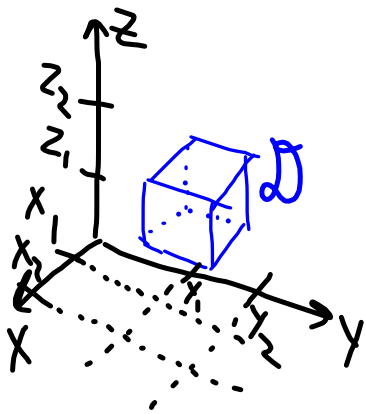


## Triple integrals

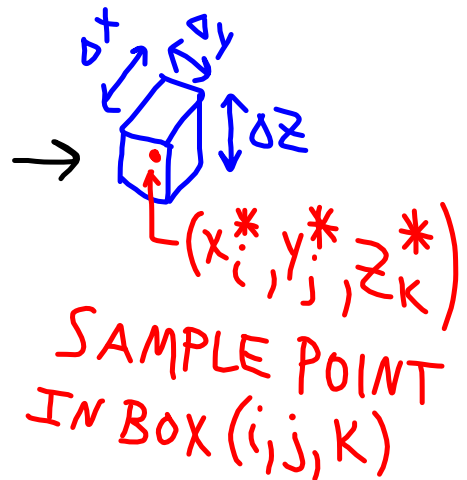
$f = f(x, y, z)$  now has a 3-D domain.

Consider a box domain  $\mathcal{D}$ ,

$$\mathcal{D} = \{(x, y, z) \mid x_1 \leq x \leq x_2, y_1 \leq y \leq y_2, z_1 \leq z \leq z_2\}.$$



Look at a  
Small volume  
 $\Delta x \Delta y \Delta z = \Delta V$   
inside  $\mathcal{D}$



Form a Riemann sum over the boxes

$$\sum_{i,j,k=1}^N f(x_i^*, y_j^*, z_k^*) \Delta x \cdot \Delta y \cdot \Delta z$$

Now shrink the boxes and consider the limit as the number of boxes goes to  $\infty$ :

$$\iiint_{\mathcal{D}} f(x, y, z) dx dy dz = \lim_{N \rightarrow \infty} \Delta x \Delta y \Delta z \sum_{i,j,k=1}^N f(x_i^*, y_j^*, z_k^*)$$

Calculation of triple integrals: Fubini's applies for continuous integrands.

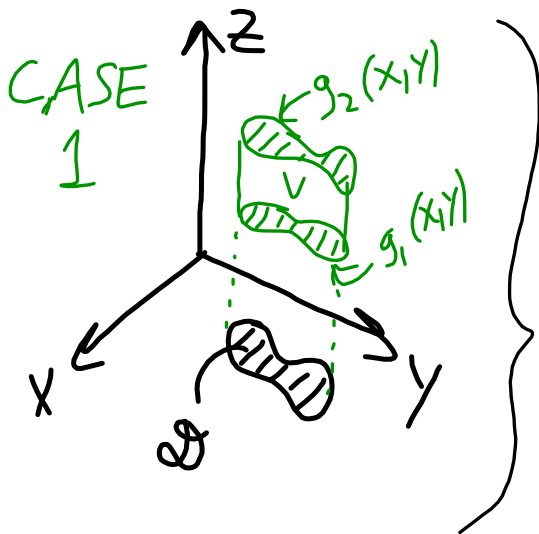
EX: Let  $\mathcal{D} = \{(x, y, z) \mid 0 \leq x \leq 3, 0 \leq y \leq 2, 0 \leq z \leq 1\}$ .

Find  $\iiint_{\mathcal{D}} xyz \, dV$ .

$$\begin{aligned} &= \int_0^1 \int_0^2 \int_0^3 xyz \, dx \, dy \, dz = \int_0^1 \int_0^2 \left[ \frac{1}{2} x^2 yz \right]_{x=0}^{x=3} dy \, dz \\ &= \int_0^1 \int_0^2 \frac{9}{2} yz \, dy \, dz = \int_0^1 \frac{9}{4} y^2 z \Big|_{y=0}^{y=2} dz \end{aligned}$$

$$= \int_0^1 9z \, dz = \frac{9}{2} z^2 \Big|_0^1 = \frac{9}{2}.$$

Integration over domains bounded between surfaces; there are three cases.



