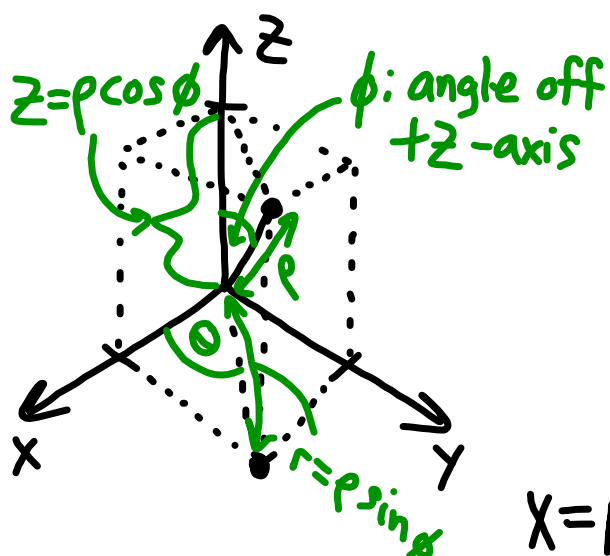


Spherical coordinates



ρ : distance to origin

θ : angle off +x-axis in the xy-plane

ϕ : angle off +z-axis

$$0 \leq \rho, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi.$$

Claim: $x^2 + y^2 + z^2 = \rho^2$.

Proof: $x^2 + y^2 + z^2 = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta$
 $+ \rho^2 \cos^2 \phi$
 $= \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + \rho^2 \cos^2 \phi$
 $= \rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi$
 $= \rho^2 (\sin^2 \phi + \cos^2 \phi) = \rho^2$.

□

EX: Given $(-\sqrt{3/2}, -\sqrt{3/2}, -1)$ in Cartesian coordinates, find the spherical coordinates.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\frac{3}{2} + \frac{3}{2} + 1} = \sqrt{4} = 2 = \rho$$

$$z = \rho \cos \phi = 2 \cos \phi = -1 \Rightarrow \cos \phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$$

$$x = \rho \cos \theta \sin \phi = 2 \cos \theta \sin\left(\frac{2\pi}{3}\right) = \cos \theta \sqrt{3} = \frac{-\sqrt{3}}{\sqrt{2}}$$

$$\text{So } \cos \theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4} \text{ or } \theta = \frac{5\pi}{4}$$

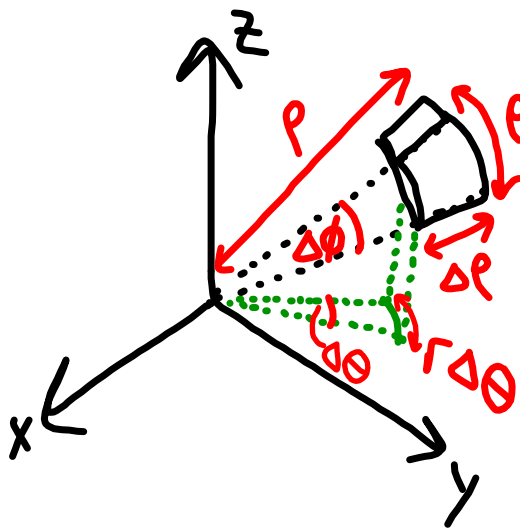
$$y = -\frac{\sqrt{3}}{\sqrt{2}} = \rho \sin \phi \sin \theta = 2 \frac{\sqrt{3}}{2} \sin \theta$$

$$\Rightarrow -\frac{1}{\sqrt{2}} = \sin \theta, \text{ thus}$$

$$\theta = \frac{5\pi}{4}$$

$$\text{since } \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Differential volume: think about a small change in each of the coordinates $\Delta\rho$, $\Delta\theta$, $\Delta\phi$



$$\begin{aligned} \Delta V &= (\rho \Delta\phi)(\Delta\rho)(r \Delta\theta) \\ &= (\rho \Delta\phi)(\Delta\rho)(\rho \sin\phi \Delta\theta) \\ &= \rho^2 \sin\phi \Delta\rho \Delta\theta \Delta\phi \end{aligned}$$

