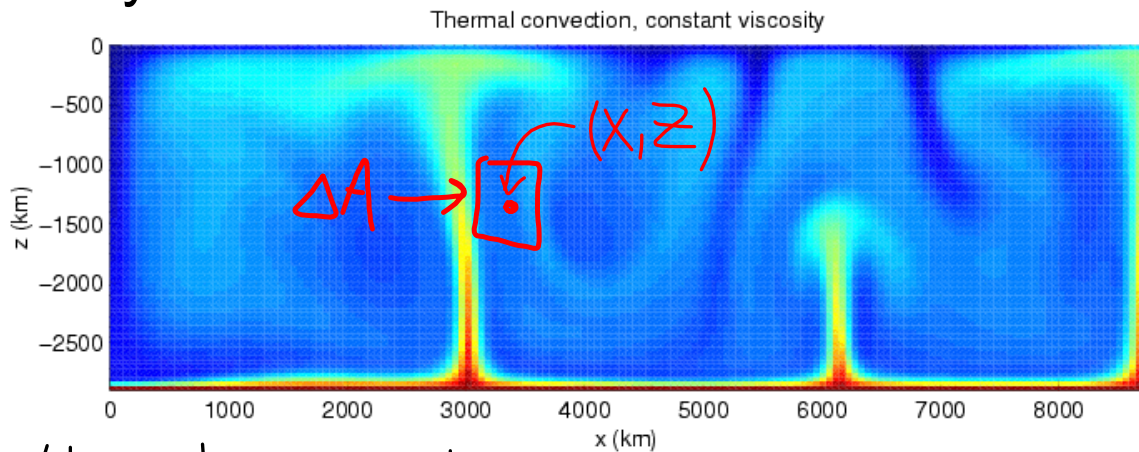


Density



E : thermal energy in region ΔA
 \Rightarrow Thermal energy density is $E/\Delta A$ "energy per unit area"

$$\lim_{\Delta A \rightarrow 0} \frac{E}{\Delta A} = \rho(x, z) : \text{"local thermal energy density"}$$

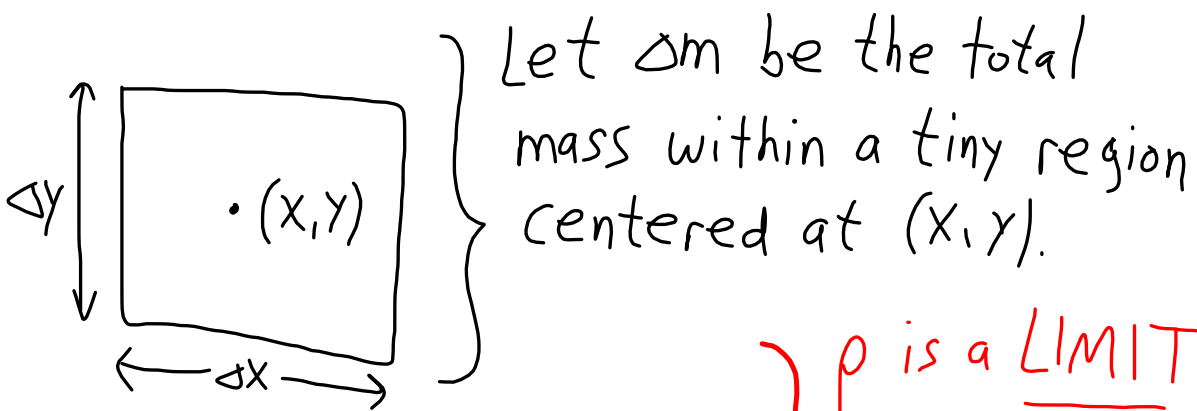
Generally, if $\rho(x,y)$ is a local density function for some quantity, then the total amount of the quantity over an area A is

$$\iint_A \rho(x,y) dA.$$

For example, if ρ is mass density, then the total mass, m , is $m = \iint_A \rho dA.$

NOTE: Point-wise densities are a mathematical tool and MODEL, not a "real" thing. For example, what does mass density at a point mean? There is no physical description of mass at a single point; even sub-atomic particles are considered to take up some non-zero volume of space.

To clarify the point...

