

Fundamental Theorem for Line Integrals

$$\vec{F} = \nabla f \quad (\text{conservative field})$$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leq t \leq b.$$

$$\begin{aligned} \frac{d}{dt} f(\vec{r}(t)) &= f_x x' + f_y y' + f_z z' \\ &= \nabla f \circ \vec{r}'(t) = \vec{F} \cdot \frac{d\vec{r}}{dt} \end{aligned}$$

$$\text{so } \int_C \vec{F} \cdot d\vec{r} = \int_c \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_a^b \frac{d}{dt} f(\vec{r}(t)) dt.$$

The Fundamental Theorem of Calculus says

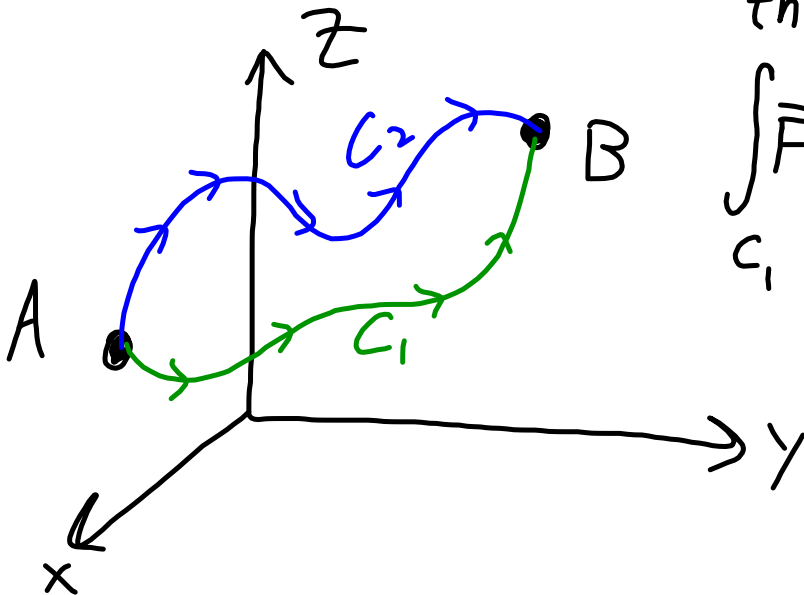
$$\int_a^b \frac{d}{dt} f(\vec{r}(t)) dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Summary:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Fundamental Theorem for line integrals.

Consequence



If \vec{F} is CONSERVATIVE
then

$$\int_{C_1} \vec{F} \cdot d\vec{r}_1 = \int_{C_2} \vec{F} \cdot d\vec{r}_2$$

$$= f(B) - f(A).$$

Path-independent.

Gravity has a conservative force field

$$\vec{F} = \frac{-mMG}{|\vec{r}|^3} \vec{r} ; \vec{r} = \langle x, y, z \rangle.$$

Take $f = \frac{mMG}{|\vec{r}|}$, then $\vec{F} = \nabla f$.

$$f_x = \frac{\partial}{\partial x} \left(\frac{mMG}{(x^2+y^2+z^2)^{1/2}} \right) = -\frac{1}{2} \frac{mMG (2x)}{(x^2+y^2+z^2)^{3/2}} = \frac{-mMG x}{|\vec{r}|^3}.$$

$$f_y = \frac{-mMG}{|\vec{r}|^3} y, \quad f_z = \frac{-mMG}{|\vec{r}|^3} z.$$

EX : Find the work done by gravity in moving a point mass (m) from $(-1, 0, 1)$ to $(1, 1, 2)$.

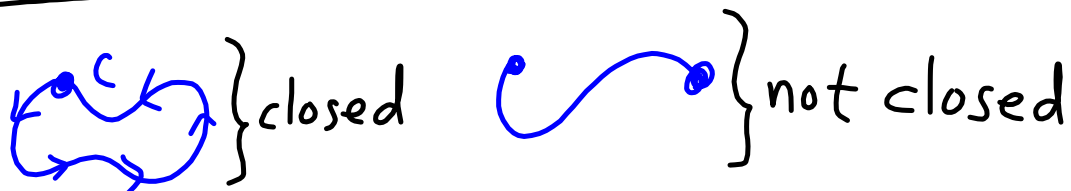
$$W = \int_C \vec{F} \cdot d\vec{r} = f(1, 1, 2) - f(-1, 0, 1) \\ = mMg \left(\frac{1}{\sqrt{1+1+4}} - \frac{1}{\sqrt{1+1}} \right)$$

