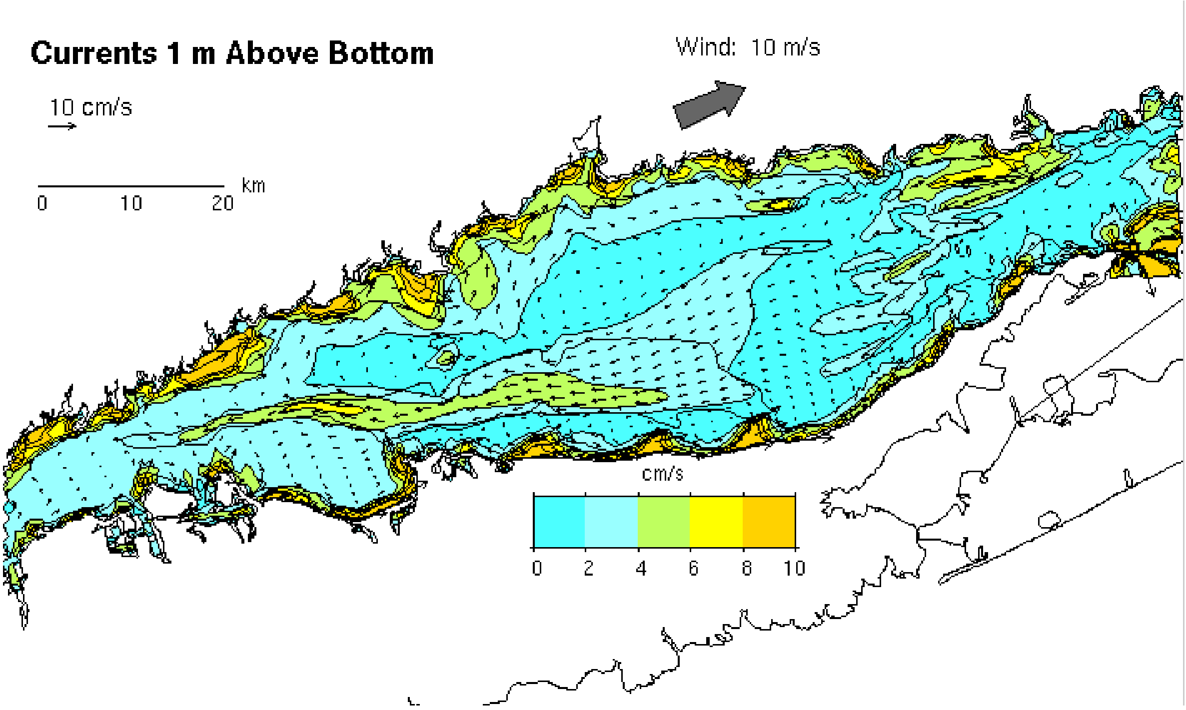
Vector fields - we describe these as vector-valued functions that (1) depend on *n* variables and (2) have *n* components.

$$2D: \vec{F}(x,y) = \langle f_1(x,y), f_2(x,y) \rangle$$

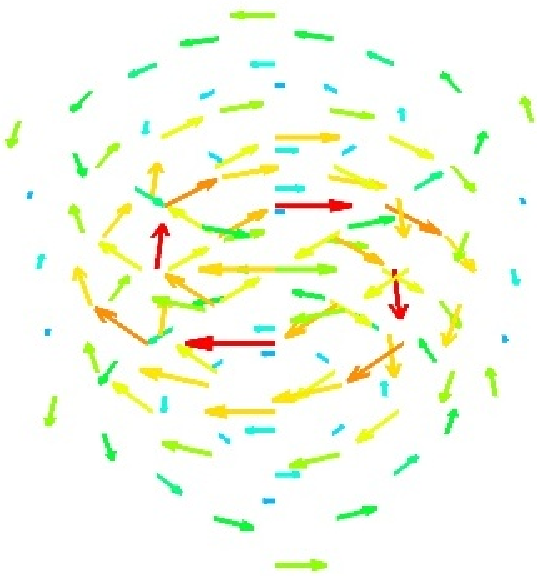
$$3D: \vec{F}(x,y,z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$$

At each point (x,y) there is a vector.



Near-bottom currents in Long Island Sound

Hurricane-type wind pattern in 3D



Representations:

1) $\vec{F} = \langle f_1, f_2, f_3 \rangle$.

2) $\vec{F} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$.

Sketching:

1) Pick some points in the domain.

