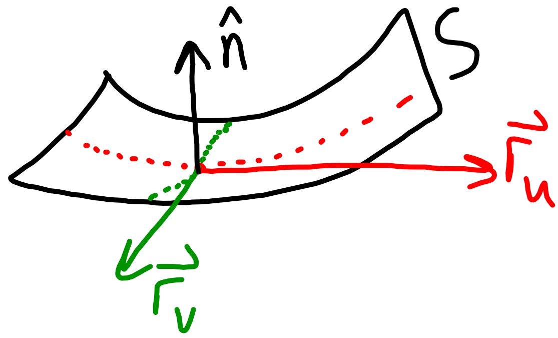


"outward" orientation OR "positive"



"inward" or "negative" orientation

How to get $\hat{n}$



$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

If you calculate $\hat{n}$ this way and it points the "wrong" way then multiply by -1!

$$-\hat{n} = \frac{-\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{\vec{r}_v \times \vec{r}_u}{|\vec{r}_v \times \vec{r}_u|}.$$

For the natural parameterization,

$$\vec{r}(x,y) = \langle x, y, z(x,y) \rangle \quad \begin{cases} \vec{r}_x = \langle 1, 0, z_x \rangle \\ \vec{r}_y = \langle 0, 1, z_y \rangle \end{cases}$$

$$\vec{r}_x \times \vec{r}_y = \langle -z_x, -z_y, 1 \rangle$$

"upward"

$$\Rightarrow \boxed{\hat{n} = \frac{1}{\sqrt{1+(z_x)^2+(z_y)^2}} \langle -z_x, -z_y, 1 \rangle}$$

Surface integrals for vector fields

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \underbrace{(\vec{F}\cdot\hat{n})}_{\text{scalar, so we know what this integral means}}\, dS$$

notation

Parameterize S by $\vec{r}(u,v)$, domain D, then

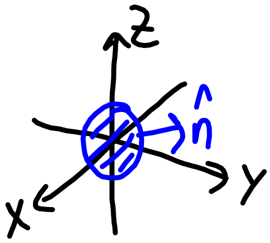
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot \underbrace{\left(\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\right)}_{\hat{n}} \underbrace{|\vec{r}_u\times\vec{r}_v|\,du\,dv}_{dS}$$

$$\Rightarrow \boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv}.$$

Summary of integrals:

	LINE	SURFACE
Scalar f	$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\lvert\vec{r}\,'(t)\rvert\,dt$	$\iint_S f\,dS = \iint_D f(\vec{r}(u,v))\,\lvert\vec{r}_u \times \vec{r}_v\rvert\,du\,dv$
Vector $\vec{F}$	$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$	$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv.$

EX: Let S be the unit sphere with positive orientation. Find $\iint_S \vec{F} \cdot d\vec{S}$ if $\vec{F} = \langle 0, 0, 1 \rangle$.



We want $\hat{n}$ pointing outward:

$$\left.\begin{aligned} x &= \cos\theta \sin\phi \\ y &= \sin\theta \sin\phi \\ z &= \cos\phi \end{aligned}\right\}$$

We calculated previously

$$\vec{r}_\theta \times \vec{r}_\phi = -\cos\theta \sin^2\phi \, \vec{i} - \sin\theta \sin^2\phi \, \vec{j} - \sin\phi \cos\phi \, \vec{k}.$$

We don't really need $\hat{n}$ but will look at it for illustration... $|\vec{r}_\theta \times \vec{r}_\phi| = \sin\phi$

$$\hat{n} = \frac{\vec{r}_\theta \times \vec{r}_\phi}{|\vec{r}_\theta \times \vec{r}_\phi|} = \langle -\cos\theta\sin\phi, -\sin\theta\sin\phi, -\cos\phi \rangle$$

$$= -\langle x, y, z \rangle \Big\} \text{ points "inward"}$$

So we use $-\vec{r}_\theta \times \vec{r}_\phi = \vec{r}_\phi \times \vec{r}_\theta$ in the integral:

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi} \underbrace{\langle 0,0,1 \rangle \cdot (\vec{r}_\phi \times \vec{r}_\theta)}_{\sin\phi\cos\phi} \, d\phi \, d\theta$$

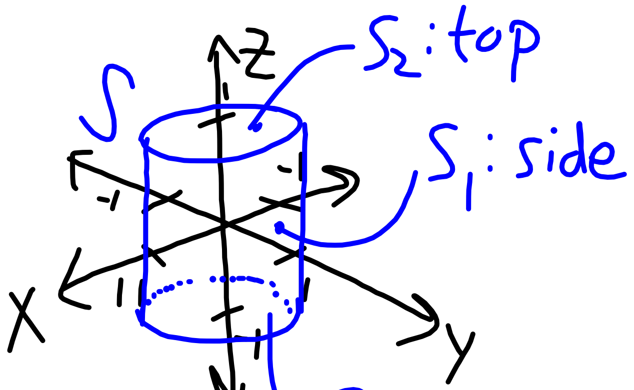
$$= \int_0^{2\pi}\int_0^{\pi} \sin\phi\cos\phi \, d\phi \, d\theta = 2\pi \int_0^{\pi} \frac{1}{2}\frac{d}{d\phi}\sin^2\phi \, d\phi$$

$$= \frac{2\pi}{2}\sin^2\phi\Big|_0^{\pi} = \boxed{0}.$$

*Subtlety: note we did <u>not</u> stick in $\rho^2\sin\phi$ here, but applied the surface integral formula. Using a <u>parameterization</u>, not change of variables formula.

EX: Let S be the surface of the cylinder $\{(x,y,z) \mid x^2+y^2 \le 1, -1 \le z \le 1\}$, with positive orientation. Find $\iint_S \vec{F}\cdot d\vec{S}$ if $\vec{F} = \langle x,y,z \rangle$.

$S = S_1 \cup S_2 \cup S_3$



On S_1 : $\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$

$0 \le \theta \le 2\pi, \ -1 \le z \le 1$

$$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle$$

$$\vec{r}_z = \langle 0, 0, 1 \rangle$$

$$\vec{r}_\theta \times \vec{r}_z = \underbrace{\langle \cos\theta, \sin\theta, 0 \rangle}_{= \langle x, y, 0 \rangle, \text{ "outward"}}$$

$$\vec{F}(\vec{r}(\theta, z)) = \langle \cos\theta, \sin\theta, z \rangle$$

$$\vec{F} \cdot (\vec{r}_\theta \times \vec{r}_z) = \cos^2\theta + \sin^2\theta = 1.$$

Thus $\iint_{S_1} \vec{F}\cdot d\vec{S}_1 = \int_0^{2\pi}\int_{-1}^{1} 1\, dz\, d\theta = 4\pi$

On S_2 (top) : $\vec{r}(x,y) = \langle x, y, 1\rangle$, $\underbrace{x^2+y^2 \le 1}_{\mathcal{D}}$

$$\left.\begin{aligned} \vec{r}_x &= \langle 1, 0, 0\rangle \\ \vec{r}_y &= \langle 0, 1, 0\rangle \end{aligned}\right\} \underbrace{\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1\rangle}_{\text{"outward"}}$$

$$\iint_{S_2} \vec{F}\cdot d\vec{S}_2 = \iint_{\mathcal{D}} \langle x, y, 1\rangle \cdot \langle 0,0,1\rangle\, dx\,dy = \iint_{\mathcal{D}} 1\, dx\,dy = \overbrace{\pi}^{\text{Area of circle}}.$$

$\underline{\underline{\text{On } S_3}}$: $\vec{r}(x,y) = \langle x, y, -1 \rangle$

$$\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle \Big\} \text{ points wrong way}$$

$$\Rightarrow \iint_{S_3} \vec{F} \cdot d\vec{S}_3 = \iint_{D} \langle x, y, -1 \rangle \cdot \langle 0, 0, -1 \rangle \, dx\,dy$$

$$= \iint_{D} 1 \, dx\,dy = \underline{\underline{\pi}}.$$

Sum over 3 surfaces: $\iint_{S} \vec{F} \cdot d\vec{S} = 4\pi + \pi + \pi = \boxed{6\pi.}$

Stoke's Theorem: relates the integral of the tangential part of a vector field over a curve *C*, which bounds a surface *S*, to the integral of the CURL of the field over the surface.