only need endpoints

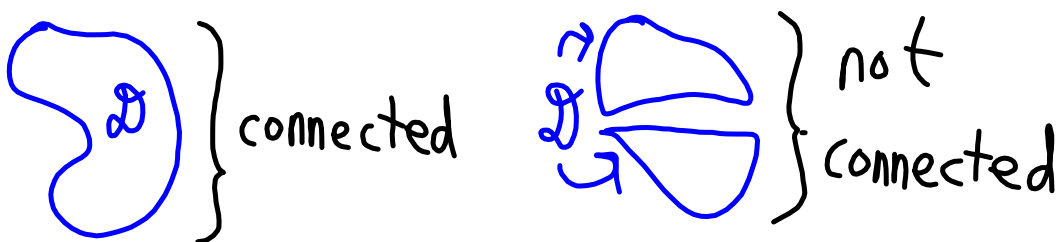
$$= mMg \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{2}} \right)$$

How can we tell if a field is conservative? To discuss this, we need to define some things.

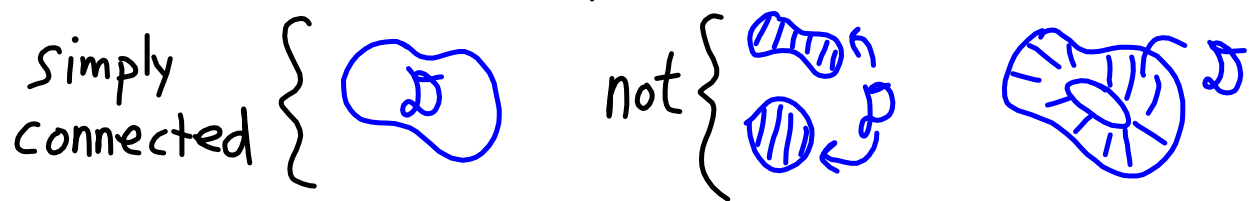
Closed curve: Initial point = end-point



Connected region: You can get from any point A in \mathcal{D} to any other point B in \mathcal{D} without leaving \mathcal{D} .



Simply-connected region: A connected region (say D) such that any simple curve contained in D only bounds subsets of D .



OPEN region D : D does not include any points on the boundary.

$$\{(x,y) \mid x^2 + y^2 < 1\} \leftarrow \text{open}$$

$$\{(x,y) \mid x^2 + y^2 \leq 1\} \leftarrow \text{not open}$$

A nice condition to see if you have a 2D conservative field (3D discussed later on).

If : 1) \mathcal{D} is open & simply-connected
2) $\vec{F} = P\vec{i} + Q\vec{j}$; P, Q are C^1 -functions
3) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on \mathcal{D}

then^{*} \vec{F} is CONSERVATIVE on \mathcal{D} .

* \vec{F} could still be conservative otherwise, but these conditions guarantee it.

EX: Determine if \vec{F} is conservative.

$$a) \vec{F} = \underbrace{xy}_{P} \vec{i} - \underbrace{xy}_{Q} \vec{j} \quad \left. \begin{array}{l} \mathcal{D} \text{ not specified} \\ \Rightarrow \mathcal{D} = \mathbb{R}^2. \end{array} \right\}$$

$$\begin{array}{l} \frac{\partial P}{\partial y} = x \\ \frac{\partial Q}{\partial x} = -y \end{array} \quad \left\{ \begin{array}{l} \text{not always} \\ \text{equal} \end{array} \right. \Rightarrow \vec{F} \text{ not conservative.}$$

$$b) \vec{F} = \underbrace{x^2}_{P} \vec{i} + \underbrace{y^2}_{Q} \vec{j}.$$

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x} \Rightarrow \text{CONSERVATIVE.}$$

$$c) \vec{F} = \underbrace{y}_{P} \vec{i} + \underbrace{x}_{Q} \vec{j}$$

$$\frac{\partial Q}{\partial x} = 1 = \frac{\partial P}{\partial y} \Rightarrow \text{CONSERVATIVE.}$$

How to find f such that $\vec{F} = \nabla f$.

