

Maximums and minimums

Let $f(x,y)$ be defined on $D \subset \mathbb{R}^2$.

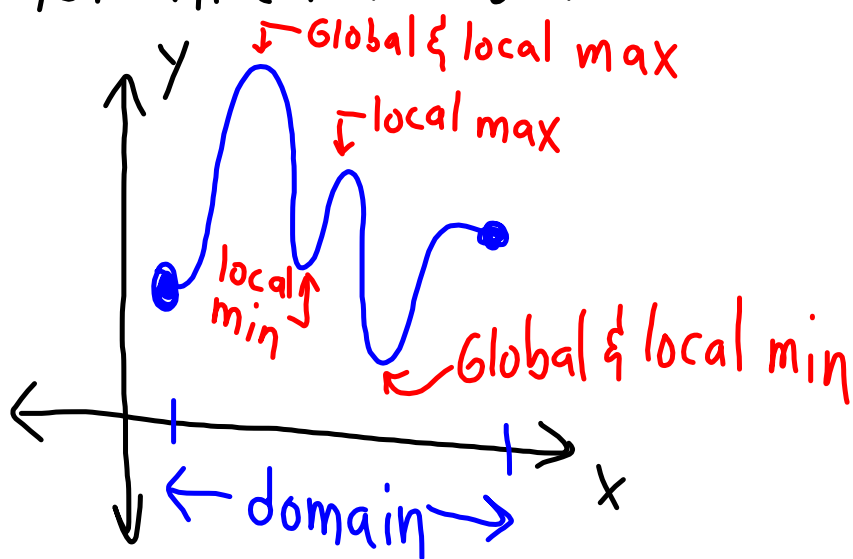
Then for (a,b) in D ,

- Local max @ $(a,b) \Leftrightarrow f(x,y) \leq f(a,b)$
holds for all (x,y) near (a,b) .

- Global max/absolute max

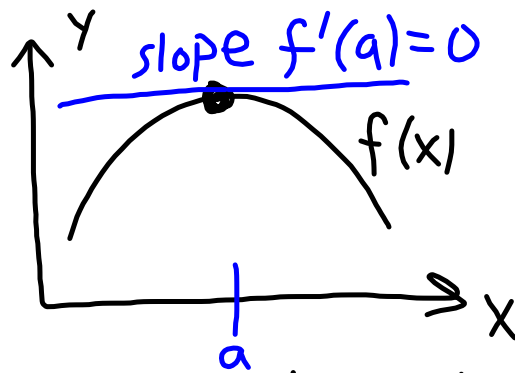
$\Leftrightarrow f(x,y) \leq f(a,b)$ for all (x,y) in D .

- Local min $\Leftrightarrow f(x,y) \geq f(a,b)$
holds for all (x,y) near (a,b)
- Global/absolute min if this holds
for all (x,y) in \mathcal{D} .

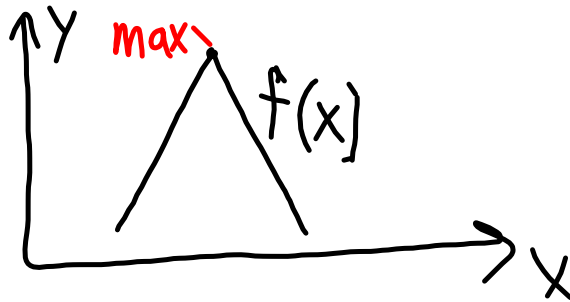


When derivatives exist at a max/min, they must be zero.