* Very important result.

* Requires the curve to have positive orientation with respect to the surface.

Theorem: (Stoke's)

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}}$$

Notation: If C is a closed curve and is the boundary of S, a surface, then

$$C = \partial S \leftarrow \text{not a derivative}$$

If a volume V is bounded by surface S,

$$S = \partial V.$$

Stoke's again:

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

or $$\boxed{\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \int_{\partial S} \vec{F} \cdot d\vec{r}}.$$

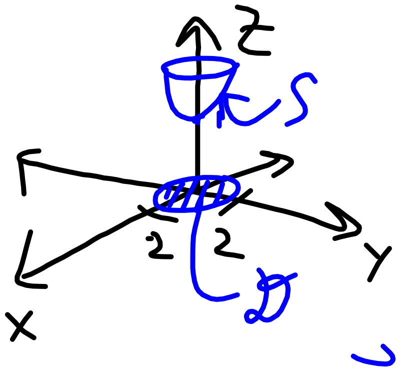
We will pick up here next time.

Practice!

#1 Let $S = \{(x,y,z) \mid x^2+y^2 \le 4,\ z = 1+x^2+y^2\}$,

with upward orientation.

Find $\iint_S \vec{F} \cdot d\vec{S}$, if $\vec{F} = \langle -x, -y, 0 \rangle$.



$$\vec{r}(x,y) = \langle x, y, 1+x^2+y^2 \rangle$$

$$\vec{r}_x = \langle 1, 0, 2x \rangle$$

$$\vec{r}_y = \langle 0, 1, 2y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle -2x, -2y, 1 \rangle$$

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = \langle -x, -y, 0 \rangle \cdot \langle -2x, -2y, 1 \rangle$$

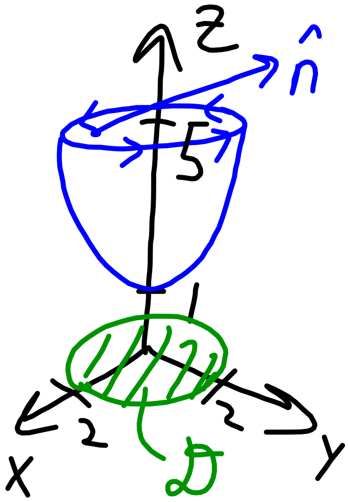
$$= 2(x^2+y^2).$$

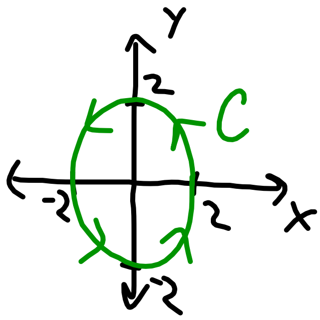
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D 2(x^2+y^2)\,dA$$

$$= 2\int_0^{2\pi}\int_0^2 r^2\, r\,dr\,d\theta = 2\cdot 2\pi\cdot \frac{1}{4} r^4 \Big|_0^2$$

$$= 2^4\pi = \boxed{16\pi}$$

#2 Parameterize ∂S in the last problem with positive orientation.





$$\left.\begin{aligned} z &= 1+x^2+y^2 \\ &= 1+r^2 = 1+4 = 5 \\ x &= r\cos\theta = 2\cos\theta \\ y &= 2\sin\theta \\ z &= 5 \end{aligned}\right\}$$

$$\boxed{\begin{aligned} &\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 5\rangle. \\ &\quad 0 \le \theta \le 2\pi. \end{aligned}}$$