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Integration formula:

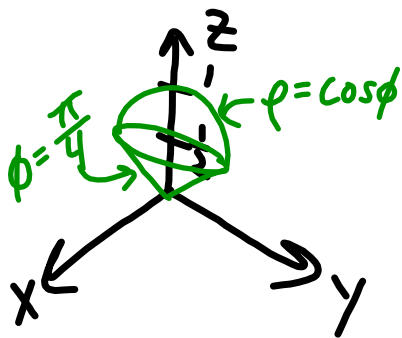
$$\iiint_V f(x, y, z) dV = \iiint_V f(\rho, \theta, \phi) \rho^2 \sin \phi d\rho d\theta d\phi.$$

Ex: Derive the formula for the volume of a sphere of radius $\rho = r$.

$$|V| = \iiint_V dV = \int_0^{2\pi} \int_0^{\pi} \int_0^r \rho^2 \sin \phi d\rho d\phi d\theta$$

$$\begin{aligned}
 \Rightarrow |V| &= \int_0^{2\pi} \int_0^{\pi} \int_{\rho=0}^{\rho=r} \frac{1}{3} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^{\pi} \frac{1}{3} r^3 \sin\phi \, d\phi \, d\theta \\
 &= \frac{1}{3} r^3 \int_0^{2\pi} (-\cos\phi) \Big|_0^{\pi} d\theta \\
 &= \frac{1}{3} r^3 \int_0^{2\pi} 2 \, d\theta = \frac{4\pi r^3}{3} .
 \end{aligned}$$

EX: Find the volume of a solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below $x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$.



Think "spherical"

radius $\frac{1}{2}$

$$\text{Cone: } z = \sqrt{\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)}$$

$$= \rho \sin \phi = \rho \cos \phi$$

$$\Rightarrow \sin \phi = \cos \phi \Rightarrow \phi = \frac{\pi}{4}$$

Upper boundary...

$$x^2 + y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4} \Rightarrow \rho^2 = z = \rho \cos \phi$$

$$\Rightarrow \rho = \cos \phi$$

"cone"

Limits of integration: $0 \leq \theta \leq 2\pi$
 $0 \leq \rho \leq \cos \phi$
 $0 \leq \phi \leq \pi/4$

$$|V| = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3} \cos^3 \phi \sin \phi \, d\phi \, d\theta$$

$$\int_0^{2\pi} \int_0^{\pi/4} -\frac{1}{12} \frac{\partial}{\partial \phi} \cos^4 \phi \, d\phi \, d\theta = \int_0^{2\pi} -\frac{1}{12} \left[\left(\frac{1}{\sqrt{2}}\right)^4 - 1 \right] d\theta$$

$$\Rightarrow |V| = 2\pi \left(-\frac{1}{12}\right) \left(-\frac{3}{4}\right) = \frac{3\pi}{24} = \boxed{\frac{\pi}{8}}.$$

Change of variables

Recall $\int_a^b f(x)dx$ is sometimes evaluated

(Chain Rule)

via a change $x = g(u) \Rightarrow dx = g'(u)du$

If g^{-1} exists,

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)du.$$

} Extend
to multiple
variables?

Transformations

We want some $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$T(u, v) = (x, y)$$

$$\left. \begin{array}{l} x = g(u, v) \\ y = h(u, v) \end{array} \right\} \text{we often write}$$

$$\begin{array}{l} x = x(u, v) \\ y = y(u, v). \end{array}$$

T : "transformation"