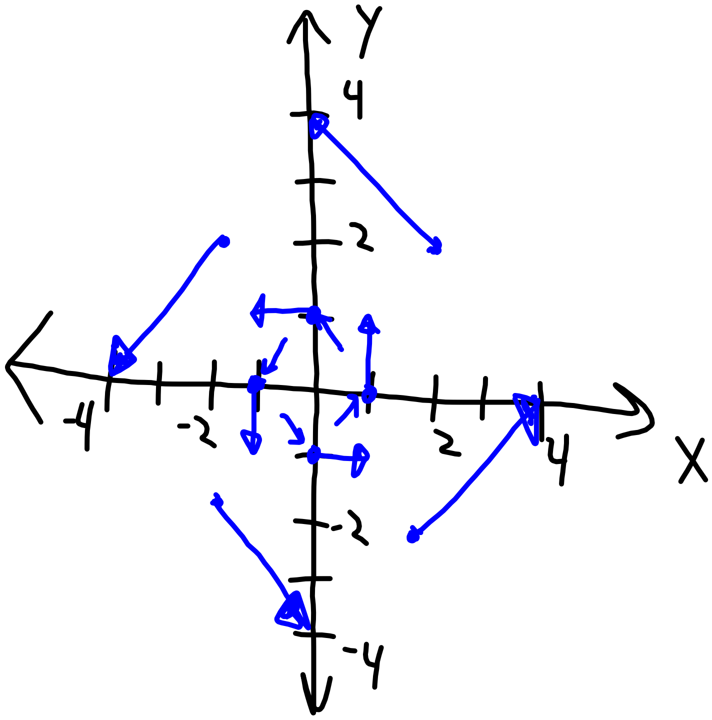
2) Calculate the corresponding vector at each point.

3) Plot each vector with it's tail at that point.

EX: Sketch the vector field

$$\vec{F} = -y\,\vec{i} + x\,\vec{j}\,.$$

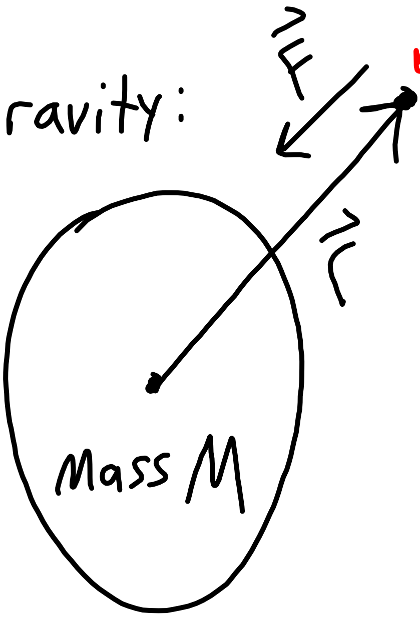
(x,y)	$\vec{F}$
$(1,0)$	$\langle 0,1\rangle$
$(0,1)$	$\langle -1,0\rangle$
$(-1,0)$	$\langle 0,-1\rangle$
$(0,-1)$	$\langle 1,0\rangle$
$\vdots$	$\vdots$
$(\frac{1}{2},\frac{1}{2})$	$\langle -\frac{1}{2},\frac{1}{2}\rangle$
$\vdots$	$\vdots$
$(2,2)$	$\langle -2,2\rangle$



Force fields

Gravity:

mass m at (x, y, z) ; $\vec{r} = \langle x, y, z \rangle$

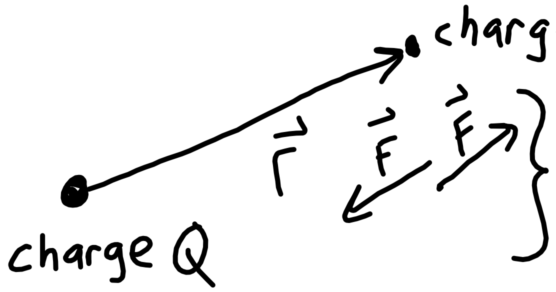


- Direction of $\vec{F}$ is $-\hat{r}$
- $|\vec{F}| = \dfrac{mMG}{|\vec{r}|^2}$
- $\vec{F} = |\vec{F}|(-\hat{r}) = \dfrac{-mMG}{|\vec{r}|^2} \cdot \dfrac{\vec{r}}{|\vec{r}|}$

$$\boxed{\vec{F} = \left(\frac{-mMG}{|\vec{r}|^3}\right)\vec{r}}.$$

$G \approx 6.6738 \, \dfrac{N \cdot m^2}{Kg^2}$.

Electric force field (Coulomb's Law)



$\vec{F}$ depends on sign(qQ)

$$\vec{F} = \frac{\varepsilon q Q}{|\vec{r}|^3}\vec{r}$$

ε is a constant; depends on situation

- $qQ > 0 \Rightarrow$ repulsion, $\vec{F}$ in $\vec{r}$-direction
- $qQ < 0 \Rightarrow$ attraction, $\vec{F}$ in $(-\vec{r})$-direction.