Best to learn by example here...

$$\text{Let } \vec{F} = \underbrace{(y^2 + 2xy - 1)}_{\partial f / \partial x} \vec{i} + \underbrace{(x^2 + 2xy - 1)}_{\partial f / \partial y} \vec{j}$$

Integrate... $\int \frac{\partial f}{\partial x} dx = f(x,y) = xy^2 + x^2y - x + g(y)$

↑ indefinite... we can add ↑
in any $g(y)$ since $\frac{\partial}{\partial x} g(y) = 0$

$$\vec{F} = \underbrace{(y^2 + 2xy - 1)}_{\partial f / \partial x} \vec{i} + \underbrace{(x^2 + 2xy - 1)}_{\partial f / \partial y} \vec{j}$$

So $f(x, y) = xy^2 + x^2y - x + g(y) \dots$

$$\begin{aligned} \text{hence } x^2 + 2xy - 1 &= \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (xy^2 + x^2y - x + g(y)) \\ &= 2xy + x^2 + g'(y) \end{aligned}$$

$$\Rightarrow -1 = g'(y) \Rightarrow -y = g(y)$$

↑ could add + C... let C=0.

Answer: $f(x,y) = xy^2 + x^2y - x - y$.

Summary: For $\vec{F} = P\vec{i} + Q\vec{j} \dots$

- 1) Integrate $\int P dx$
- 2) Add on $g(y)$ arbitrary... $f(x,y) = \int P dx + g(y)$
- 3) Get $g(y)$ by setting
$$Q = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left(\int P dx + g(y) \right); \text{ solve for } g(y).$$

Method in 3D: $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$
 $= f_x\vec{i} + f_y\vec{j} + f_z\vec{k}.$

EX: Let $P = y + z$, $Q = x + z^2$, $R = x + 2yz - 1.$

$$\int (y+z) dx = xy + xz + g(y, z) = f(x, y, z)$$

\uparrow depends on y & z now.

$$Q = x + z^2 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (xy + xz + g(y, z)) = x + \frac{\partial g}{\partial y}.$$

$$R = x + 2yz - 1 = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (xy + xz + g(y, z)) = x + \frac{\partial g}{\partial z}.$$

Solve these for $g_y, g_z \dots$

$$\frac{\partial g}{\partial y} = z^2 \quad \& \quad \frac{\partial g}{\partial z} = 2yz - 1$$

$$\Rightarrow g(y, z) = \int \frac{\partial g}{\partial y} dy = \int z^2 dy = yz^2 + h(z)$$

$$\frac{\partial g}{\partial z} = 2yz - 1 = \frac{\partial}{\partial z} (yz^2 + h(z)) = 2yz + h'(z)$$

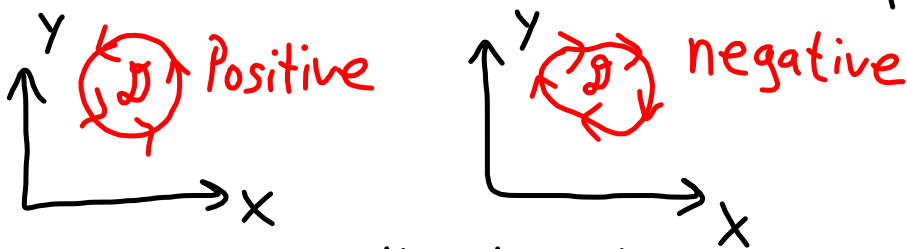
$$\Rightarrow h'(z) = -1 \Rightarrow h(z) = -z$$

$$\Rightarrow g = yz^2 - z$$

$$\Rightarrow f(x, y, z) = xy + xz + g = xy + xz + yz^2 - z$$

Green's Theorem: allow you to switch between line integrals over a closed curve and double integrals over the region bounded by the curve.

The curve must be **POSITIVELY** oriented.



If an observer walks along the curve in the indicated direction, positive orientation means \iint is to the "left".

Green's Theorem: Let C be a curve that

1) is positively oriented

2) simple, closed

3) piecewise-smooth

4) encloses a region D .