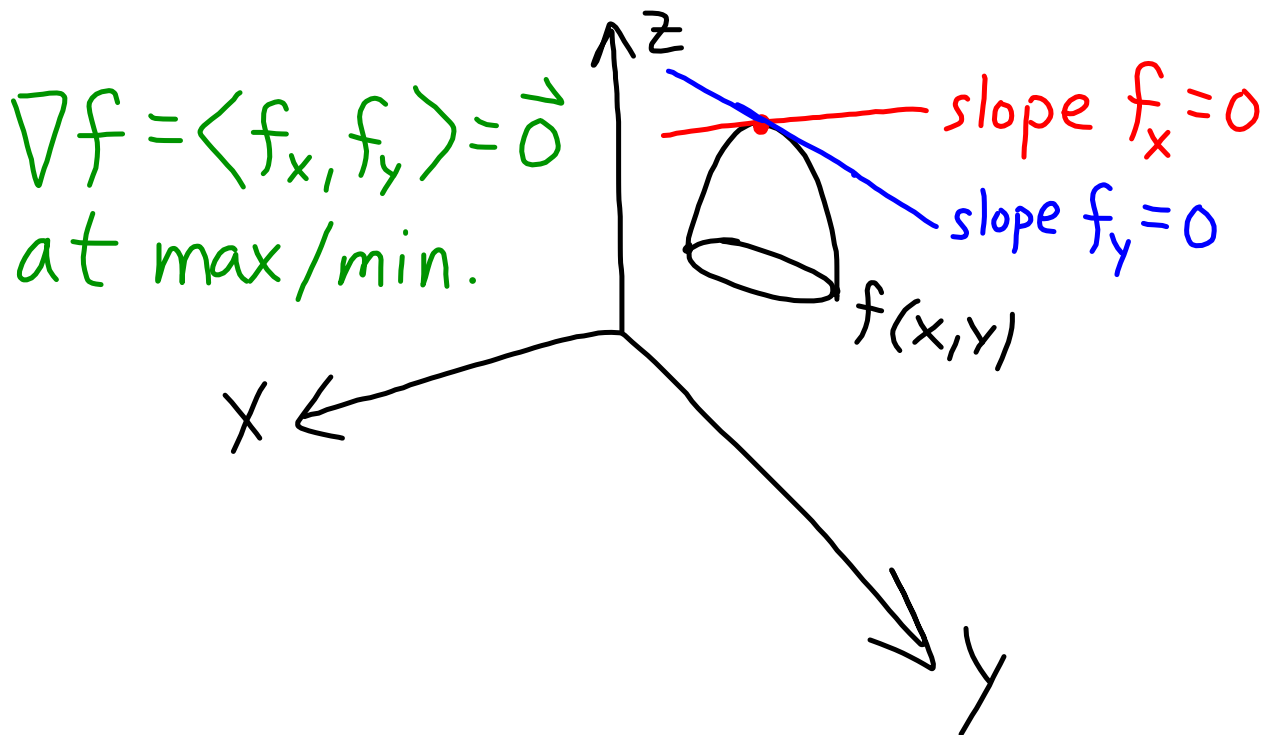
Case 1 : derivative = 0



Case 2 : no derivative



With multiple variables, the partial derivatives will be zero when they exist at max/min points.



... similarly for more variables

$$f(x, y, z) \text{ has MAX/MIN} \\ \text{at } (a, b, c) \Rightarrow \nabla f = \langle f_x, f_y, f_z \rangle \\ = \langle 0, 0, 0 \rangle \text{ at } (a, b, c)$$

Since ∇f may not exist at MAX/MIN points
we check all **CRITICAL POINTS**, i.e.

where $\nabla f = \vec{0}$ or $\nabla f = \text{D.N.E.}$

EX: Find all critical points of
 $f(x,y) = x^2 + xy - \frac{1}{4}y^2$.

$$f_x = 2x + y = 0$$

$$f_y = x - \frac{1}{2}y = 0$$

(0,0) only critical pt.

$$\frac{f_x - 2f_y = 2y = 0 \Rightarrow y = 0 \Rightarrow x = 0.$$

*Note f decreases in y -direction but increases in x -direction at (0,0)

\Rightarrow SADDLE POINT.