Gradient fields

Recall for $f:\mathbb{R}^3 \to \mathbb{R}$ we have

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

So $\vec{F} = \nabla f$ defines a "gradient vector field".

In this case:

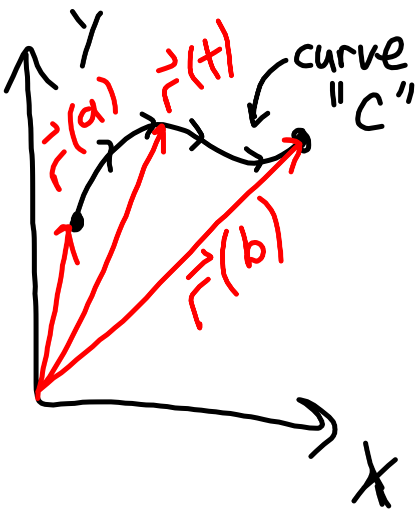
1) f is called a "potential function" for $\vec{F}$.

2) $\vec{F}$ is a "conservative" vector field.

Line integrals

$\vec{r}(t)$: parameterization of a curve $(a \le t \le b)$

s : arc length



Let $f(x,y) = f(x(t), y(t))$
$= f(\vec{r}(t))$ on the curve C.

LINE INTEGRAL w.r.t. arc-length

$$\int_C f(x,y)\,ds = ?$$

Recall $s = \int_a^t |\vec{r}\,'(\tau)|\,d\tau$

$$\Rightarrow \frac{ds}{dt} = |\vec{r}\,'(t)|$$

In terms of differentials: $ds = |\vec{r}\,'(t)|\,dt$

$$\Rightarrow \boxed{\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}.$$

So if $\vec{r} = \langle x(t), y(t) \rangle$

then $\vec{r}\,' = \langle \frac{dx}{dt}, dy/dt \rangle$

$$\Rightarrow \int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

<u>3D</u>: $\int_C f(x,y,z)\,ds =$

$$\int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Application: find the center of gravity for a "wire"

$\rho(x,y)$: density of a wire

$\vec{r}(t)$: parameterization of wire, $a \le t \le b$

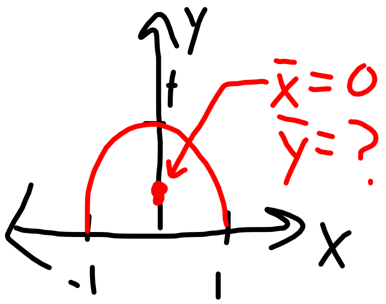
Then $\text{mass} = m = \int_a^b \rho(x(t),y(t))\sqrt{(x')^2+(y')^2}\,dt$

$$\bar{x} = \frac{1}{m}\int_a^b x(t)\,\rho(x(t),y(t))\sqrt{(x')^2+(y')^2}\,dt$$

$$\bar{y} = \frac{1}{m}\int_a^b y(t)\,\rho(x(t),y(t))\sqrt{(x')^2+(y')^2}\,dt.$$

EX: Let a wire have mass density $\rho = 1 - \frac{1}{2}y^2$ and position $x^2+y^2=1$ for $y \geq 0$.

Find the center of mass.



Parameterize the curve!

$x=\cos(t) \quad y=\sin(t)$

$x'=-\sin(t) \quad y'=\cos(t)$

$(x')^2+(y')^2 = \sin^2(t)+\cos^2(t)=1$

$$m = \int_0^{\pi} \left(1-\frac{1}{2}\sin^2(t)\right)\sqrt{1}\,dt = \pi - \frac{1}{2}\int_0^{\pi} \sin^2(t)\,dt$$

$$\Rightarrow m = \pi - \frac{1}{2}\int_0^{\pi} \frac{1}{2}\big(1-\cos(2t)\big)\,dt \quad \text{(Trig. identity)}$$

$$= \pi - \frac{1}{4}\left[\pi - \int_0^{\pi}\cos(2t)\,dt\right] = \frac{3\pi}{4} + \frac{1}{4}\cdot\frac{1}{2}\underbrace{\sin(2t)\Big|_0^{\pi}}_{0}$$

So $\boxed{m = 3\pi/4}$.