We can talk about the partial derivatives

$$\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$$

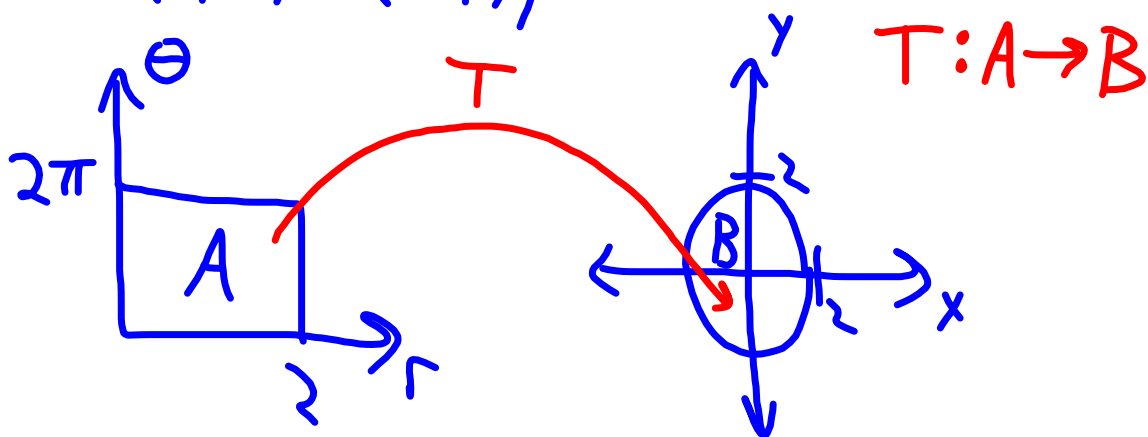
When these are all CTS, we say
"T is a C^1 -transformation".

We will work with C^1 -transformations.

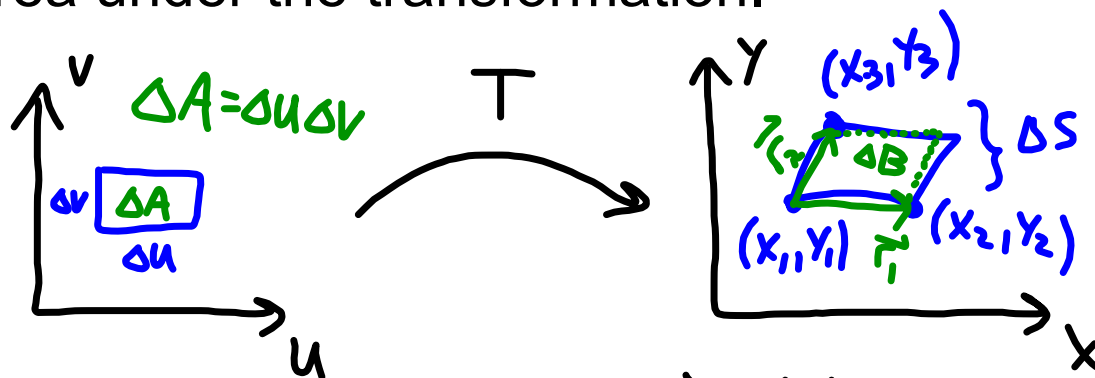
EX: Find a transformation T from a rectangle to $B = \{(x, y) \mid x^2 + y^2 \leq 4\}$.

Take $x = r \cos \theta$, $y = r \sin \theta$

$$T(r, \theta) = (x, y)$$



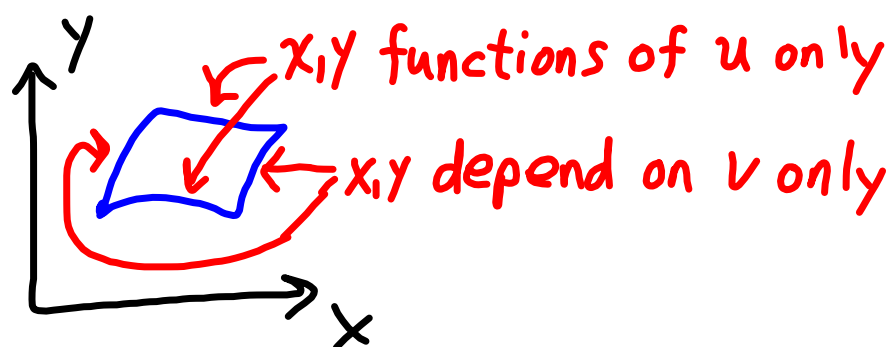
Let's sketch out a derivation for the differential of area under the transformation.