- Domain of  $f$  is  $V \subset \mathbb{R}^3$
- Domain of  $g_1, g_2$  is  $D \subset \mathbb{R}^2$

$$\iiint_V f \, dV = \iint_D \left[ \int_{g_1(x,y)}^{g_2(x,y)} f \, dz \right] dx \, dy.$$

EX: Let  $V$  be the volume below  $z = 1 + x^2 + y^2$  above  $\mathcal{D} = \{(x, y) \mid 0 \leq x \leq 1, 1 \leq y \leq 2\}$  in the  $xy$ -plane. Find  $\iiint_V xy \, dV$ .

$$= \int_0^1 \int_1^2 \int_0^{1+x^2+y^2} xy \, dz \, dy \, dx = \int_0^1 \int_1^2 xy z \Big|_{z=0}^{z=1+x^2+y^2} \, dy \, dx$$

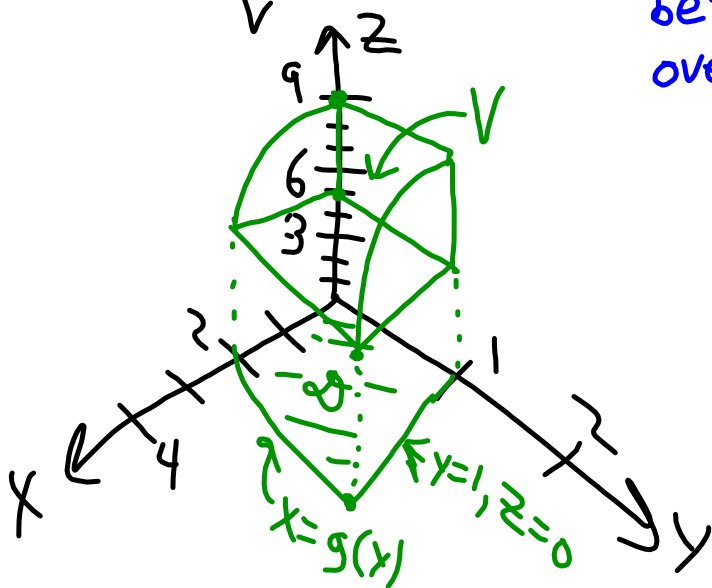
$$= \int_0^1 \int_1^2 xy(1+x^2+y^2) \, dy \, dx = \int_0^1 \int_1^2 xy + x^3y + xy^3 \, dy \, dx$$

$$= \int_0^1 \left[ \frac{1}{2} xy^2 + \frac{1}{2} x^3 y^2 + \frac{1}{4} xy^4 \right]_{y=1}^{y=2} \, dx$$

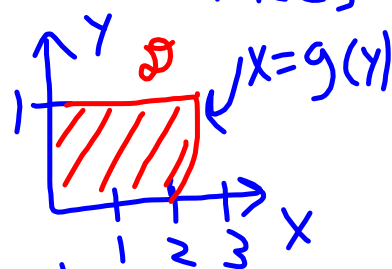
$$\begin{aligned} &= \int_0^1 \left( \frac{3}{2}x + \frac{3}{2}x^3 + \frac{15}{4}x \right) dx \\ &= \left[ \frac{3}{4}x^2 + \frac{3}{8}x^4 + \frac{15}{8}x^2 \right]_0^1 \\ &= \frac{3}{4} + \frac{3}{8} + \frac{15}{8} \\ &= \frac{6+3+15}{8} = \frac{24}{8} = \boxed{3}. \end{aligned}$$

EX: Let  $V$  be the region bounded by  
 $x=0, y=0, y=1, z=5-y, z=9-x^2$ .

Find  $\iiint_V 2dV$ .