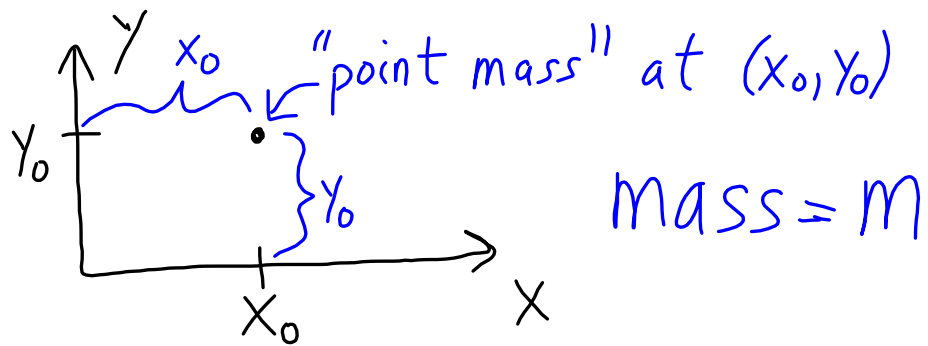


Let Δm be the total mass within a tiny region centered at (x, y) .

$$\rho(x, y) = \lim_{\Delta x, \Delta y \rightarrow 0} \frac{\Delta m}{\Delta x \cdot \Delta y}$$

ρ is a LIMIT at each point. It just models the behavior of mass near a particular point.

Moments

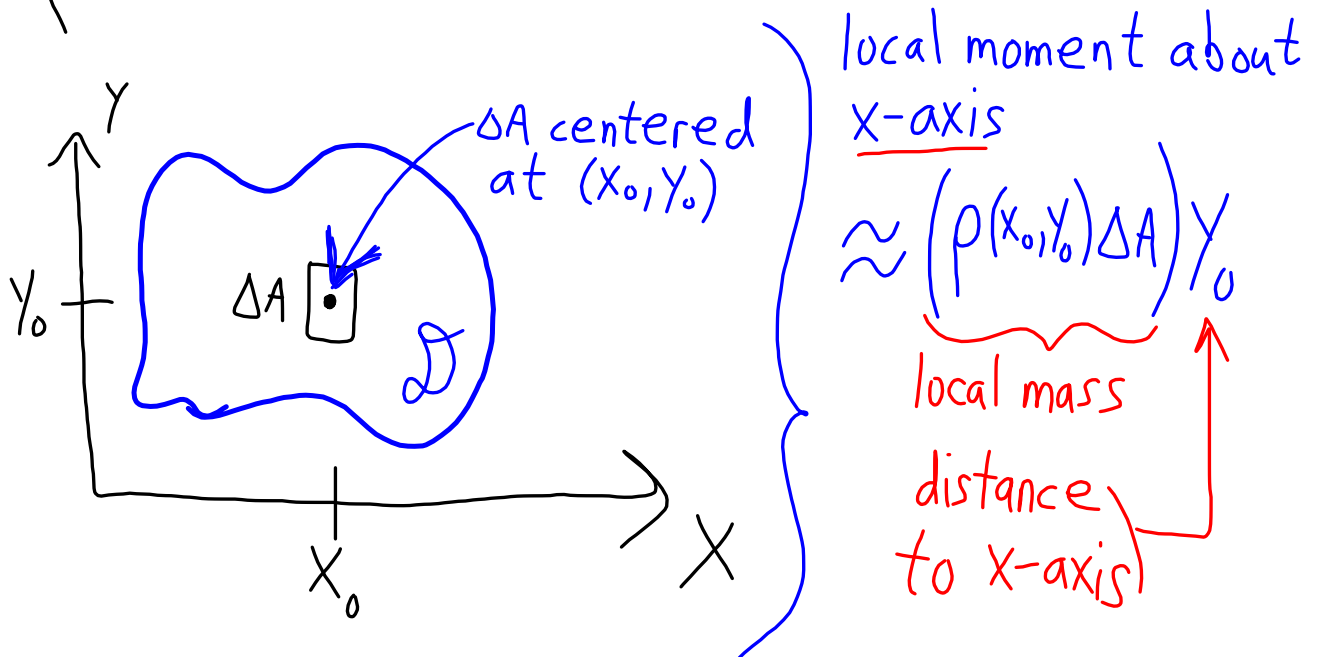


$$X\text{-moment} \circ my_0$$

$$Y\text{-moment} \circ mx_0$$

Now consider a solid body as a collection/
continuum of particles.

