

Welcome to MATH 2110Q
Multivariable Calculus

Our goal is to extend calculus tools to account for multiple dimensions. For example, the Navier-Stokes equations for an incompressible fluid with 3D in space + 1D in time are:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \nabla p - \nu \Delta \vec{u} = \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

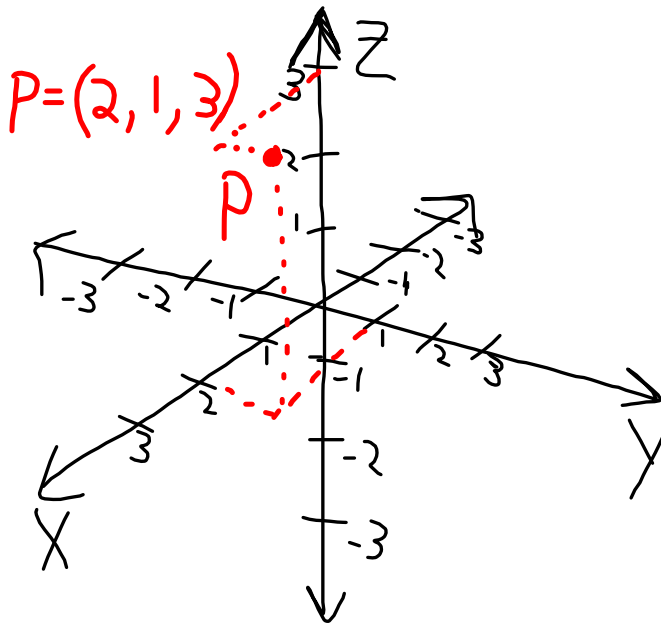
time
space

where $\vec{u} = \vec{u}(x, y, z, t)$ is a "vector".

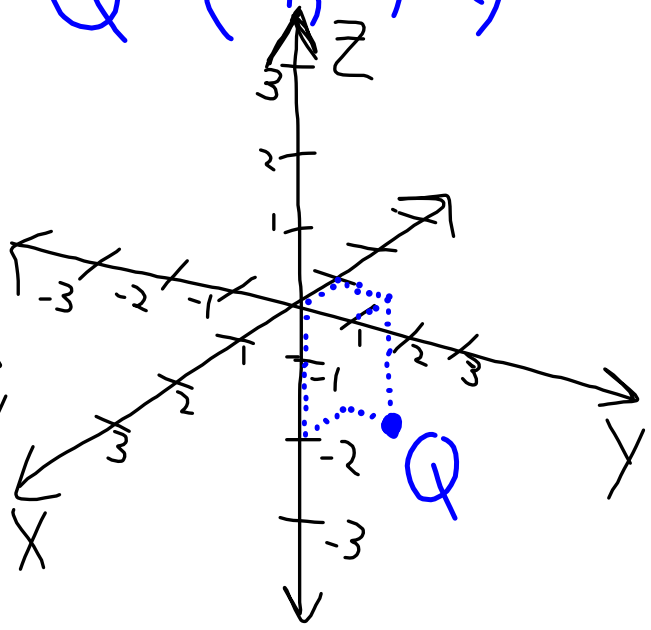
You should be able to "read" these equations at the end of the course.

Let us stop for a bit and talk about the syllabus and other details of the course.