Look at  $V$  as bounded  
between two surfaces  
over  $\mathcal{D}$ :



$g(y)$  is where

$$5-y=z=9-x^2$$

$$\Rightarrow x^2 = y+9-5 = y+4$$

$$g(y) = \sqrt{y+4}$$

$$= 2 \int_0^1 \int_0^{\sqrt{y+4}} \int_{5-y}^{9-x^2} dz dx dy$$

$$= 2 \int_0^1 \int_0^{\sqrt{y+4}} (9-x^2-5+y) dx dy = 2 \int_0^1 \int_0^{\sqrt{y+4}} (4-x^2+y) dx dy$$

$$= 2 \int_0^1 \left[ 4x - \frac{1}{3}x^3 + xy \right]_{x=0}^{x=\sqrt{y+4}} dy = 2 \int_0^1 \left( 4\sqrt{y+4} - \frac{1}{3}(y+4)^{3/2} + y\sqrt{y+4} \right) dy$$

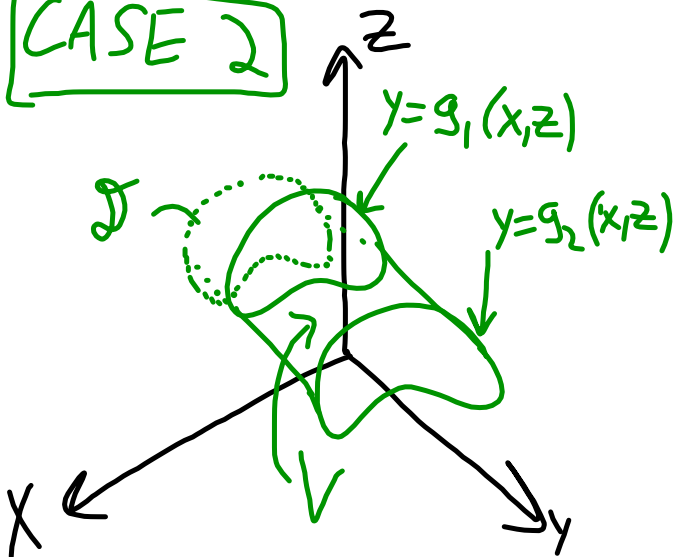
$$= 2 \int_0^1 \left( (y+4)\sqrt{y+4} - \frac{1}{3}(y+4)^{3/2} \right) dy = 2 \int_0^1 \frac{2}{3}(y+4)^{3/2} dy$$



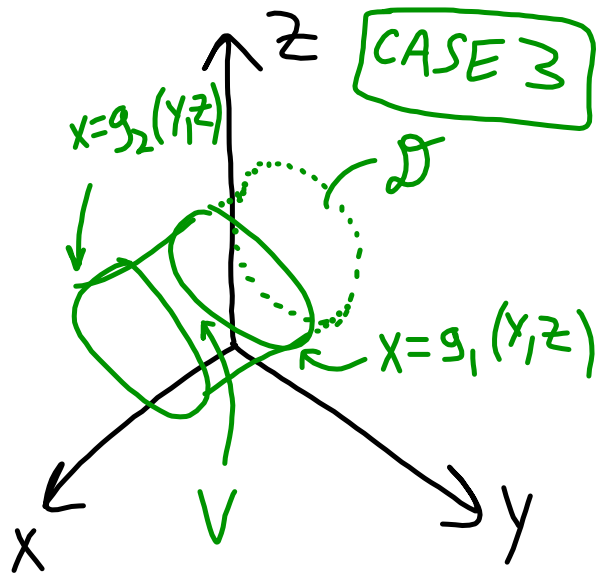
$$= \frac{4}{3} \cdot \frac{2}{5} \cdot (y+4)^{5/2} \Big|_0^1 = \frac{8}{15} (5^{5/2} - 4^{5/2})$$