The region $\Delta S \approx \Delta B = |\vec{\Gamma}_1 \times \vec{\Gamma}_2|$

$$\vec{\Gamma}_1 = \langle x_2 - x_1, y_2 - y_1 \rangle$$

$$\vec{\Gamma}_2 = \langle x_3 - x_1, y_3 - y_1 \rangle$$



So, e.g., $x_2 - x_1 \approx \frac{\partial x}{\partial u} \cdot \Delta u$ (Mean Value Thm.)

$y_2 - y_1 \approx \frac{\partial y}{\partial u} \cdot \Delta u$

$x_3 - x_1 \approx \frac{\partial x}{\partial v} \cdot \Delta v$

$y_3 - y_1 \approx \frac{\partial y}{\partial v} \cdot \Delta v$

$$\Rightarrow \vec{r}_1 = \Delta u \langle X_u, Y_u \rangle$$

$$\vec{r}_2 = \Delta v \langle X_v, Y_v \rangle$$

$$\vec{r}_1 \times \vec{r}_2 = \Delta u \Delta v \begin{vmatrix} X_u & Y_u \\ X_v & Y_v \end{vmatrix} \left. \begin{array}{l} \text{2D-cross product} \\ \text{equal} \end{array} \right\}$$

* Notation: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} X_u & X_v \\ Y_u & Y_v \end{vmatrix} = X_u Y_v - X_v Y_u$

is called the "Jacobian of T"

$$\text{Thus, } \Delta S \approx \Delta B = |\vec{r}_1 \times \vec{r}_2| = \Delta u \Delta v \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$\Rightarrow \Delta B = \Delta A \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

($\Delta A = \delta u \cdot \delta v$)

$$\Rightarrow dx dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

$$\iint_S f(x,y) dx dy = \iint_A f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv.$$

$S \xleftarrow{T} A \xrightarrow{\quad} S$ — A

"Change of variables formula"

As usual, there are restrictions. Some conditions to look out for:

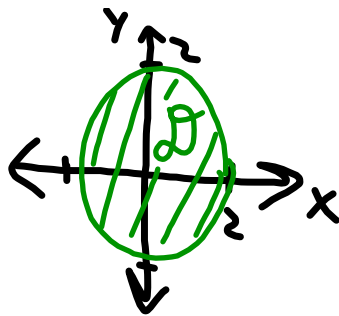
* f is CTS.

* T is C^1

* Jacobian $\neq 0$

Two uses for a change of variables: 1) work with an easier domain of integration and 2) simplify the complexity of the integrand.

EX: Find $\iint_D x^2 + y^2 dA$,



Polar: $x = r \cos \theta$

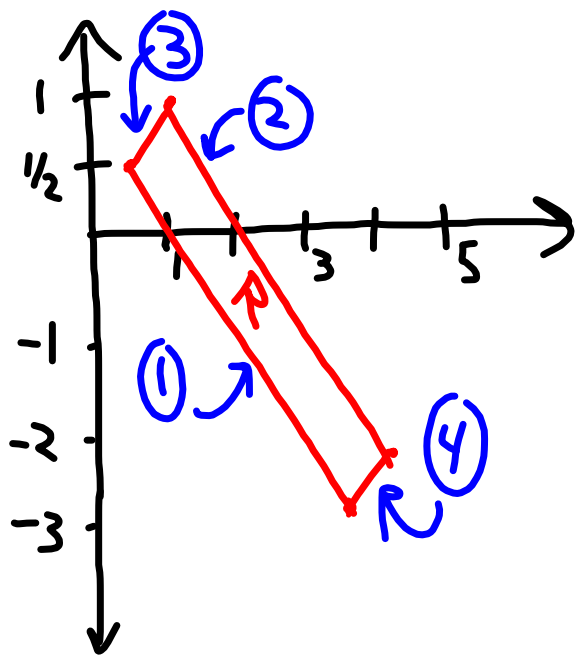
$y = r \sin \theta$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} X_r & X_\theta \\ Y_r & Y_\theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r(\cos^2 \theta + \sin^2 \theta) = r$$

$$\Rightarrow \iint_D x^2 + y^2 dx dy = \int_0^{2\pi} \int_0^2 r^2 \cdot r dr d\theta = 2\pi \cdot \frac{1}{4} r^4 \Big|_0^2 = 8\pi.$$

