

Curl: an important differential operator on vector fields.

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}, \quad \vec{F} = \vec{F}(x, y, z).$$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \left. \begin{array}{l} \text{Think} \\ \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \end{array} \right\}$$

$$\Rightarrow \nabla \times \vec{F} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}.$$

EX: Find  $\text{Div } \vec{F}$  if  $\vec{F} = \cos(xz)\vec{i} + xy\vec{j} + yz\vec{k}$ .

$$\text{Div } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(xz) & xy & yz \end{vmatrix}$$

$$= \left( \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(xy) \right) \vec{i} + \left( \frac{\partial}{\partial z} \cos(xz) - \frac{\partial}{\partial x}(yz) \right) \vec{j} + \left( \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(\cos(xz)) \right) \vec{k}$$

$$= z\vec{i} - x\sin(xz)\vec{j} + y\vec{k}.$$

Theorem : If  $f(x, y, z)$  is a  $C^2$ -function  
(has continuous 2<sup>nd</sup>-order derivatives)  
then

$$\nabla \times (\nabla f) = 0.$$

Proof : The  $C^2$ -requirement allows application  
of Clairaut's Theorem (symmetry of mixed  
partials). So, direct calculation of

$\nabla \times \nabla f$  gives...

$$\begin{aligned} \nabla \times \nabla f &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \vec{i} + \left( \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \vec{j} \\ &\quad + \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \vec{k} \end{aligned}$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}.$$

□

Recall if  $\vec{F} = \nabla f$  then  $\vec{F}$  is called CONSERVATIVE. It follows that

$$\vec{F} \text{ conservative} \Rightarrow \nabla \times \vec{F} = 0,$$

(so long as  $f$  is  $C^2$ ). Conversely, if

1)  $\vec{F}$  is  $C^1$  and 2)  $\nabla \times \vec{F} = 0$ , then

$\vec{F}$  is conservative.

EX: Determine whether or not

$$\vec{F} = \langle x \cos(x+y) + \sin(x+y), x \cos(x+y) \rangle$$

is conservative.

Extend to 3D:  $P = x \cos(x+y) + \sin(x+y)$

$$Q = x \cos(x+y)$$

$$R = 0$$

$$\nabla \times \vec{F} = \langle 0, 0, Q_x - P_y \rangle; \text{ check if } Q_x = P_y \dots$$

$$Q_x = \cos(x+y) - x \sin(x+y)$$

$$P_y = -x \sin(x+y) + \cos(x+y)$$

(conservative).

\*  $Q_x = P_y$  was already derived as a condition for 2D vector fields to be conservative.

EX: Is  $\vec{F} = x\vec{i} + y\vec{j} - z^2\vec{k}$  conservative?

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & -z^2 \end{vmatrix} = \left( \frac{\partial}{\partial y}(-z^2) - \frac{\partial}{\partial z}(y) \right) \vec{i} \\ + \left( \frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(-z^2) \right) \vec{j} \\ + \left( \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \right) \vec{k}$$

$$= \vec{0}$$

$\Rightarrow$  conservative.



## Divergence operator\*

\* Produces a scalar, unlike  $\nabla \times \vec{F}$ .

$$\begin{aligned}\operatorname{div}(\vec{F}) &= \nabla \cdot \vec{F} \\ &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle \\ &= P_x + Q_y + R_z.\end{aligned}$$

EX: Find  $\nabla \cdot \vec{F}$ , if  $\vec{F} = \langle x^2yz, y^2z, z \rangle$ .

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(z)$$

$$= 2xyz + 2y + 1.$$

Theorem: If  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ , where  $P, Q, \& R$  are  $C^2$ , then

$$\nabla \cdot (\nabla \times \vec{F}) = 0.$$

Proof:  $\nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$$\Rightarrow \nabla \cdot (\nabla \times \vec{F}) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

$$= 0.$$

↑  
Clairaut

□

Laplace operator : Let  $f(x, y, z)$  be scalar-valued. Then

$$\begin{aligned} \nabla \cdot (\nabla f) &= \nabla^2 f = \nabla \cdot \langle f_x, f_y, f_z \rangle \\ &= f_{xx} + f_{yy} + f_{zz} . \end{aligned}$$

$\nabla^2$  : Laplace operator, also known

as  $\Delta$  ... as in  $\Delta f = f_{xx} + f_{yy} + f_{zz}$ .