EX: Find the maxima of
 $f(x,y) = -x^2 + 2x - y^2 - 4y - 5.$

$$\left. \begin{array}{l} f_x = -2x + 2 = 0 \quad (x=1) \\ f_y = -2y - 4 = 0 \quad (y=-2) \end{array} \right\} \text{CRIT. PT. } (1, -2)$$

Complete the square...

$$f(x,y) = -(x-1)^2 - (y+2)^2 \leq 0 = f(1, -2)$$

holds for all $(x,y) \Rightarrow$ Global max @ $(1, -2).$

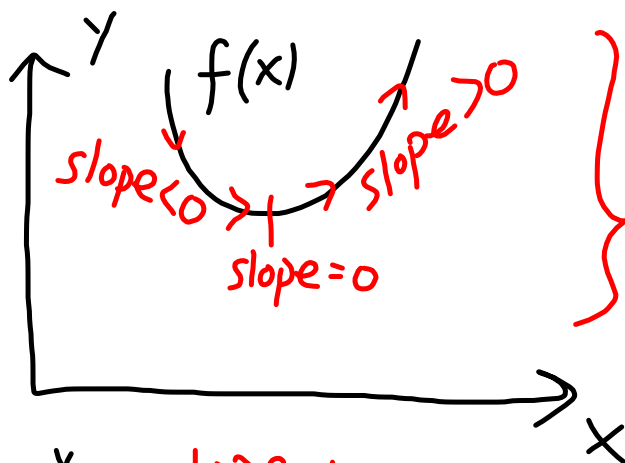
The Second Derivative Test gives a more convenient way to classify critical points.

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

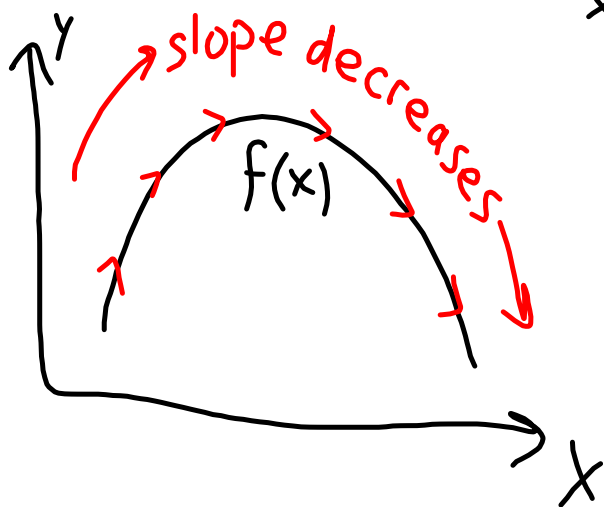
(1) $D > 0, f_{xx} > 0 \Rightarrow$ Local min.

(2) $D > 0, f_{xx} < 0 \Rightarrow$ Local max.

(3) $D < 0 \Rightarrow$ Saddle point.



f_x increasing
 $\Rightarrow f_{xx} > 0$ at MIN.



... and $f_{xx} < 0$
at MAX.

EX: Classify all critical points of
 $f(x,y) = x^4 + y^4 - 4xy + 1$.

$$f_x = 4x^3 - 4y = 0 \quad f_y = 4y^3 - 4x = 0$$

$$\Rightarrow y = x^3 \longrightarrow x^9 - x = 0$$

$$\Rightarrow x(x^8 - 1) = 0$$

$$\Rightarrow x = 0, \pm 1.$$

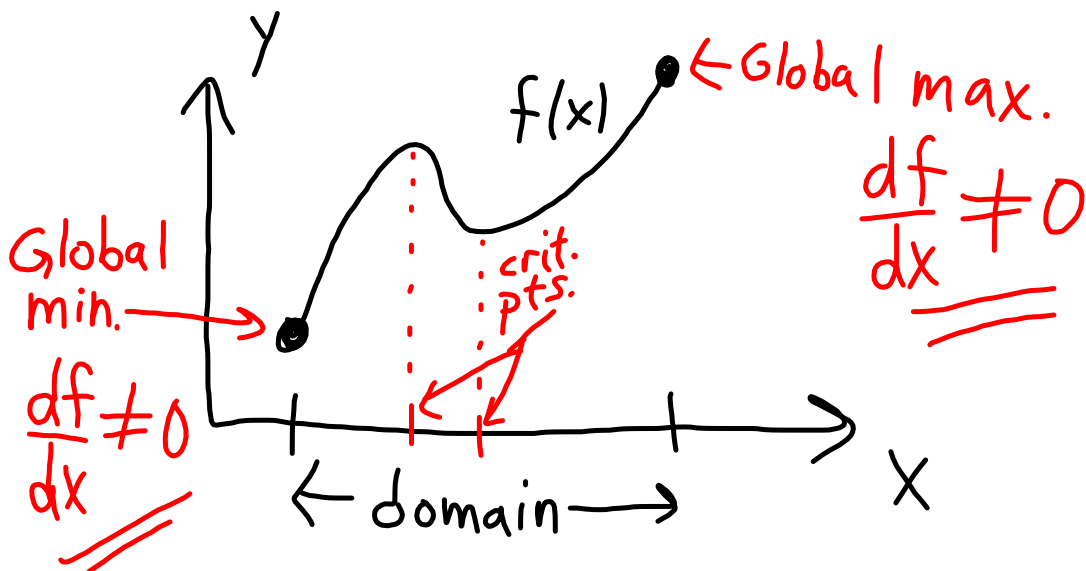
CRIT. PTS. $\{(0,0), (-1,-1), (1,1)\}$.