$\uparrow 2^5 = 32$

CASE 2

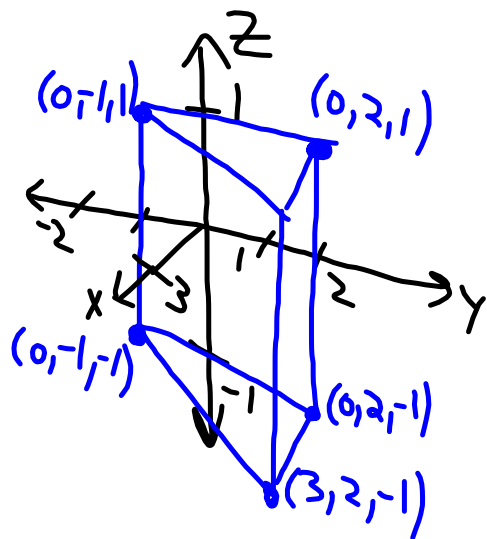


CASE 3

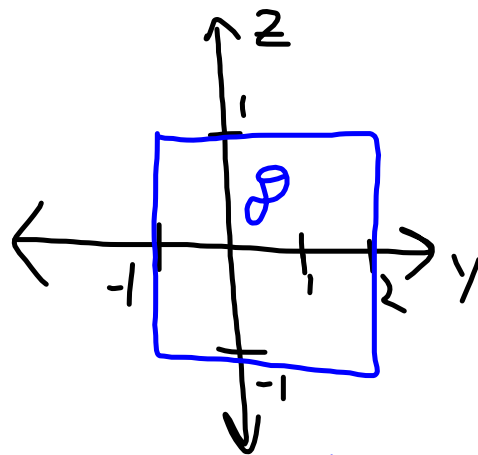


EX: Find  $\iiint_V x+y+z dV$ , where  $V$  is the region bounded by  $x=0, x=1+y, y=2, z=-1, z=1$ .

Sketch  $V$ ...



Let  $\mathcal{D} \subset yz$ -plane...



$V$  between  $x=0$  &  $x=1+y$ ...

$$\begin{aligned}
\int_{-1}^2 \int_{-1}^1 \int_0^{1+y} (x+y+z) dx dz dy &= \int_{-1}^2 \int_{-1}^1 \left[ \frac{1}{2}x^2 + x(y+z) \right]_0^{1+y} dz dy \\
&= \int_{-1}^2 \int_{-1}^1 \left[ \frac{1}{2}(1+y)^2 + (1+y)(y+z) \right] dz dy \\
&= \int_{-1}^2 \left[ \frac{1}{2}(1+y)^2 z + \frac{1}{2}(1+y)(y+z)^2 \right]_{z=-1}^{z=1} dy \\
&= \int_{-1}^2 \left( (1+y)^2 + \frac{1}{2}(1+y)^3 - \frac{1}{2}(1+y)(y-1)^2 \right) dy
\end{aligned}$$

$$= \int_{-1}^2 (1+y)^2 + \frac{1}{2}(1+y) \underbrace{[(1+y)^2 - (1-y)^2]}_{a^2 - b^2 = (a-b)(a+b)} dy$$

$$(1+y)^2 - (1-y)^2 = (1+y-1+y)(1+y+1-y) \\ = 4y$$

$$= \int_{-1}^2 (1+y)^2 + 2y(1+y) dy = \int_{-1}^2 (1+y)^2 + 2y + 2y^2 dy$$

$$= \left[ \frac{1}{3}(1+y)^3 + y^2 + \frac{2}{3}y^3 \right]_{-1}^2 = 9 + 4 + \frac{16}{3} - 1 + \frac{2}{3}$$

$$= \boxed{18}$$

Center of mass in 3D

$\rho = \rho(x, y, z)$ : density in space

$$m = \iiint_V \rho \, dV \quad \text{: total mass in } V$$

$$\text{Moments: } M_{xy} = \iiint_V z \rho \, dV \quad \text{- about } xy\text{-plane}$$

$$M_{xz} = \iiint_V y \rho \, dV \quad \text{- about } xz\text{-plane}$$

$$M_{yz} = \iiint_V x \rho \, dV \quad \text{- about } yz\text{-plane}$$

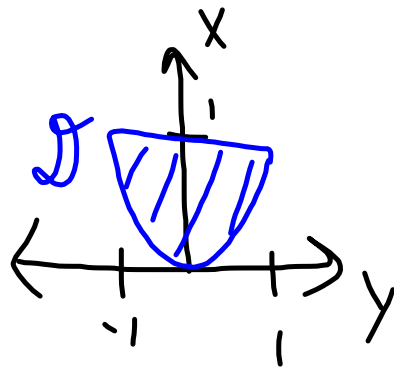
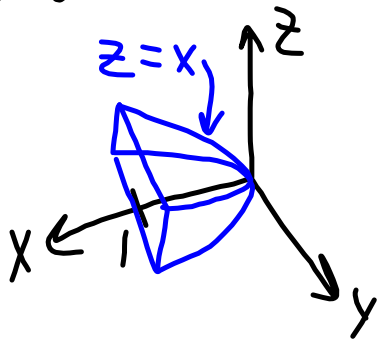
Center of mass is  $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}$$