Apply to vector fields  $\vec{F} = \langle P, Q, R \rangle$ :

$$\Delta \vec{F} = (\Delta P)\vec{i} + (\Delta Q)\vec{j} + (\Delta R)\vec{k}.$$

EXTRA NOTATION :  $\vec{u} = \langle u_1, u_2, u_3 \rangle$

Each  $u_j = u_j(x_1, x_2, x_3)$ ,  $j = 1, 2, 3$ .

$$\Rightarrow (\nabla \vec{u})_{ij} = \frac{\partial u_j}{\partial x_i}, \quad i, j = 1, 2, 3$$

So  $D\vec{u}$  is a  $3 \times 3$  array :

$$D\vec{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

We extend the dot product:

$$(\vec{u} \cdot D\vec{u})_j = \sum_{i=1}^3 u_i (Du)_{ij} = \sum_{i=1}^3 u_i \frac{\partial u_j}{\partial x_i}, \quad j=1,2,3.$$

Application : Navier-Stokes equations  
of incompressible fluid flow.

$\vec{u}$  : fluid velocity at point  $(x_1, x_2, x_3)$

$p$  : pressure

$\nu$  : kinematic viscosity

The following equations are derived  
from conservation laws.

Conservation of momentum:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \nabla p = \nu \Delta \vec{u} .$$

Conservation of mass:

$$\nabla \cdot \vec{u} = 0 .$$

divergence measures rate at which  
fluid flows "out" from a point.

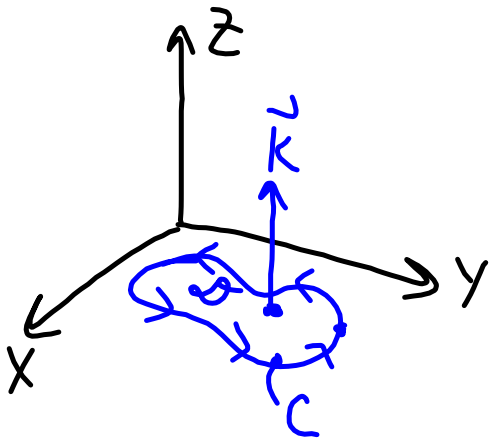


Connections...

- $\nabla \times \vec{u} = \omega$  is the "vorticity", which measures the tendency of the fluid to "rotate" near a point.
- $\nu \Delta u$  measures dissipation of momentum (converts to heat) via molecular friction.

## Green's Theorem in vector form

"Green's in 3D"



Recall

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y) dA.$$

$$\text{But } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} \\ = \langle 0, 0, Q_x - P_y \rangle.$$

Thus

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k} dA.$$

- \* Green's Theorem relates the integral of the *tangential part* of the field along  $C$  to the integral of the *vertical part* of the curl over  $D$ .
- \* We will see this come up again with Stoke's Theorem later in the course.
- \* We now come up with a similar identity involving the *normal component* of the field, integrated along the curve  $C$ .

◦ "Normal part of  $\vec{F}$ " is  $\vec{F} \cdot \hat{n}$ .

Let  $\vec{r}(t)$  be a parameterization of  $C$ ,

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}.$$

Derived previously...

$$\vec{T}(t) = \frac{x'(t)}{|\vec{r}'(t)|} \vec{i} + \frac{y'(t)}{|\vec{r}'(t)|} \vec{j}$$

$$\hat{n}(t) = \frac{y'(t)}{|\vec{r}'(t)|} \vec{i} + \frac{-x'(t)}{|\vec{r}'(t)|} \vec{j}$$

Note  
 $\vec{T} \cdot \hat{n} = 0$

It follows that

$$\int_C \vec{F} \cdot \hat{n} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \underbrace{\left\langle \frac{y'}{|\vec{r}'|}, \frac{-x'}{|\vec{r}'|} \right\rangle}_{\hat{n}} \underbrace{|\vec{r}'| \, dt}_{ds}$$

$$= \int_a^b \left( P \frac{dy}{dt} - Q \frac{dx}{dt} \right) dt$$

$$= \int_a^b P y'(t) dt - \int_a^b Q x'(t) dt = \int_C P dy - Q dx$$

$$\text{so } \int_C \vec{F} \cdot \hat{n} \, ds = \int_C P \, dy - Q \, dx$$

$$= \iint_D \frac{\partial}{\partial x} (P) - \frac{\partial}{\partial y} (-Q) \, dA$$

$$\text{(Green's)} = \iint_D (P_x + Q_y) \, dA = \iint_D (\nabla \cdot \vec{F}) \, dA.$$

$$\boxed{\int_C (\vec{F} \cdot \hat{n}) \, ds = \iint_D (\nabla \cdot \vec{F}) \, dA.}$$

USED  
VERY  
OFTEN.