$$\begin{array}{l}
 f_{xx} = 12x^2 \\
 f_{yy} = 12y^2 \\
 f_{xy} = -4
 \end{array}
 \left. \vphantom{\begin{array}{l} f_{xx} = 12x^2 \\ f_{yy} = 12y^2 \\ f_{xy} = -4 \end{array}} \right\}
 \begin{array}{l}
 D = f_{xx} f_{yy} - (f_{xy})^2 \\
 D(0,0) = -16 < 0 \\
 D(-1,-1) = 144 - 16 > 0, f_{xx} > 0. \\
 D(1,1) = 144 - 16 > 0, f_{xx} > 0.
 \end{array}$$

$(0,0) \Rightarrow$ SADDLE PT.

$(1,1)$ & $(-1,-1) \Rightarrow$ LOCAL MIN.

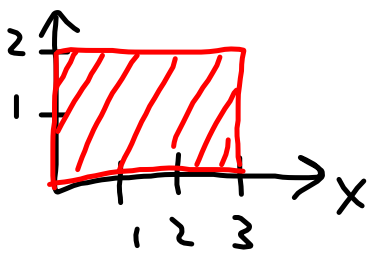
Looking at critical points gives us local extrema and sometimes global extrema; not generally on a closed and bounded domain.



Identifying GLOBAL extrema takes work:

- (1) Find f -values at all critical points *inside* the domain.
- (2) Find max/min f -values along the domain boundary.
- (3) Take MAX/MIN of all f -values from steps (1) & (2).

EX: Find the global extrema of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 2$.



(1) Find f @ CRIT. PTS.

$$f_x = 0 = 2x - 2y \quad (x=y)$$

$$f_y = 0 = 2 - 2x \quad (x=1 \Rightarrow y=1)$$

$$f(1,1) = 1 - 2 + 2 =$$

1

$$f = x^2 - 2xy + 2y$$

(2) Max/min along boundary.

- $x=0 \Rightarrow f=2y, 0 \leq y \leq 2 \Rightarrow 0 \leq f \leq 4$
- $x=3 \Rightarrow f=9-6y+2y=9-4y \Rightarrow 1 \leq f \leq 9$
- $y=0 \Rightarrow f=x^2, 0 \leq x \leq 3 \Rightarrow 0 \leq f \leq 9$
- $y=2 \Rightarrow f=x^2-4x+4=(x-2)^2 \Rightarrow 0 \leq f \leq 4$

(3)

$f=1$ from step (1) \Rightarrow

$$\text{MIN} = 0$$

$$\text{MAX} = 9.$$

Ex: Find the global max/min for
 $f(x,y) = 8x - 4x^2 - y^2$ on $\mathcal{D} = \{(x,y) \mid x^2 + y^2 \leq 4, x \geq 0\}$.

Domain is a bit trickier now...