EX: Let R be the rectangle in the xy -plane with corners $(\frac{1}{2}, \frac{1}{2})$, $(1, 1)$, $(\frac{1}{2} + \pi, \frac{1}{2} - \pi)$, $(1 + \pi, 1 - \pi)$. Find $\iint_R \frac{x-y}{x+y} dA$.



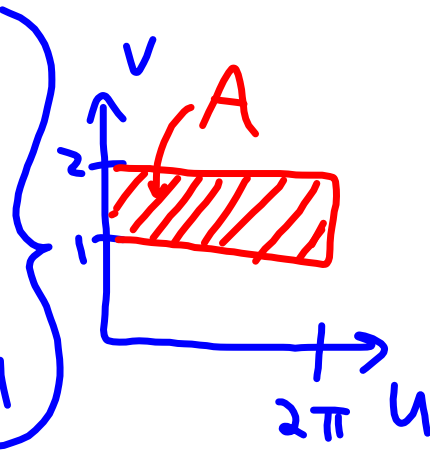
Idea: $u = x - y$
 $v = x + y$

① $y = 1 - x$ ② $y = 2 - x$

③ $y = x$ ④ $y = x - 2\pi$

$x+y \neq 0$
over R

Note

$$\begin{aligned} \textcircled{1} &\Rightarrow x+y=1=v \\ \textcircled{2} &\Rightarrow x+y=2=v \\ \textcircled{3} &\Rightarrow x-y=0=u \\ \textcircled{4} &\Rightarrow x-y=2\pi=u \end{aligned}$$


$$T(u,v) = (x,y) \Rightarrow T: A \rightarrow \mathbb{R}^2.$$

Need $x = x(u,v)$ $y = y(u,v)$... "invert" the

equations

$$\begin{aligned} u &= x-y &\Rightarrow & u-v = -2y &\Rightarrow & x = \frac{u+v}{2} \\ v &= x+y &\Rightarrow & u+v = 2x &\Rightarrow & y = \frac{v-u}{2} \end{aligned}$$

$$\text{Jacobian: } \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

(Note $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$, T is C^1 ...)

$$\iint_R \frac{x-y}{x+y} dA = \int_1^2 \int_0^{2\pi} \frac{u}{v} \cdot \frac{1}{2} \cdot du dv = \frac{1}{2} \int_1^2 \left. \frac{1}{2} \frac{u^2}{v} \right|_{u=0}^{u=2\pi} dv$$

$$= \frac{1}{4} \int_1^2 \frac{1}{v} (4\pi^2) dv = \pi^2 \ln(2).$$

Change of variables for triple integrals

Let $T: S \rightarrow R$ in \mathbb{R}^3 ; $T(u, v, w) = (x, y, z)$.

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$\iiint_R f(x, y, z) dV = \iiint_S f(u, v, w) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw.$$

Spherical coordinates...