EX: Let  $\vec{F} = \frac{y}{1+x^2} \vec{i} - \frac{x}{1+x^2} \vec{j}$  on  
 $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$ . Show  $\iint_D \nabla \cdot \vec{F} dA = 0$ .

$$\begin{aligned} \text{Note } \nabla \cdot \vec{F} &= \frac{\partial}{\partial x} \left( \frac{y}{1+x^2} \right) + \frac{\partial}{\partial y} \left( \frac{-x}{1+x^2} \right) \\ &= \frac{-2xy}{(1+x^2)^2}. \end{aligned}$$

You can vaguely see the integral should be zero, but this is easily shown by:

$$\iint_D \vec{D} \cdot \vec{F} dA = \int_C \vec{F} \cdot \hat{n} ds, \text{ since}$$

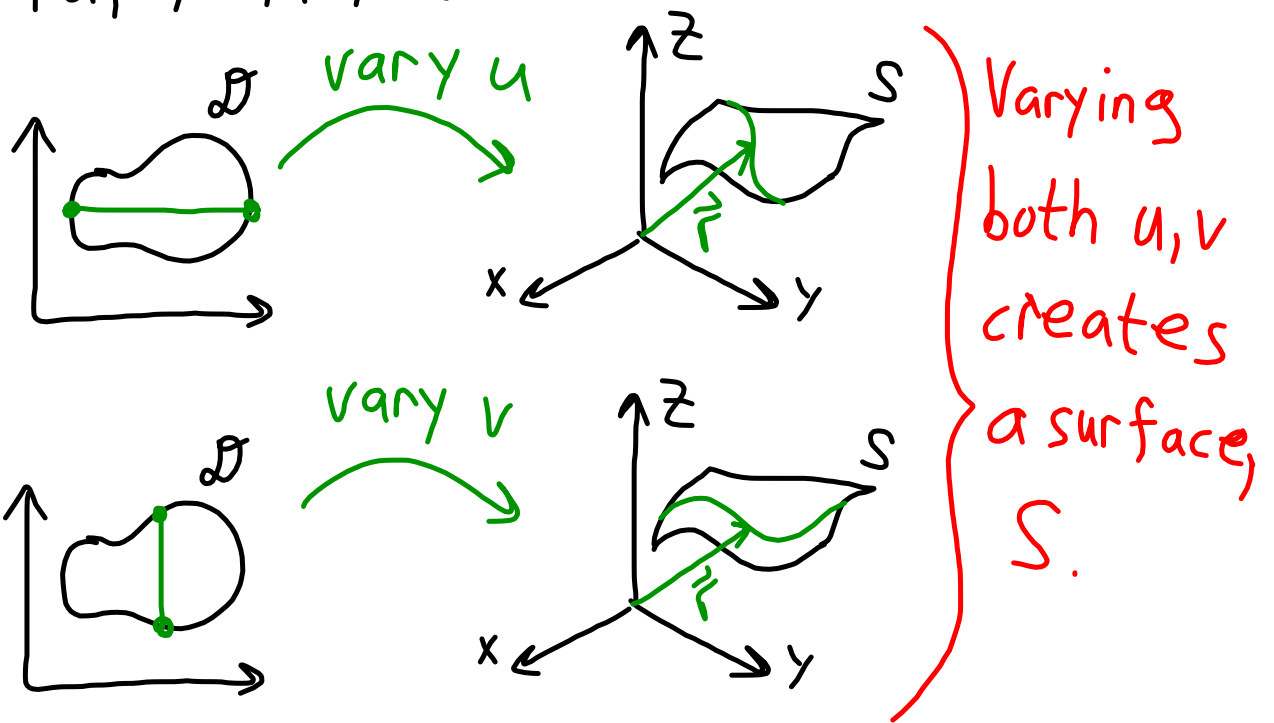
$$\hat{n} = \langle x, y \rangle \text{ on } C \text{ (unit circle)}$$

$$\Rightarrow \vec{F} \cdot \hat{n} = \frac{y}{1+x^2} x - \frac{x}{1+x^2} y = 0.$$