EX: Let  $\rho \equiv \text{constant}$  on  $V$ ,

$$V = \{(x, y, z) \mid -1 \leq y \leq 1, y^2 \leq x \leq 1, 0 \leq z \leq x\}.$$

Find the center of mass.



$$\text{Mass: } m = \iiint_V \rho \, dV = \int_{-1}^1 \int_{y^2}^1 \int_0^x \rho \, dz \, dx \, dy$$

$$= \rho \int_{-1}^1 \int_{y^2}^1 x \, dx \, dy = \frac{\rho}{2} \int_{-1}^1 x^2 \Big|_{y^2}^1 dy = \frac{\rho}{2} \int_{-1}^1 (1 - y^4) \, dy$$

$$\Rightarrow m = \frac{\rho}{2} \left( y - \frac{1}{5} y^5 \right) \Big|_{-1}^1 = \frac{\rho}{2} \left( 2 - \frac{2}{5} \right) = \frac{4\rho}{5}$$

$m$

Moments:

$$M_{xz} = 0 \quad \left( \rho \text{ constant, } V \text{ symmetric in } y, \right. \\ \left. \text{so } \bar{y} = 0 \Rightarrow M_{xz} = 0 \right).$$

$$\begin{aligned}
M_{xy} &= \rho \int_{-1}^1 \int_{y^2}^1 \int_0^x z \, dz \, dx \, dy \\
&= \rho \int_{-1}^1 \int_{y^2}^1 \frac{1}{2} x^2 \, dx \, dy = \frac{\rho}{6} \int_{-1}^1 x^3 \Big|_{y^2}^1 \, dy \\
&= \frac{\rho}{6} \int_{-1}^1 (1 - y^6) \, dy = \frac{\rho}{6} \left( y - \frac{1}{7} y^7 \right) \Big|_{-1}^1 \\
&= \frac{\rho}{6} \left( 2 - \frac{2}{7} \right) = \boxed{\frac{2\rho}{7}}.
\end{aligned}$$

$M_{xy}$



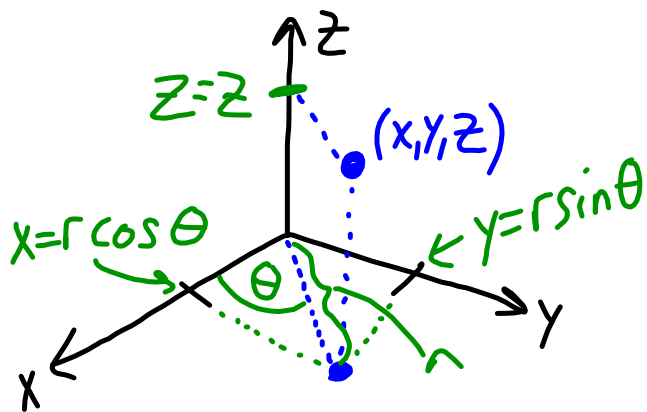
$$M_{yz} = \int_{-1}^1 \int_{y^2}^1 \int_0^x x \rho \, dz \, dx \, dy = \frac{4\rho}{7}.$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{4\rho}{7} \cdot \frac{5}{4\rho} = \frac{5}{7} \quad (\text{Exercise})$$

$$\bar{y} = 0$$

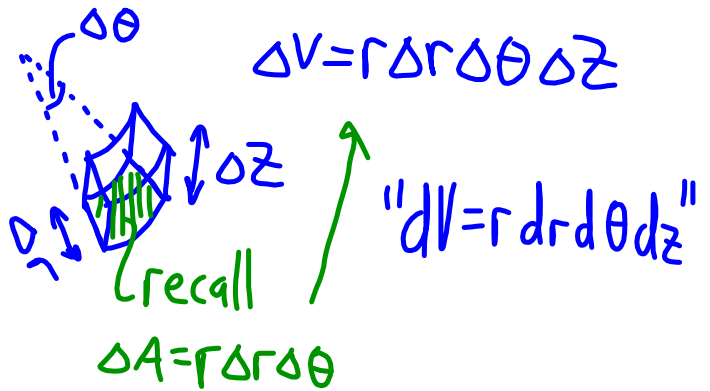
$$\bar{z} = \frac{M_{xy}}{m} = \frac{2\rho}{7} \cdot \frac{5}{4\rho} = \frac{5}{14}.$$

## Cylindrical coordinates



$$\begin{aligned}
 x &= r \cos \theta \\
 y &= r \sin \theta \\
 z &= z \\
 x^2 + y^2 &= r^2 \\
 \tan \theta &= y/x
 \end{aligned}$$

Differential volume?