$\rho(x, y)$: local mass density



M_x^o moment of body about x-axis

Riemann sum... $M_x \approx \sum_{i,j} \rho(x_i^*, y_j^*) \Delta A_{ij} y_j^*$

$(\Delta A_{ij} = \Delta A, \text{constant}) = \Delta A \sum_{i,j} \rho(x_i^*, y_j^*) y_j^*$

$$\Rightarrow M_x = \lim_{\Delta A \rightarrow 0} \Delta A \sum_{i,j} \rho(x_i^*, y_j^*) y_j^* = \iint_D y \rho(x, y) dA.$$

$$M_x = \iint_D y \rho \, dA$$

Moment
about
X-axis

$$M_y = \iint_D x \rho \, dA$$

Moment
about
y-axis

EX: Let a region $D = \{(x,y) \mid 0 \leq x \leq 2, 1 \leq y \leq 4\}$ represent a solid object with mass density $\rho(x,y) = 1+x^2$. Find the moments about the x - & y -axes.

$$\begin{aligned} M_x &= \int_0^2 \int_1^4 y(1+x^2) dy dx = \frac{1}{2} \int_0^2 (16-1)(1+x^2) dx \\ &= \frac{15}{2} \left[x + \frac{1}{3}x^3 \right]_0^2 = \frac{15}{2} \left(2 + \frac{8}{3} \right) = 35. \end{aligned}$$

$$M_y = \int_0^2 \int_1^4 x(1+x^2) dy dx$$

$$= 3 \int_0^2 x+x^3 dx = 3 \left[\frac{1}{2}x^2 + \frac{1}{4}x^4 \right]_0^2$$

$$= 3[2+4] = 18.$$

Center of mass

$$\bar{x} = \frac{M_y}{m} .$$

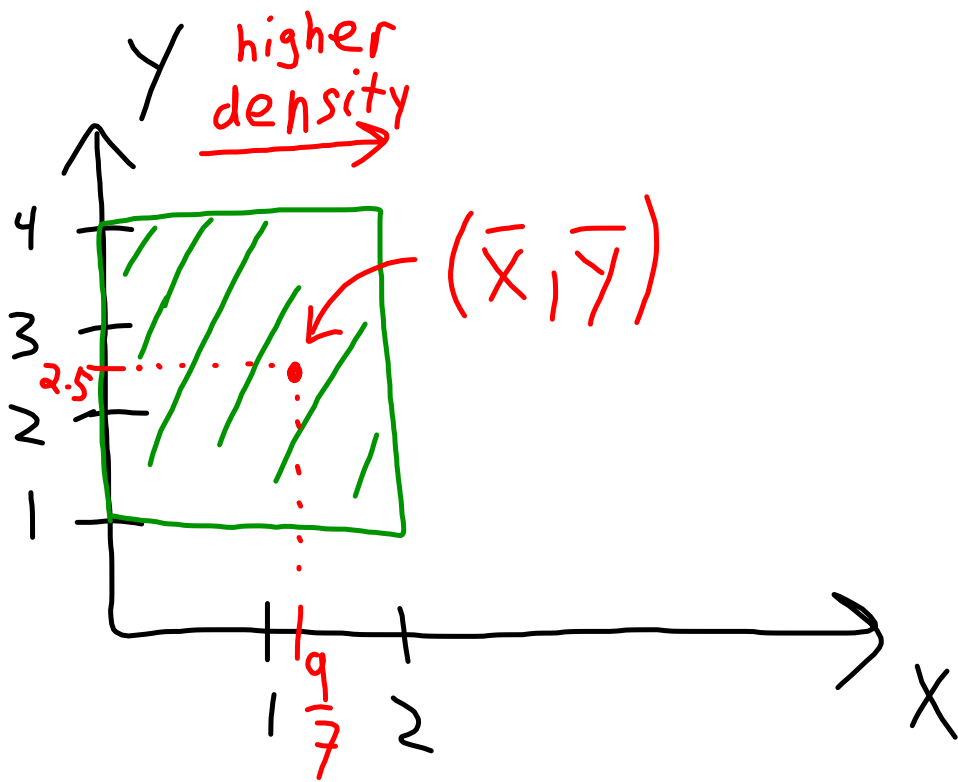
$$\bar{y} = \frac{M_x}{m} .$$

$$m = \iint_D \rho dA .$$

EX: For the previous example, find the center of mass.

$$\rho = 1+x^2 \Rightarrow m = \int_0^2 \int_1^4 (1+x^2) dy dx = 3 \int_0^2 1+x^2 dx$$
$$= 3 \left[x + \frac{1}{3}x^3 \right]_0^2 = 3 \left(2 + \frac{8}{3} \right) = 14.$$

Thus $\bar{x} = \frac{M_y}{m} = \frac{18}{14} = \frac{9}{7}$, $\bar{y} = \frac{35}{14} = \frac{5}{2}$.