## Parametric Surfaces

$$\vec{r}(u, v) = P(u, v)\vec{i} + Q(u, v)\vec{j} + R(u, v)\vec{k}.$$



EX: The cylinder  $x^2 + y^2 = 9$ ,  $-1 \leq z \leq 1$

has parameterization

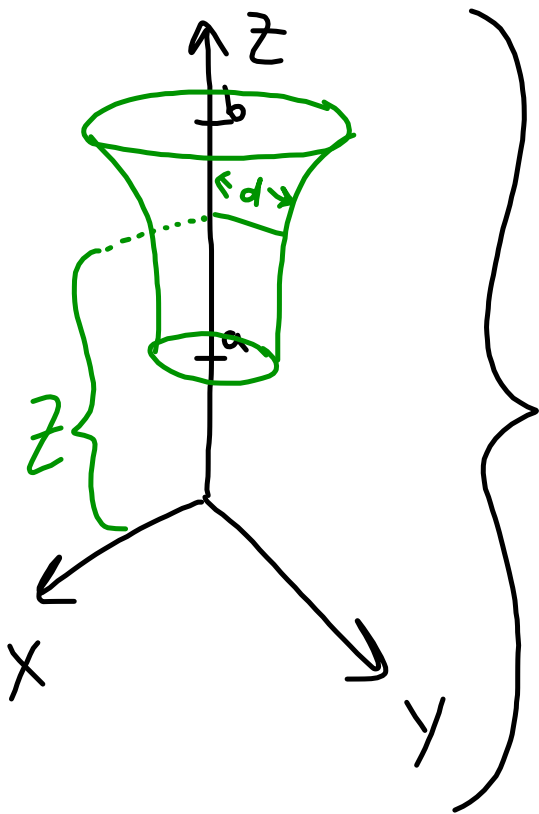
$$\left. \begin{array}{l} x = 3 \cos \theta \\ y = 3 \sin \theta \\ z = z \end{array} \right\} \begin{array}{l} 0 \leq \theta \leq 2\pi \\ -1 \leq z \leq 1 \end{array}$$

We say  $\vec{r}(\theta, z) = \langle 3 \cos \theta, 3 \sin \theta, z \rangle$ .

EX: The sphere  $x^2 + y^2 + z^2 = 4$  has  
parameterization  $x = 2 \cos \theta \sin \phi$   
 $y = 2 \sin \theta \sin \phi$   
 $z = 2 \cos \phi$

with  $0 \leq \theta \leq 2\pi$  &  $0 \leq \phi \leq \pi$ .

## Rotational surfaces



$d = d(z)$ : distance from z-axis, rotate curve around z-axis.

$$x = d(z) \cos \theta$$

$$y = d(z) \sin \theta$$

$$z = z$$

$$a \leq z \leq b, 0 \leq \theta \leq 2\pi.$$

"Natural" parameterizations

When you have a surface  $z = f(x, y)$   
then just use  $\vec{r}(x, y) = \langle x, y, f(x, y) \rangle$ .

EX: Elliptic paraboloid  $z = 4x^2 + y^2$

$$\Rightarrow \vec{r}(x, y) = \langle x, y, 4x^2 + y^2 \rangle.$$

Surface area: We showed previously when discussing surface area that  $\Delta S \approx |\vec{r}_x \times \vec{r}_y| \Delta x \Delta y$ .

Now we generalize...  $\Delta S \approx |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$   
(same derivation).

$$\text{AREA of } S = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| du dv.$$

For the natural parameterization,

$$u=x, v=y$$

$$\Rightarrow |\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

$$\text{Then } \iint_S dS = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx dy$$

is a special case.