If P, Q are C^1 on D , then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Implication for conservative fields:

$$\vec{F} = P\vec{i} + Q\vec{j} = \nabla f$$

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

$$\Rightarrow \int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0.$$

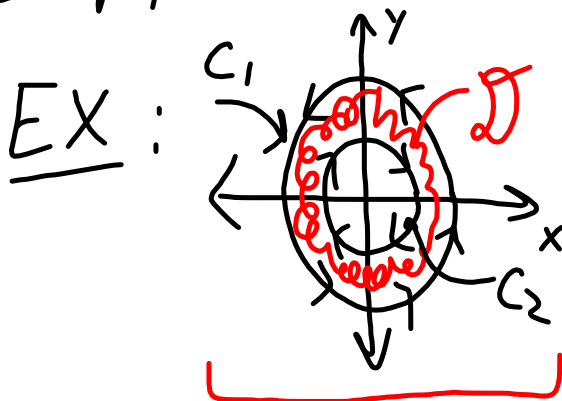
A line integral with a conservative field over a simple closed curve is always zero.

This also follows from the Fundamental Theorem for line integrals...

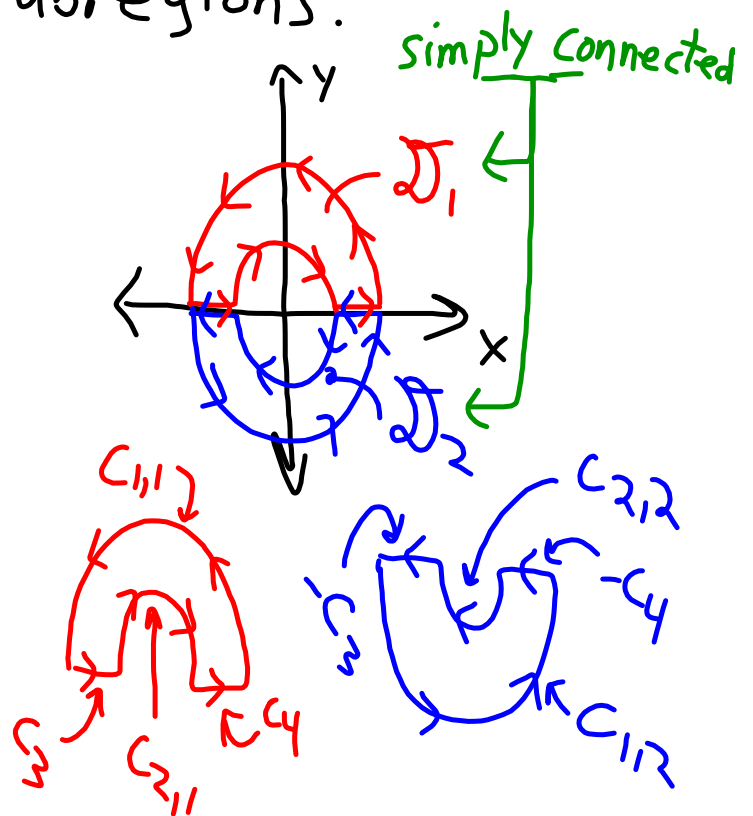
$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) \quad \left. \vphantom{\int_C \vec{F} \cdot d\vec{r}} \right\} \begin{array}{l} B=A \text{ for a closed} \\ \text{curve.} \end{array}$$

Extensions are often possible for Green's to more complicated domains.

Consider when you can break \mathcal{D} into simply-connected subregions.



Not simply connected



$$\boxed{\int_D (Q_x - P_y) dA} = \int_{D_1} (Q_x - P_y) dA + \int_{D_2} (Q_x - P_y) dA$$

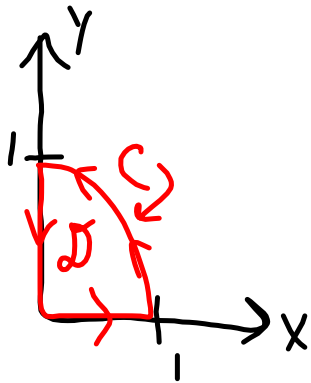
$$= \int_{C_{1,1}} P dx + Q dy + \int_{C_{2,1}} P dx + Q dy + \int_{C_3} P dx + Q dy + \int_{C_4} P dx + Q dy$$