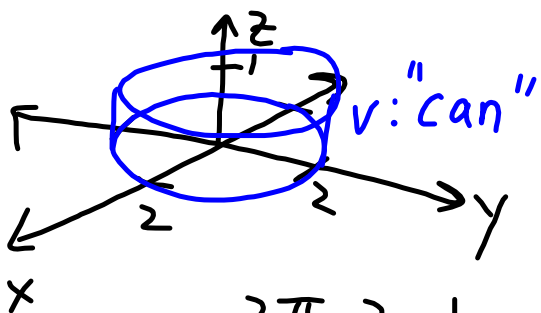


Integration formula:

$$\iiint_V f(x, y, z) dx dy dz = \iiint_V f(r, \theta, z) r dr d\theta dz.$$

EX: Note that volume of  $V$ ,  $|V|$ , is  
 $|V| = \iiint_V 1 dV$ . Find the volume of

$$V = \{(x, y, z) \mid x^2 + y^2 \leq 4, 0 \leq z \leq 1\}.$$



$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 2$$

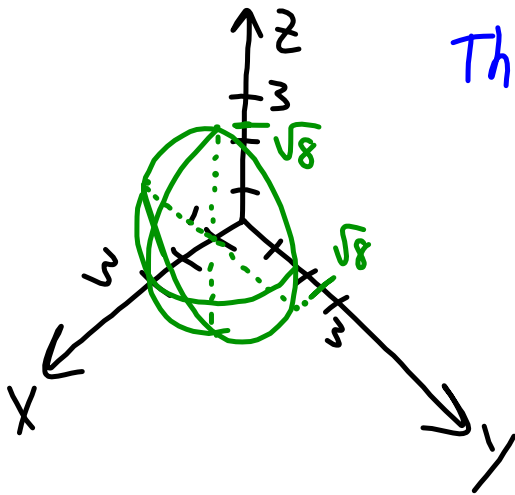
$$0 \leq z \leq 1$$

$$|V| = \int_0^{2\pi} \int_0^2 \int_0^1 r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r \, dr \, d\theta = \int_0^{2\pi} \left. \frac{1}{2} r^2 \right|_0^2 d\theta$$

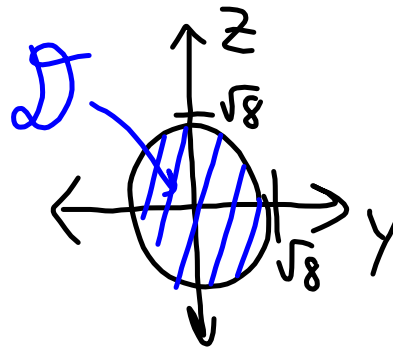
$$= 2 \cdot \int_0^{2\pi} d\theta = 4\pi.$$

\* Check:  $V = \pi r^2 h = \pi (2)^2 (1) = 4\pi.$

EX: Find the volume of the spherical cap between  $x^2 + y^2 + z^2 = 9$  and  $x = 1$  (so  $1 \leq x$ ).



Think of  $V$  between  $x=1, x=\sqrt{9-y^2-z^2}$



$$\begin{aligned}
 y &= r \cos \theta & x &= x \\
 z &= r \sin \theta & 0 &\leq r \leq \sqrt{8} \\
 & & 0 &\leq \theta \leq 2\pi
 \end{aligned}$$

$$|V| = \int_0^{2\pi} \int_0^{\sqrt{8}} \int_1^{\sqrt{9-y^2-z^2}} r \, dx \, dr \, d\theta$$