Moments of inertia: find the torque needed to generate some angular acceleration.

$$1) I_x = \iint_{\mathcal{D}} y^2 \rho \, dA \quad 2) I_y = \iint_{\mathcal{D}} x^2 \rho \, dA$$

about x-axis about y-axis

$$3) I_o = I_x + I_y = \iint_{\mathcal{D}} (x^2 + y^2) \rho \, dA.$$

} rotation around origin

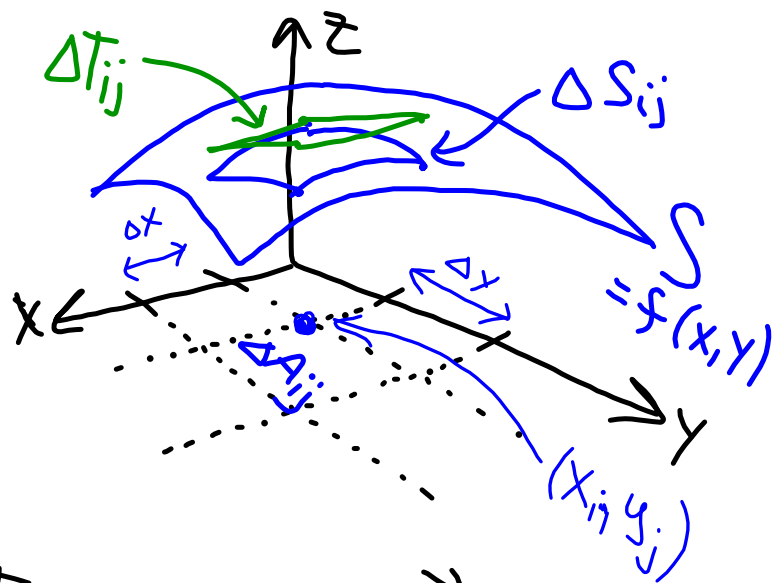
Comments

- * The required torque increases with distance.
- * The required torque increases with density, but not as quickly as with distance.
- * Think of walking a tight-rope... people hold a long pole rather than something smaller but heavier.
- * Abstractly, the n-th moment of $f(x,y)$ with respect to variable ω is

$$\iint_S \omega^n f dA.$$

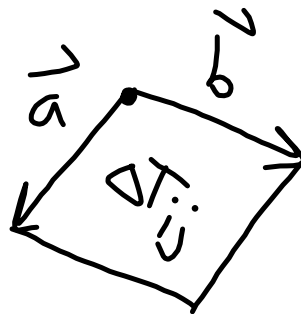
Surface area

ΔT_{ij} : piece of tangent plane to f at (x_{ij}, y_{ij}) lying above ΔA_{ij} .



Formula for ΔT_{ij} :

$$\Delta T_{ij} = |\vec{a} \times \vec{b}|$$



Note \vec{a} lies along tangent line in the
X-direction & \vec{b} " " in y-direction;

$$\vec{a} = \langle \Delta x, 0, \Delta x \cdot f_x \rangle = \Delta x \langle 1, 0, f_x \rangle$$

$$\vec{b} = \langle 0, \Delta y, \Delta y \cdot f_y \rangle = \Delta y \langle 0, 1, f_y \rangle$$

$$\Rightarrow \vec{a} \times \vec{b} = \Delta x \cdot \Delta y \langle -f_x, -f_y, 1 \rangle$$

$$|\vec{a} \times \vec{b}| = \Delta T_{ij} = \Delta x \Delta y \sqrt{1 + (f_x)^2 + (f_y)^2}$$