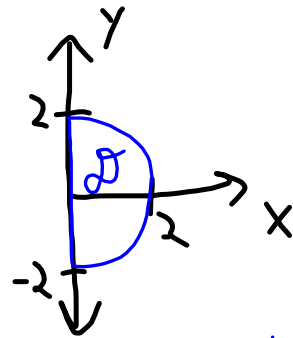
$$f_x = 8 - 8x = 0 \Leftrightarrow x = 1$$

$$f_y = -2y = 0 \Leftrightarrow y = 0$$

$$f(1,0) = 8 - 4 = 4 \leftarrow \text{candidate for max/min}$$

Two boundaries to check... on $x=0, -2 \leq y \leq 2$

$$\Rightarrow f(0,y) = -y^2 \text{ so } -4 \leq f \leq 0,$$



then $-4 \leq f \leq 4$ so far. To check the curved boundary, parameterize $f(x,y)$ somehow... note $x^2 + y^2 = 4$ on this part of the boundary, so $y^2 = 4 - x^2$

$$\Rightarrow f = 8x - 4x^2 - (4 - x^2) = 8x - 3x^2 - 4 = g(x)$$

with $0 \leq x \leq 2$.

(calc 1): Now find extreme values of $g = 8x - 3x^2 - 4$ on domain $0 \leq x \leq 2 \dots$

$$\text{Ends: } g(0) = -4, g(2) = 0.$$

$$g = 8x - 4x^2 - (4 - x^2) = 8x - 3x^2 - 4$$

$$\Rightarrow g'(x) = 8 - 6x = 0 \Leftrightarrow x = 4/3$$

$$\text{Check } g\left(\frac{4}{3}\right) = 8 \cdot \frac{4}{3} - 3\left(\frac{4}{3}\right)^2 - 4 = \frac{32 - 16 - 12}{3}$$

$$\boxed{\text{MAX} = 4, \text{MIN} = -4}$$

$$= 4/3.$$

Lagrange Multipliers: used to maximize or minimize a function under constraints.

- Maximize the volume of a wooden crate given a fixed amount of wood to use.
- Find the minimum distance to a planet along a fixed, elliptical orbit.
- Analogous problems abound in engineering applications.

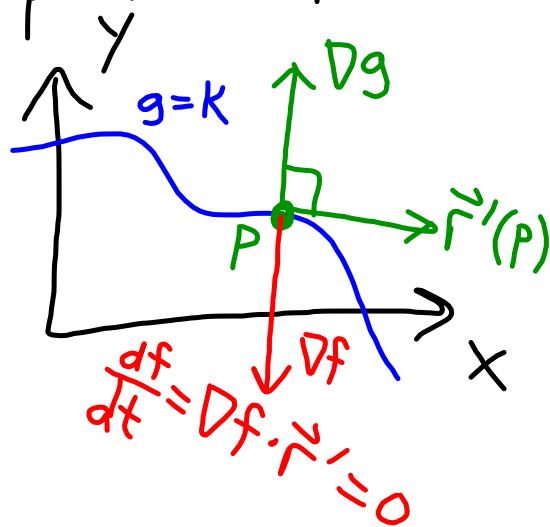
Abstractly, we consider

$f(x, y)$: A function to minimize
or maximize

$g(x, y) = K$: "Constraint equation"
restricts (x, y) allowed
(think "design parameters")

* First step: identify these for your problem.

A picture for the idea...



Let $f(P)$ be a MAX/MIN of f along curve $g(x,y)=k$.

Since $g=k \Rightarrow$ level curve we know $\nabla g \perp \vec{r}'$, where $\vec{r}(t)$ is a parameterization of the curve.

Along the curve, $f=f(x(t),y(t)) \Rightarrow \frac{df}{dt}(P) = 0$.

chain Rule... $f_x x' + f_y y' = \nabla f \cdot \vec{r}' = 0$.

In summary, at an extreme value of f along $g=k$, we have

$$\nabla f = \lambda \nabla g \quad \text{"} \nabla f \parallel \nabla g \text{"}$$

↑
scalar,
"Lagrange multiplier"

$$g(x,y) = k \quad \left. \vphantom{g(x,y) = k} \right\} \begin{array}{l} \text{Solve this with} \\ \nabla f = \lambda \nabla g. \end{array}$$

Let $f = f(x, y, z) \dots$ $g(x, y, z) = K$ constraint.
To find the MAX or MIN of f , solve

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z}$$

$$g(x, y, z) = K$$

solving for x, y, z, λ

4 variables

4 equations

EX: Find the minimum distance from the origin to $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

- We are minimizing $d = \sqrt{x^2 + y^2}$.
- Equivalently, use $f(x, y) = x^2 + y^2$ (easier).
- Constraint: $g(x, y) = \frac{x^2}{4} + \frac{y^2}{9} = 1$.