$$\bar{y} = \frac{1}{m}\int_0^{\pi}\sin(t)\left(1-\frac{1}{2}\sin^2(t)\right)dt = \frac{4}{3\pi}\left[-\cos(t)\Big|_0^{\pi} - \frac{1}{2}\int_0^{\pi}\sin^3(t)\,dt\right]$$

$$= \frac{4}{3\pi}\left[2 - \frac{1}{2}\int_0^{\pi}\sin(t)\big(1-\cos^2(t)\big)\,dt\right]$$

$$\bar{y} = \frac{4}{3\pi}\left[2 - \frac{1}{2}\int_0^{\pi} \sin(t)\,dt + \frac{1}{2}\int_0^{\pi} \sin(t)\cos^2(t)\,dt\right]$$

$$= \frac{4}{3\pi}\left[2 + \frac{1}{2}\cos(t)\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \frac{-1}{3}\frac{d}{dt}\cos^3(t)\,dt\right]$$

$$= \frac{4}{3\pi}\left[2 - \frac{2}{2} - \frac{1}{6}\left(\cos^3(t)\right)\Big|_0^{\pi}\right]$$

$$\Rightarrow \bar{y} = \frac{4}{3\pi}\left[1 + \frac{1}{3}\right] = \frac{4}{3\pi}\cdot\frac{4}{3} = \boxed{\frac{16}{9\pi}}.$$

(and $\bar{x} = 0$).

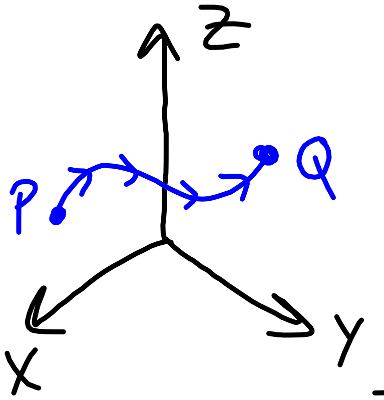
We will sometimes see integrals over a line with respect to x or y ...

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

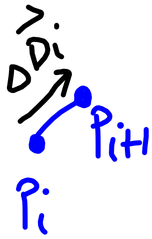
$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Notation: $\int_C f_1\,dx + \int_C f_2\,dy = \int_C f_1\,dx + f_2\,dy$.

WORK: We will calculate work done in moving a particle through a force field from point P to Q.



Consider moving over a small segment of the curve...



Work is $\vec{F}\cdot\vec{D}$, in this case $\approx \vec{F}(t_i)\cdot\Delta\vec{D}_i$

$$\Delta\vec{D}_i \cong \Delta s\,\vec{T}(t_i)$$

unit tangent

arc length

$$\approx \vec{F}(t_i)\cdot\vec{T}(t_i)\Delta s$$

Total work $\approx \sum_i \vec{F}(t_i) \cdot \vec{T}(t_i) \Delta s$

Pass to the limit as $\Delta s \to 0$...

Total work $= W = \int_C \vec{F} \cdot \vec{T} \, ds$

$\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ & $ds = |\vec{r}\,'(t)| dt$

$$\Rightarrow \boxed{W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt}.$$

Now let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\vec{F} = \langle f_1, f_2, f_3 \rangle$$

$$\Rightarrow W = \int_a^b \langle f_1, f_2, f_3 \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

$$= \int_a^b f_1 x'(t) \, dt + \int_a^b f_2 y'(t) \, dt + \int_a^b f_3 z'(t) \, dt$$

$$= \int_C f_1 \, dx + \int_C f_2 \, dy + \int_C f_3 \, dz = \int_C f_1 \, dx + f_2 \, dy + f_3 \, dz.$$

SUMMARY of main formulas

1) $\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

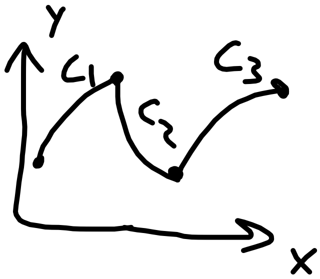
2) $W = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

3) $W = \int_C \vec{F}\cdot d\vec{r} = \int_C f_1\,dx + f_2\,dy + f_3\,dz$.

Two notations, same thing.

Two comments:

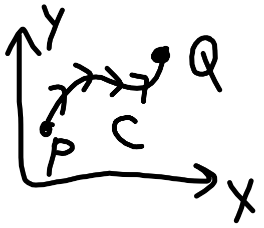
1) If a curve has "kinks", break the integral up into multiple integrals over the smooth parts.



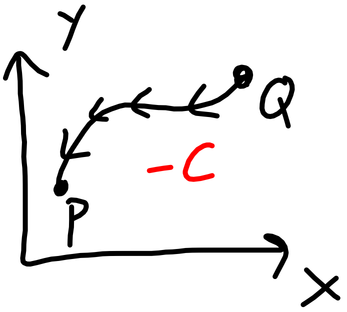
$$C = C_1 \cup C_2 \cup C_3$$

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds\,.$$

2) Direction of motion/"orientation" of a parameterization matters.