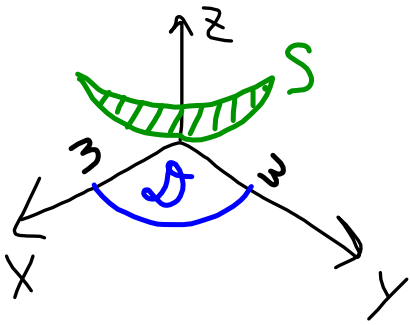
$$\Rightarrow S \approx \sum_{ij} \Delta T_{ij} = \Delta x \Delta y \sum_{ij} \sqrt{1 + f_x(x_{ij}, y_{ij})^2 + f_y(x_{ij}, y_{ij})^2}$$

Let $\Delta T_{ij} = \Delta x \cdot \Delta y \rightarrow 0 \dots$

Surface area is a double integral

$$S = \lim_{\Delta x \Delta y \rightarrow 0} \sum_{ij} \Delta T_{ij} = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2} \, dA.$$

Ex: Calculate the surface area of the paraboloid $z = x^2 + y^2$ over the domain $D = \{(x, y) \mid x^2 + y^2 \leq 9, x \geq 0, y \geq 0\}$.



$$z_x = 2x \dots z_y = 2y$$

$$\text{Polar: } 0 \leq \theta \leq \pi/2$$

$$0 \leq r \leq 3$$

$$z_x^2 + z_y^2 = 4(x^2 + y^2) = 4r^2$$

$$S = \int_0^{\pi/2} \int_0^3 \sqrt{1 + 4r^2} r dr d\theta \quad \left. \begin{array}{l} u = 4r^2 \\ \frac{1}{8} du = r dr \end{array} \right\}$$

$$S = \frac{1}{8} \int_0^{\pi/2} \int_{u=0}^{u=4 \cdot 3^2=36} \sqrt{1+u} \, du \, d\theta$$

$$= \frac{1}{8} \left(\frac{\pi}{2} \right) \left(\frac{2}{3} (1+u)^{3/2} \Big|_0^{36} \right)$$
$$= \frac{\pi}{24} (37^{3/2} - 1).$$

Practice!

(#1) Let a solid disc have radius $r=3$ and density $\rho(x,y) = e^{-x^2-y^2}$. Find (a) the total mass and (b) center of mass.

$$\int_0^{2\pi} \int_0^3 e^{-r^2} r dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^9 e^{-u} du d\theta$$

$u=r^2$
 $\frac{1}{2} du = r dr$

$$\begin{aligned} &= \frac{1}{2} \int_0^{2\pi} \left[\int_0^q e^{-u} du \right] d\theta \\ &= \frac{1}{2} \int_0^q e^{-u} du \cdot \int_0^{2\pi} d\theta \\ &= \pi \left[-e^{-u} \right]_0^q = \pi (1 - e^{-q}) \end{aligned}$$

(b) center of mass

$$M_x = \int_0^{2\pi} \int_0^3 r \sin \theta e^{-r^2} r dr d\theta = \int_0^{2\pi} \int_0^3 r^2 e^{-r^2} \sin \theta dr d\theta$$

↑ don't try this,
switch order

$$= \int_0^3 \int_0^{2\pi} r^2 e^{-r^2} \sin \theta d\theta dr = 0, \text{ since } \int_0^{2\pi} \sin \theta d\theta = 0.$$

Similarly, $M_y = 0$

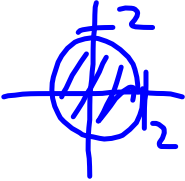
$$\Rightarrow \bar{x} = \bar{y} = 0.$$

#2 The portion of a cone above the xy -plane is given as

$$z = \frac{1}{2} \sqrt{x^2 + y^2}.$$

Find the surface area lying above

$$D = \{(x, y) \mid x^2 + y^2 \leq 4\}.$$
$$z_x = \frac{2x}{4\sqrt{x^2 + y^2}} = \frac{x}{2\sqrt{x^2 + y^2}} \quad z_y = \frac{y}{2\sqrt{x^2 + y^2}}$$
$$z_x^2 + z_y^2 = \frac{1}{4}$$



$$\begin{aligned}
 S &= \int_0^{2\pi} \int_0^2 \sqrt{1 + \frac{1}{4}} r \, dr \, d\theta \\
 &= \frac{\sqrt{5}}{2} \cdot 2\pi \int_0^2 r \, dr = \pi\sqrt{5} \left. \frac{r^2}{2} \right|_0^2 \\
 &= \boxed{2\pi\sqrt{5}}.
 \end{aligned}$$