These cancel

$$+ \int_{C_{1,2}} P dx + Q dy + \int_{C_{2,2}} P dx + Q dy + \int_{-C_3} P dx + Q dy + \int_{-C_4} P dx + Q dy$$

$$= \int_{C_1} P dx + Q dy + \int_{C_2} P dx + Q dy = \boxed{\int_C P dx + Q dy}$$

EX: Often a line integral may be tedious so we apply Green's... let D be the region by $x \geq 0, y \geq 0, x^2 + y^2 \leq 1$ and C is the boundary, positively oriented. Find $\int_C xy dx + y^2 dy$.



$$\begin{aligned} \text{Green's} \Rightarrow \int_C xy dx + y^2 dy &= \iint_D (Q_x - P_y) dA \\ &= \iint_D (-x) dA \quad (\text{polar}) \end{aligned}$$

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \int_0^1 (-r \cos \theta) r dr d\theta \\ &= - \left(\int_0^{\frac{\pi}{2}} \cos \theta d\theta \right) \left(\int_0^1 r^2 dr \right) \\ &= - \sin \theta \Big|_0^{\frac{\pi}{2}} \cdot \frac{1}{3} r^3 \Big|_0^1 = \boxed{-\frac{1}{3}}. \end{aligned}$$

EX: Let $f(x,y) = xy(1-x^2-y^2)$.

Find $\iint_{\mathcal{D}} \frac{\partial f}{\partial x} dA$ if \mathcal{D} is the region bounded

by $x^2+y^2=1$. So think $\iint_{\mathcal{D}} \frac{\partial f}{\partial x} dA = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

or $Q=f, P=0$

$$\Rightarrow \iint_{\mathcal{D}} \frac{\partial f}{\partial x} dA = \int_C P dx + Q dy = \int_C f dy, \text{ where}$$

$$x^2+y^2=1 \text{ on } C \Rightarrow f=0 \text{ on } C.$$

Area: Let D be bounded by a simple, closed curve C . Then

$$|D| = \iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,

e.g.

$$Q = x \quad \& \quad P = 0$$

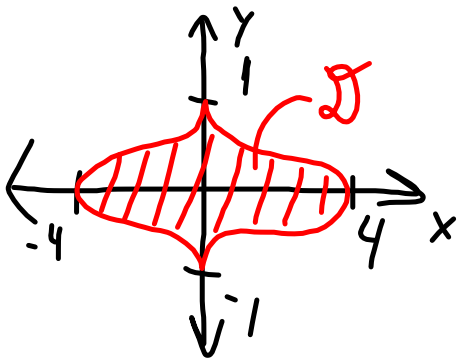
$$Q = 0 \quad \& \quad P = -y$$

$$Q = \frac{1}{2}x \quad \& \quad P = -\frac{1}{2}y$$

Green's

$$\begin{aligned} |D| &= \int_C P dx + Q dy \\ &= \int_C x dy = - \int_C y dx \\ &= \frac{1}{2} \int_C x dy - y dx. \end{aligned}$$

EX: Find the area enclosed by the curve $\vec{r}(t) = \langle 4\cos^3(t), \sin(t) \rangle, 0 \leq t \leq 2\pi$.



$$|D| = \iint_D dA \quad ? \quad \text{Use Green's}$$

$$|D| = \int_C x dy \quad \left. \begin{array}{l} x = 4\cos^3(t) \\ y = \sin(t) \\ dy = y'(t) dt \\ = \cos(t) dt \end{array} \right\}$$

$$\Rightarrow |D| = \int_0^{2\pi} 4\cos^3(t) \cdot \cos(t) dt$$

We've seen this before...

$$\cos^4(t) = \frac{1}{4} \left(\frac{3}{2} + 2\cos(2t) + \frac{1}{2}\cos(4t) \right)$$

$$\begin{aligned} \Rightarrow |D| &= \int_0^{2\pi} \left(\frac{3}{2} + 2\cos(2t) + \frac{1}{2}\cos(4t) \right) dt \\ &= \frac{3}{2} \int_0^{2\pi} dt + 2 \int_0^{2\pi} \cos(2t) dt + \frac{1}{2} \int_0^{2\pi} \cos(4t) dt \\ &\Rightarrow |D| = \frac{3}{2} \cdot 2\pi = \boxed{3\pi}. \end{aligned}$$

Practice!