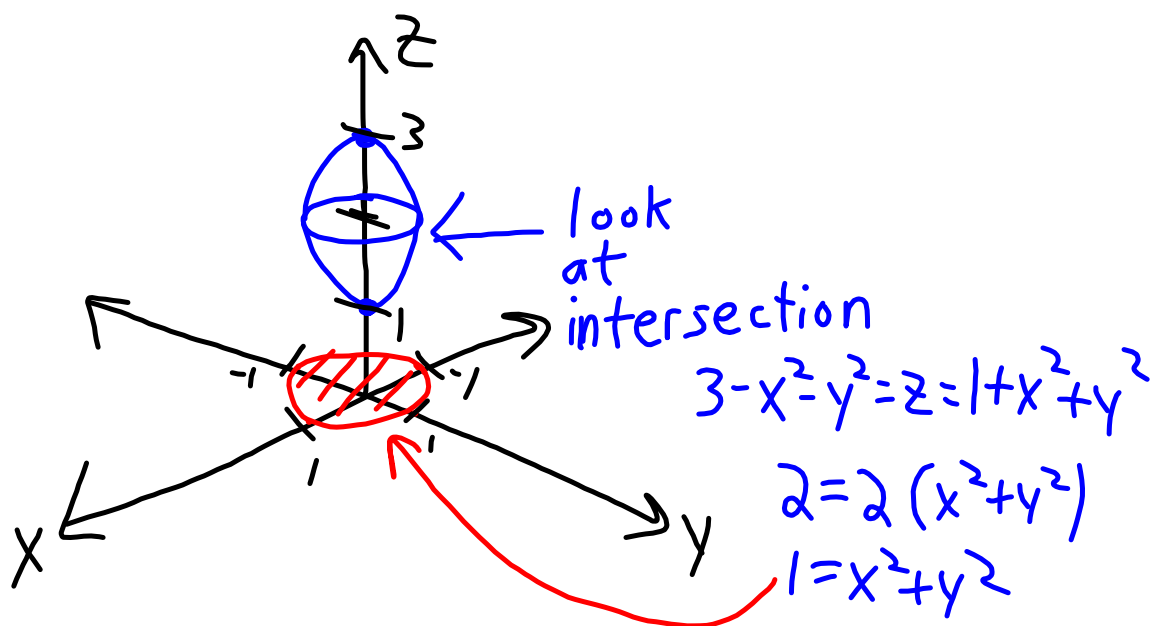
$$y^2 + z^2 = r^2 \Rightarrow |V| = \int_0^{2\pi} \int_0^{\sqrt{8}} \int_1^{\sqrt{9-r^2}} r \, dx \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{8}} r \sqrt{9-r^2} - r \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{8}} r \sqrt{9-r^2} \, dr \, d\theta - \int_0^{2\pi} \int_0^{\sqrt{8}} r \, dr \, d\theta$$

$\leftarrow u = 9 - r^2$   
 $du = -2r \, dr$   
 $-\frac{1}{2} du = r \, dr$

$$\begin{aligned}
|V| &= \int_0^{2\pi} \int_{u=9}^{u=1} -\frac{1}{2} \sqrt{u} \, du \, d\theta - \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^{\sqrt{8}} \, d\theta \\
&= \int_0^{2\pi} \int_1^9 \frac{1}{2} \sqrt{u} \, du \, d\theta - 4 \cdot 2\pi \\
&= \int_0^{2\pi} \left[ \frac{1}{3} u^{3/2} \right]_1^9 \, d\theta - 8\pi = \frac{2\pi}{3} (3^3 - 1) - 8\pi \\
&= \frac{(52-24)\pi}{3} = \frac{28\pi}{3}
\end{aligned}$$

Practice Problem : Find the volume of the region  $V$  bounded by  $z = 3 - x^2 - y^2$  above and  $z = 1 + x^2 + y^2$  below.





The volume is bounded between two surfaces over the domain  $x^2 + y^2 \leq 1$ .

Cylindrical:  $z = 3 - x^2 - y^2 = 3 - r^2$

$$z = 1 + x^2 + y^2 = 1 + r^2$$

$$\Rightarrow V = \int_0^{2\pi} \int_0^1 \int_{1+r^2}^{3-r^2} r \, dz \, dr \, d\theta = \left( \int_0^{2\pi} d\theta \right) \left( \int_0^1 (3-r^2-1-r^2) dr \right)$$

$$\Rightarrow V = 2\pi \int_0^1 (2r - 2r^3) dr = 2\pi \left[ r^2 - \frac{1}{2}r^4 \right]_0^1 = \frac{2\pi}{2} = \boxed{\pi}.$$