EX: Find the surface area of a sphere of radius  $a$ .  
 $x = a \cos \theta \sin \phi$   
 $y = a \sin \theta \sin \phi$   
 $z = a \cos \phi$

$$\vec{r}_\theta = \langle -a \sin \theta \sin \phi, a \cos \theta \sin \phi, 0 \rangle$$

$$\vec{r}_\phi = \langle a \cos \theta \cos \phi, a \sin \theta \cos \phi, -a \sin \phi \rangle$$

$$\begin{aligned} \vec{r}_\theta \times \vec{r}_\phi &= -a^2 \cos \theta \sin^2 \phi \vec{i} - a^2 \sin \theta \sin^2 \phi \vec{j} \\ &\quad - a^2 (\underbrace{\sin^2 \theta + \cos^2 \theta}_1) \sin \phi \cos \phi \vec{k}. \end{aligned}$$



$$|\vec{\Gamma}_\theta \times \vec{\Gamma}_\phi| = \left( a^4 \cos^2 \theta \sin^4 \phi + a^4 \sin^2 \theta \sin^4 \phi + a^4 \sin^2 \phi \cos^2 \phi \right)^{1/2}$$

$$= a^2 \left( \sin^4 \phi + \sin^2 \phi \cos^2 \phi \right)^{1/2}$$

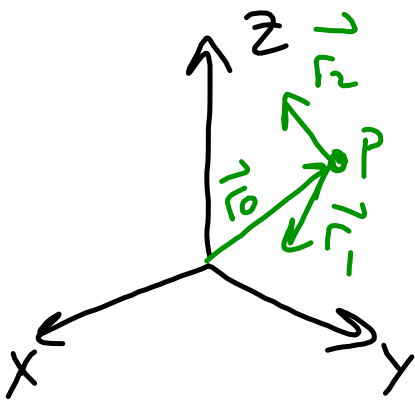
$$= a^2 \left( \sin^2 \phi (\sin^2 \phi + \cos^2 \phi) \right)^{1/2}$$

$$= a^2 \sin \phi.$$

$$\text{Area} = \int_0^\pi \int_0^{2\pi} a^2 \sin \phi \, d\theta \, d\phi = 2\pi a^2 (-\cos \phi) \Big|_0^\pi = 4\pi a^2.$$

Practice!

#1 Find a parameterization of the plane passing through  $(0, -1, 5)$  and containing  $\langle 2, 1, 4 \rangle$  &  $\langle -3, 2, 5 \rangle$ .



$$\vec{r} = \vec{r}_0 + u\vec{r}_1 + v\vec{r}_2$$

$$\vec{r}_0 = \vec{p} = \langle 0, -1, 5 \rangle$$

$$\vec{r}_1 = \langle 2, 1, 4 \rangle$$

$$\vec{r}_2 = \langle -3, 2, 5 \rangle$$

$$\Rightarrow \vec{r}(u, v) = \langle 2u - 3v, -1 + u + 2v, 5 + 4u + 5v \rangle.$$

#2 Is  $\vec{F} = \langle xy^2z, y+z, 1 \rangle$  conservative?

$$\nabla \times \vec{F} = \vec{0}?$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2z & y+z & 1 \end{vmatrix}$$

$$= \langle -1, xy^2, -2xyz \rangle \neq \langle 0, 0, 0 \rangle$$

$\Rightarrow$  NOT conservative.

#3 Find the area of the surface  
with parameterization

$$\begin{aligned}x &= u^2 \\y &= uv \\z &= \frac{1}{2}v^2\end{aligned}$$

$$0 \leq u \leq 1 \quad \& \quad 0 \leq v \leq 2.$$

$$\vec{r}_u = \langle x_u, y_u, z_u \rangle = \langle 2u, v, 0 \rangle$$

$$\vec{r}_v = \langle x_v, y_v, z_v \rangle = \langle 0, u, v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle v^2, -2uv, 2u^2 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(2u^2 + v^2)^2}$$

$$= 2u^2 + v^2$$

$$\text{Area} = \int_0^2 \int_0^1 (2u^2 + v^2) \, du \, dv = \int_0^2 \left[ \frac{2}{3}u^3 \Big|_0^1 + uv^2 \Big|_0^1 \right] dv$$

$$= \int_0^2 \frac{2}{3} + v^2 \, dv$$

$$= \frac{2}{3} \cdot 2 + \frac{1}{3} v^3 \Big|_0^2 = \frac{4}{3} + \frac{8}{3} = \frac{12}{3}$$

$$= \boxed{4.}$$

#4 Calculate  $\nabla \cdot \vec{F}$ ,

$$\vec{F} = \langle 1 - zx, -2yx^2z^3, ze^{xz} + y \rangle.$$



$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(1 - zx) \\ &\quad + \frac{\partial}{\partial y}(-2y^2x^3) \\ &\quad + \frac{\partial}{\partial z}(ze^{xz} + y) \end{aligned}$$

$$= -z - 4yx^3 + e^{xz} + xze^{xz}.$$