(#1) Is $\vec{F} = \cos(y)\vec{i} - (x\sin(y)+1)\vec{j}$ conservative? If so, find f such that $\vec{F} = \nabla f$ and calculate

$\int_C \vec{F} \cdot d\vec{r}$, where C is a simple curve from $(0,0)$ to (π, π) .

$$P = \cos(y) \quad , \quad Q = -x \sin(y) - 1$$

$$\frac{\partial Q}{\partial x} = -\sin(y) = \frac{\partial P}{\partial y} \Rightarrow \text{conservative.}$$

$$\int \cos(y) dx = x \cos(y) + g(y) = f(x, y)$$

$$\begin{aligned} \Rightarrow -x \sin(y) - 1 &= \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x \cos(y) + g(y)) \\ &= -x \sin(y) + g'(y) \end{aligned}$$

$$\Rightarrow -1 = g'(y) \Rightarrow -y = g(y)$$

$$\Rightarrow \boxed{f(x, y) = x \cos(y) - y}.$$

$$\int_C \vec{F} \cdot d\vec{r} = f(\pi, \pi) - f(0, 0)$$
$$= \pi \cos(\pi) - \pi - 0$$
$$= \boxed{-2\pi}$$

(#2) Find $\int_C (x+y)dx + x^2y dy$ if C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$, positively oriented.

Method 1:

Apply Green's...

$$P = x+y$$

$$Q = x^2y$$

$$\iint_D 2xy - 1 dA$$

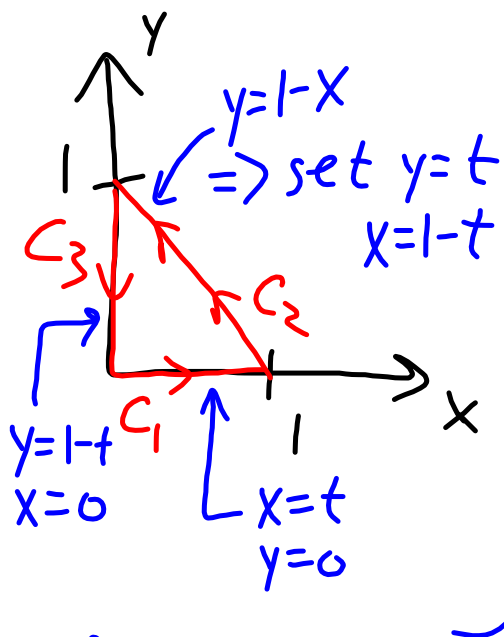
$$= \int_0^1 \int_0^{1-x} 2xy - 1 dy dx$$

$$\int_0^1 \int_0^{1-x} 2xy - 1 \, dy \, dx$$

$$= \int_0^1 \left[xy^2 - y \right]_0^{1-x} dx = \int_0^1 x(1-x)^2 - (1-x) dx$$

$$= \int_0^1 x - 2x^2 + x^3 - 1 + x dx$$

$$= \int_0^1 x + x - 2x^2 - 1 dx = \left[\frac{1}{4}x^4 + x^2 - \frac{2}{3}x^3 - x \right]_0^1$$
$$= \frac{1}{4} + 1 - \frac{2}{3} - 1 = \frac{-5}{12}.$$



Method 2: line integral.

These parameterizations work for $0 \leq t \leq 1$; they give proper orientation.

$$\int_C (x+y)dx + x^2y dy = \int_{C_1} (\dots) + \int_{C_2} (\dots) + \int_{C_3} (\dots)$$

$$\int_{C_1} (x+y)dx + x^2ydy = \int_0^1 t dt = \frac{1}{2} t^2 \Big|_0^1 = \frac{1}{2}$$

$\begin{matrix} dx = x'(t)dt \\ = dt \end{matrix}$
 $\begin{matrix} y=0 \end{matrix}$

$$\int_{C_3} (x+y)dx + x^2ydy = 0$$

$\begin{matrix} dx=0 \\ x=0 \end{matrix}$

$$\int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{1}{2} - \frac{11}{2} = \frac{-5}{2}$$

$$\int_{C_2} (x+y)dx + x^2ydy = -\int_0^1 dt + \int_0^1 t(1-t)^2 dt$$

$$= -1 + \int_0^1 (t - 2t^2 + t^3) dt = -1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{-11}{12}$$