Now apply the method; solve

$$f_x = 2x = \lambda \frac{1}{2}x = \lambda g_x \quad (x=0 \text{ or } \lambda=4)$$

$$f_y = 2y = \lambda \frac{2}{9}y = \lambda g_y \quad (y=0 \text{ or } \lambda=9)$$

$$g = \frac{x^2}{4} + \frac{y^2}{9} = 1 \quad \left. \vphantom{g} \right\} x, y \text{ not both } 0$$

- $x=0 \Rightarrow \lambda=9, y^2=9$ so $y = \pm 3$

$$d(0, \pm 3) = 3$$

MIN

- $y=0 \Rightarrow \lambda=4, x^2=4 \Rightarrow x = \pm 2, d(\pm 2, 0) = \boxed{2}$

EX: Find the minimum distance from $(0,0,1)$ to the hyperboloid $x^2 + \frac{1}{4}y^2 - z^2 = 1$.

Use $f(x,y,z) = x^2 + y^2 + (z-1)^2$

and $g = x^2 + \frac{1}{4}y^2 - z^2 = 1$.

Solve:
$$\left\{ \begin{array}{l} 2x = 2x\lambda \rightarrow x=0 \text{ or } \lambda=1 \\ 2y = \frac{1}{2}y\lambda \rightarrow y=0 \text{ or } \lambda=4 \\ 2(z-1) = -2z\lambda \rightarrow (1+\lambda)z=1 \\ x^2 + \frac{1}{4}y^2 - z^2 = 1 \rightarrow x,y \text{ not both zero...} \end{array} \right.$$

$$\text{Solve: } \begin{cases} 2x = 2x\lambda \rightarrow x=0 \text{ or } \lambda=1 \\ 2y = \frac{1}{2}y\lambda \rightarrow y=0 \text{ or } \lambda=4 \\ 2(z-1) = -2z\lambda \rightarrow (1+\lambda)z=1 \\ x^2 + \frac{1}{4}y^2 - z^2 = 1 \rightarrow x, y \text{ not both zero...} \end{cases}$$

$$\bullet x=0 \Rightarrow \lambda=4 \Rightarrow z=\frac{1}{5} \Rightarrow \frac{1}{4}y^2 - \frac{1}{25} = 1 \Rightarrow y^2 = \frac{4 \cdot 26}{25}$$

$$\Rightarrow d = \sqrt{0^2 + \frac{4 \cdot 26}{25} + \left(\frac{1}{5} - 1\right)^2} = \sqrt{\frac{120}{25}} = \frac{2}{5}\sqrt{30}$$

$$\bullet y=0 \Rightarrow \lambda=1 \Rightarrow z=\frac{1}{2} \Rightarrow x^2 - \frac{1}{4} = 1 \Rightarrow x^2 = \frac{5}{4}$$

$$\text{Thus } d = \sqrt{\frac{5}{4} + 0^2 + \left(\frac{1}{2} - 1\right)^2} = \frac{\sqrt{6}}{2} \dots$$

$$\frac{\sqrt{6}}{2} < \frac{2}{5}\sqrt{30} \Rightarrow \boxed{\text{MIN} = \frac{\sqrt{6}}{2}}.$$

Practice!

(#1) Find, classify all critical points of
 $f(x,y) = \frac{1}{2}x^4 + xy + y^2 + 1$.

$$\begin{aligned} f_x &= 2x^3 + y = 0 \\ f_y &= x + 2y = 0 \Rightarrow x = -2y \\ &\rightarrow -16y^3 + y = 0 = y(1 - 16y^2) \\ &\Rightarrow y = 0, \frac{1}{4}, -\frac{1}{4} \quad (0,0), \left(-\frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{1}{4}\right) \end{aligned}$$

$$\left. \begin{array}{l} f_{xx} = 6x^2 \\ f_{yy} = 2 \\ f_{xy} = 1 \end{array} \right\} D = 12x^2 - 1$$

$$D(0,0) = -1 < 0 \quad (\text{SADDLE})$$

$$D\left(-\frac{1}{2}, \frac{1}{4}\right) = 2 > 0, f_{xx} > 0 \Rightarrow (\text{LOCAL MIN})$$

$$D\left(\frac{1}{2}, -\frac{1}{4}\right) = 2 > 0, f_{xx} > 0 \quad (\text{LOCAL MIN})$$