$$x = \rho \cos \theta \sin \phi \quad y = \rho \sin \theta \sin \phi \quad z = \rho \cos \phi$$

$$X_\rho = \cos \theta \sin \phi \quad X_\theta = -\rho \sin \theta \sin \phi \quad X_\phi = \rho \cos \theta \cos \phi$$

$$Y_\rho = \sin \theta \sin \phi \quad Y_\theta = \rho \cos \theta \sin \phi \quad Y_\phi = \rho \sin \theta \cos \phi$$

$$Z_\rho = \cos \phi \quad Z_\theta = 0 \quad Z_\phi = -\rho \sin \phi$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \cos \theta \sin \phi \begin{vmatrix} \rho \cos \theta \sin \phi & \rho \sin \theta \cos \phi \\ 0 & -\rho \sin \phi \end{vmatrix}$$

$$+ \rho \sin \theta \sin \phi \begin{vmatrix} \sin \theta \sin \phi & \rho \sin \theta \cos \phi \\ \cos \phi & -\rho \sin \phi \end{vmatrix} + \rho \cos \theta \cos \phi \begin{vmatrix} \sin \theta \sin \phi & \rho \cos \theta \sin \phi \\ \cos \phi & 0 \end{vmatrix}$$

$$\begin{aligned}
&= \cos\theta \sin\phi (-\rho^2 \cos\theta \sin^2\phi) \\
&+ \rho \sin\theta \sin\phi (-\rho \sin\theta \sin^2\phi - \rho \sin\theta \cos^2\phi) \\
&+ \rho \cos\theta \cos\phi (-\rho \cos\theta \sin\phi \cos\phi) \\
&= -\rho^2 \sin^3\phi \cos^2\theta - \rho^2 \sin^2\theta \sin\phi \\
&\quad - \rho^2 \cos^2\theta \sin\phi \cos^2\phi \\
&= -\rho^2 \sin\phi [\sin^2\phi \cos^2\theta + \sin^2\theta + \cos^2\theta \cos^2\phi]
\end{aligned}$$

$$\begin{aligned} &= -\rho^2 \sin\phi (\cos^2\theta + \sin^2\theta) \\ &= -\rho^2 \sin\phi \end{aligned}$$

$$\text{Thus } \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| = |-\rho^2 \sin\phi| = \rho^2 \sin\phi$$

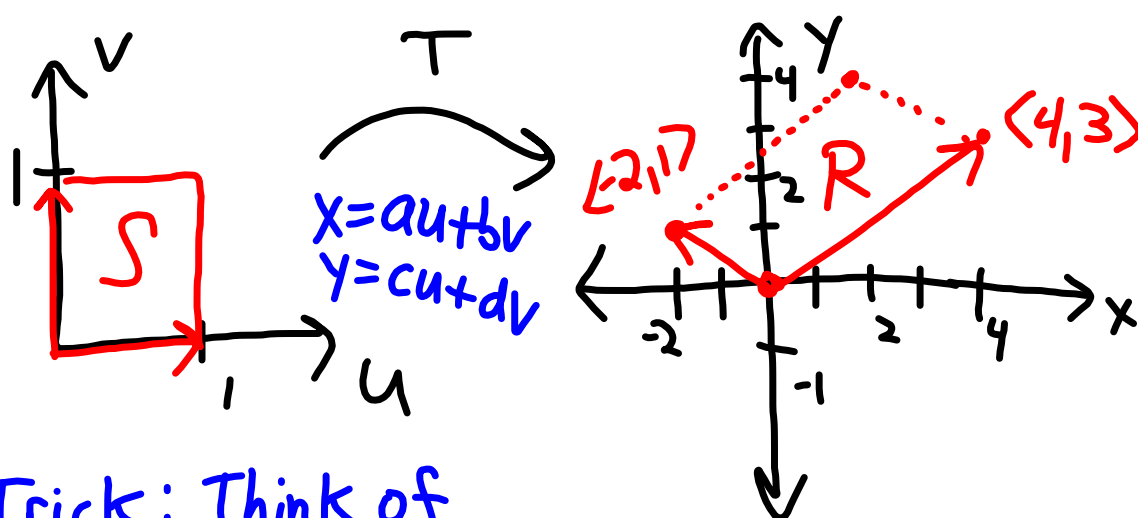
($\sin\phi > 0$)

$$\Rightarrow \int \int \int_V f(x,y,z) dV = \int \int \int_V f(\rho,\theta,\phi) \rho^2 \sin\phi d\rho d\theta d\phi.$$

Practice!