Positively oriented if moving from P to Q



Negatively oriented if moving from P to Q

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$$

If your curve is parameterized in the wrong direction you will get a sign error.

EX: Find the work done on a particle when moving from $P(1,0,0)$ to $Q(3,-1,2)$ along a straight line if $\vec{F} = \langle z, x, y \rangle$.

A line segment can be parameterized as follows:

$$\vec{r}(t) = \vec{P} + t\,\overrightarrow{PQ} = \langle 1,0,0 \rangle + t\langle 3-1, -1-0, 2-0 \rangle$$

$$= \langle 1+2t, -t, 2t \rangle, \text{ for } 0 \le t \le 1.$$

Then $\vec{r}\,'(t) = \overrightarrow{PQ} = \langle 2, -1, 2 \rangle$.

Plug into work formula...

$$W = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^1 \langle 2t, 1+2t, -t \rangle \cdot \langle 2, -1, 2 \rangle\,dt$$

$$= \int_0^1 (4t - 1 - 2t - 2t)\,dt = \int_0^1 (-1)\,dt = \boxed{-1}.$$

EX: Let a particle move along the path $y=2x^2-1$ from $(0,-1)$ to $(2,7)$. Find the work done if it is acted upon by a force $\vec{F}=-y\vec{i}+x\vec{j}$.

Parameterize: $y=y(x)$ so choose $\begin{cases} x=t \\ y=2t^2-1 \end{cases}$

with $0 \leq t \leq 2$.

$\vec{r}(t)=\langle t, 2t^2-1\rangle \Rightarrow \vec{r}\,'(t)=\langle 1, 4t\rangle.$

$$W = \int_0^2 \langle 1-2t^2, t \rangle \cdot \langle 1, 4t \rangle \, dt = \int_0^2 1-2t^2+4t^2 \, dt$$

$$W = \int_0^1 1+2t^2 \, dt = 2 + \frac{2}{3} t^3 \Big|_0^2 = \frac{6}{3} + \frac{16}{3} = \boxed{\frac{22}{3}}.$$

Practice!

#1 Let a particle move along a straight path from $(-5,-4)$ to $(1,2)$ subject to a force $\vec{F} = y^2\vec{i} + x^2\vec{j}$.

Find the work done.

$$\vec{r}(t) = \langle -5 + t(1+5), -4 + t(2+4) \rangle = \langle 6t-5, 6t-4 \rangle.$$

$$\vec{r}\,'(t) = 6\langle 1, 1 \rangle, \quad 0 \le t \le 1.$$

$$W = \int_0^1 \langle (6t-4)^2, (6t-5)^2 \rangle \cdot \langle 1,1 \rangle 6\, dt$$

$$= 6\int_0^1 (6t-4)^2 + (6t-5)^2\, dt = 6\int_0^1 6^2\left(t-\tfrac{2}{3}\right)^2 + 6^2\left(t-\tfrac{5}{6}\right)^2 dt$$

$$= 6^3 \int_0^1 (t-2/3)^2 + (t-5/6)^2\, dt = \frac{6^3}{3}\left[\left(t-\tfrac{2}{3}\right)^3 + \left(t-\tfrac{5}{6}\right)^3\right]_0^1$$

$$= \frac{6^3}{3}\left(\frac{1}{3^3} + \frac{1}{6^3} + \frac{8}{3^3} + \frac{125}{6^3}\right)$$

$$= \frac{6^3}{3}\left(\frac{3^2}{3^3} + \frac{126}{6^3}\right) = \frac{6^3}{3}\left(\frac{1}{3} + \frac{7}{12}\right)$$

$$= \frac{6^3}{3} \cdot \frac{11}{12} = \frac{6 \cdot 36 \cdot 11}{36} = \boxed{66}.$$

#2 Let $\vec{F} = -y\vec{i} + x\vec{j}$. Calculate the work done to move a particle from $(1,0)$ around the circle $x^2+y^2=1$ and back to $(1,0)$.

$x=\cos(t) \quad y=\sin(t), \quad 0\le t\le 2\pi$

$x'=-\sin(t) \quad y'=\cos(t)$

$$W=\int_0^{2\pi}\langle -\sin(t),\cos(t)\rangle\cdot\langle -\sin(t),\cos(t)\rangle\,dt$$

$$=\int_0^{2\pi}\cos^2(t)+\sin^2(t)\,dt=\int_0^{2\pi}dt=\boxed{2\pi.}$$