#2 Find the global MAX/MIN of
 $f(x, y) = x + xy - \frac{1}{3}y^3$ with $-2 \leq x \leq 2$,
 $-2 \leq y \leq 2$.

$$f_x = 1 + y = 0 \Rightarrow y = -1$$

$$f_y = x - y^2 = 0 \Rightarrow x = 1$$

$$f(1, -1) = 1 - 1 + \frac{1}{3} = \frac{1}{3} \text{ (CANDIDATE)}$$

• Check boundary...

Corners:

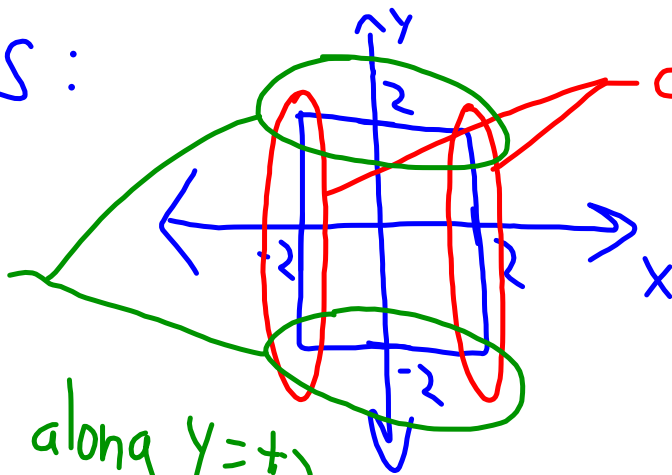
$$f(-2, 2) = \frac{14}{3}, \quad f(-2, -2) = -\frac{26}{3}$$

$$f(2, -2) = \frac{2}{3}, \quad f(2, 2) = \frac{10}{3}$$

Edges:

check
where

$$\frac{\partial f}{\partial x} = 0 \text{ along } y = \pm 2$$



check where

$$\frac{\partial f}{\partial y} = 0$$

along $x = \pm 2$

$$\bullet f_x(x, -2) = -1 \neq 0$$

$$\bullet f_x(x, 2) = 3 \neq 0$$

$$\bullet f_y(-2, y) = -2 - y^2 \neq 0$$

$$\bullet f_y(2, y) = 2 - y^2 = 0 \text{ if } y = \pm\sqrt{2}$$

$$\text{check } f(2, -\sqrt{2}) = \frac{6 - 4\sqrt{2}}{3}$$

$$f(2, \sqrt{2}) = \frac{6 + 4\sqrt{2}}{3}$$

so nothing
to check along
3 edges

Compare with
all other
f-values
now...

- Global min is $f(-2, 2) = \frac{-26}{3}$
- Global max is $f(-2, -2) = \frac{14}{3}$.

#3 Let $f(x,y,z) = x + 2y - 4z$. Minimize this over the unit sphere $x^2 + y^2 + z^2 = 1$.

$$\begin{aligned} f_x = 1 &= \lambda g_x = \lambda 2x \Rightarrow x = \frac{1}{2\lambda} \\ f_y = \lambda g_y &\Rightarrow 2 = \lambda 2y \Rightarrow y = \frac{1}{\lambda} \\ f_z = \lambda g_z &\Rightarrow -4 = \lambda 2z \Rightarrow z = \frac{-2}{\lambda} \end{aligned}$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{4}{\lambda^2} = 1$$

$$\frac{1}{4} + 1 + 4 = \lambda^2 = \frac{21}{4}$$

$$\lambda = \pm \sqrt{21}$$

$$f\left(\frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}\right) = \sqrt{21}$$

$$f\left(\frac{-1}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{4}{\sqrt{21}}\right) = -\sqrt{21} \quad (\text{MIN})$$