① Let R be a parallelogram with vertices $(0,0)$, $(4,3)$, $(2,4)$, $(-2,1)$.

Find a transformation $T: S \rightarrow R$ where S is a rectangular region in the uv -plane.



Trick: Think of mapping $\langle 1, 0 \rangle \xrightarrow{T} \langle 4, 3 \rangle$ & $\langle 0, 1 \rangle \xrightarrow{T} \langle -2, 1 \rangle$.
 (Idea from linear algebra.)

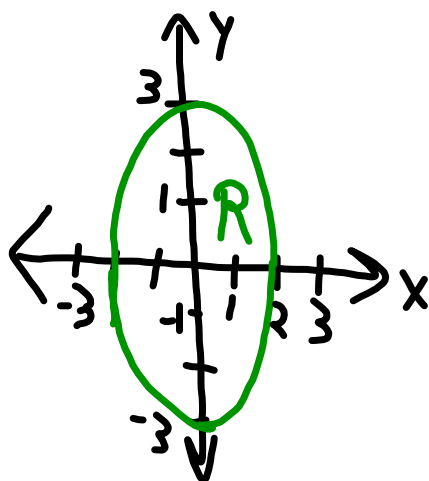
$$\begin{aligned}T(1,0) &= (a(1)+b(0), c(1)+d(0)) \\ &= (a, c) = (4, 3)\end{aligned}$$

$$\begin{aligned}T(0,1) &= (a(0)+b(1), c(0)+d(1)) \\ &= (b, d) = (-2, 1).\end{aligned}$$

$$\text{Thus } T(u,v) = (4u - 2v, 3u + v).$$

$$\begin{aligned}\text{*Check: } T(0,0) &= (0,0), T(1,0) = (4,3), T(0,1) = (-2,1) \\ T(1,1) &= (2,4). \checkmark\end{aligned}$$

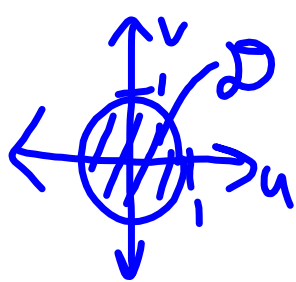
#2 Find $\iint_R x^2 dA$... R bounded by
the ellipse $9x^2 + 4y^2 = 36$. Use $\begin{cases} x=2u \\ y=3v \end{cases}$.



Plug $x=2u$, $y=3v$ into ellipse-eg'n...

$$9(2u)^2 + 4(3v)^2 = 36 \Rightarrow 36u^2 + 36v^2 = 36$$

$$\Rightarrow u^2 + v^2 = 1 \text{ (unit circle).}$$



$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 2 \cdot 3 = 6 \neq 0$$

$$\iint_R x^2 dA = \iint_D 4u^2 \cdot 6 \cdot dudv = 24 \iint_D u^2 dudv$$

$D \leftarrow \text{polar}$

$$u = r \cos \theta$$

$$v = r \sin \theta$$

$$\Rightarrow \iint_R x^2 dA = 24 \int_0^{2\pi} \int_0^1 r^2 \cos^2 \theta r dr d\theta$$

$$= 24 \int_0^{2\pi} \cos^2 \theta \left. \frac{1}{4} r^4 \right|_0^1 d\theta = \frac{24}{4} \int_0^{2\pi} \cos^2 \theta d\theta$$

$$= \boxed{6\pi}$$

$$= \pi$$

Trick to remember $\int_0^{2\pi} \cos^2 \theta d\theta = \pi$:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\int_0^{2\pi} \sin^2 \theta d\theta + \int_0^{2\pi} \cos^2 \theta d\theta = \int_0^{2\pi} 1 d\theta = 2\pi$$

same
area under curves

$$\Rightarrow 2 \int_0^{2\pi} \cos^2 \theta d\theta = 2\pi \Rightarrow \int_0^{2\pi} \cos^2 \theta d\theta = \pi.$$

$$= \int_0^{2\pi} \sin^2 \theta d\theta$$