3D Coordinates



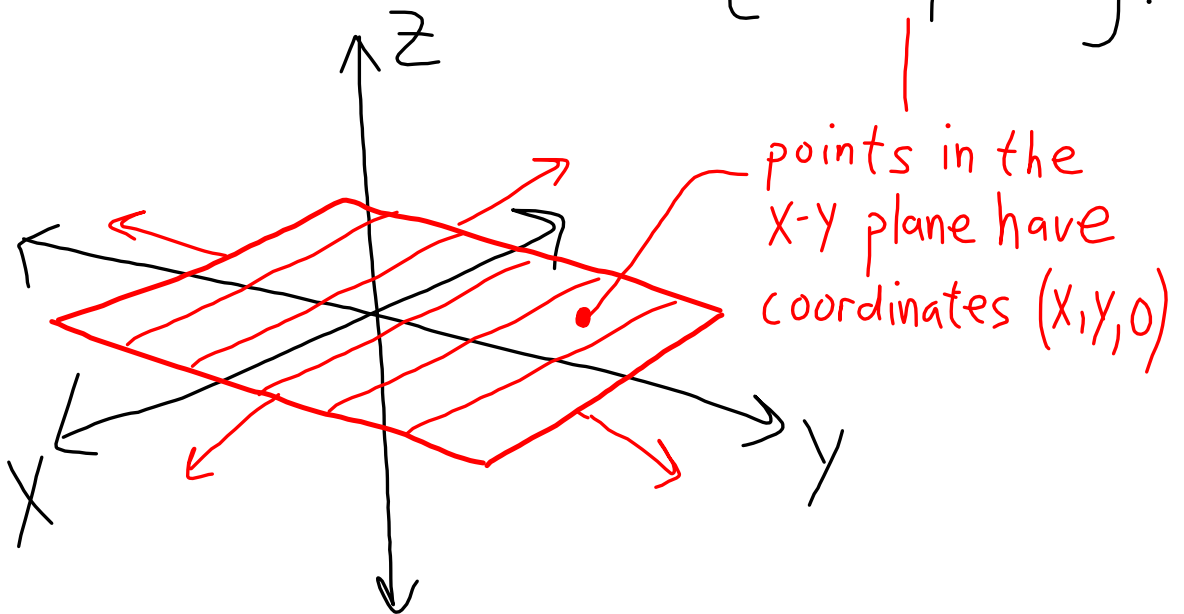
$Q = (-1, 1, -2)$



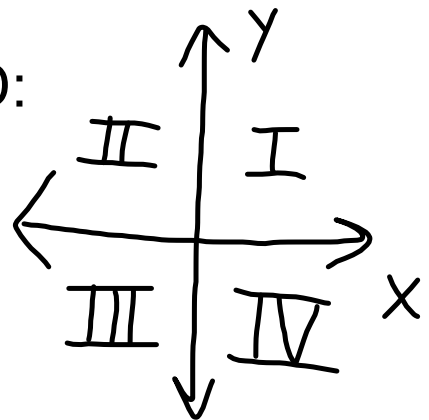
Note coordinates are always (x, y, z)

We can look at x-y, y-z, and x-z planes in 3D.

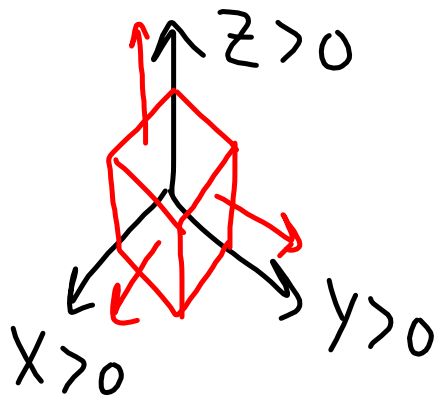
Notation... x-y plane is $\{(x,y,z) \mid z=0\}$.



Recall the four quadrants in 2D:



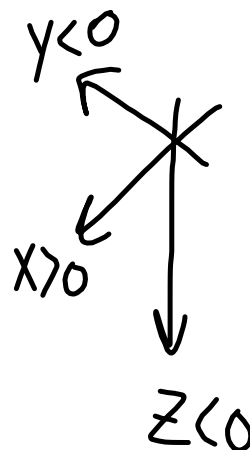
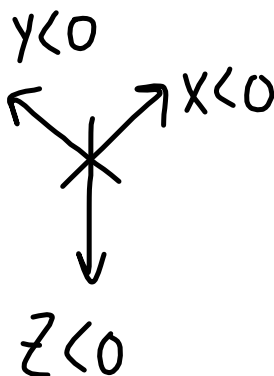
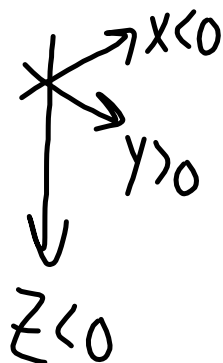
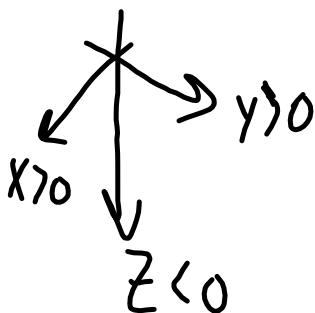
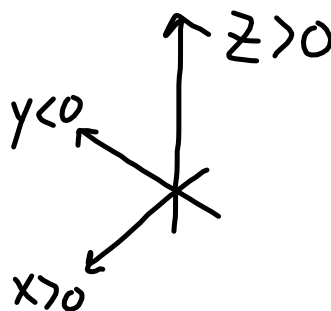
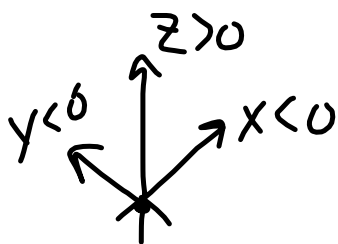
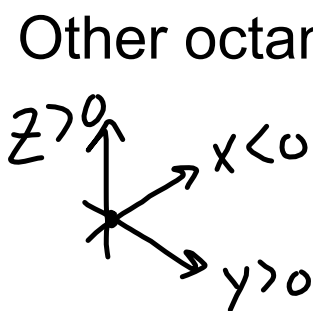
In 3D there are eight "octants". You only need to know the "first octant" by number:



First octant

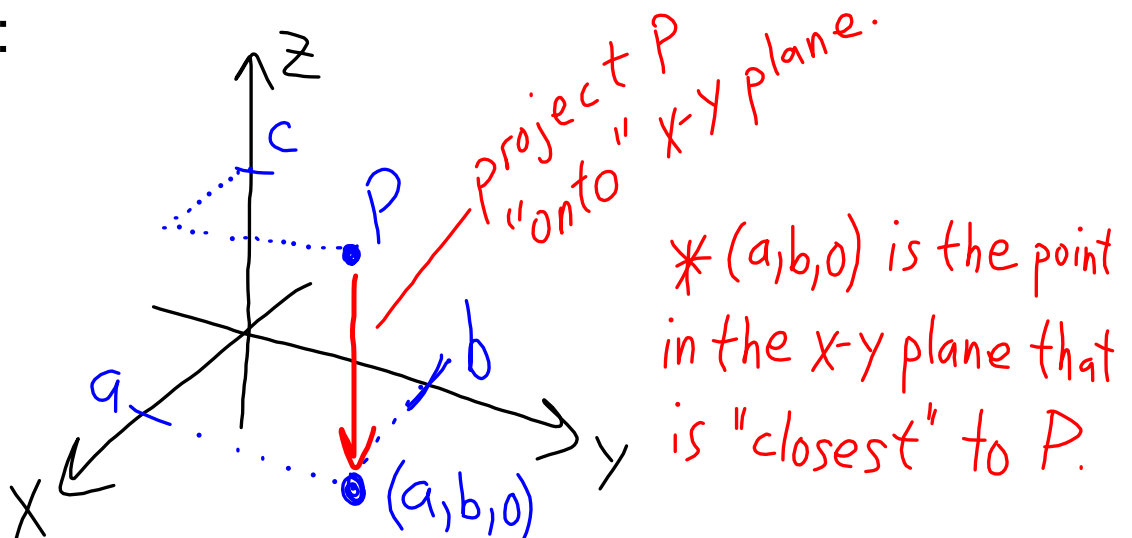
$$\{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0\}$$

Other octants:

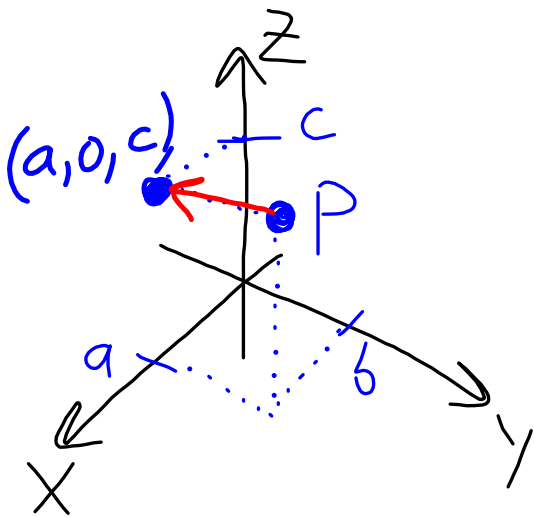


Projections: this concept can be generalized in many ways far outside the scope of this course. We will discuss "point" and "vector" projections.

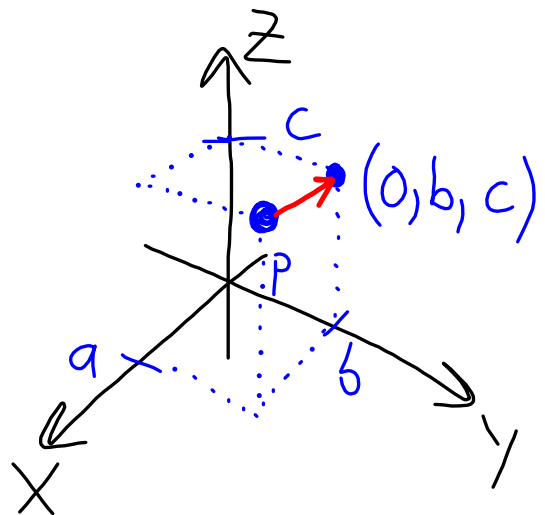
Projection of $P=(a,b,c)$ onto the x-y plane is $(a,b,0)$:



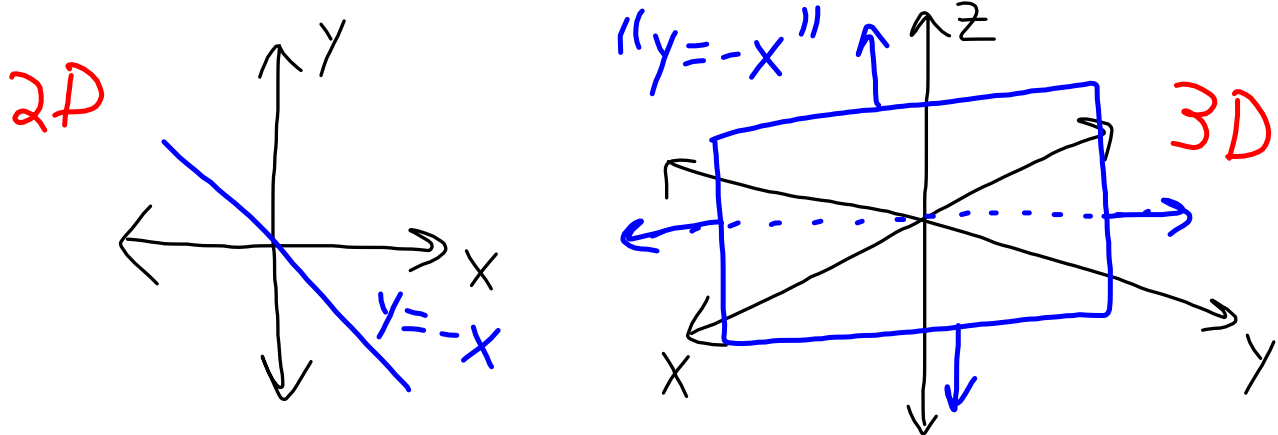
Projection of P onto
x-z plane is $(a,0,c)$



Projection of P onto
y-z plane is $(0,b,c)$



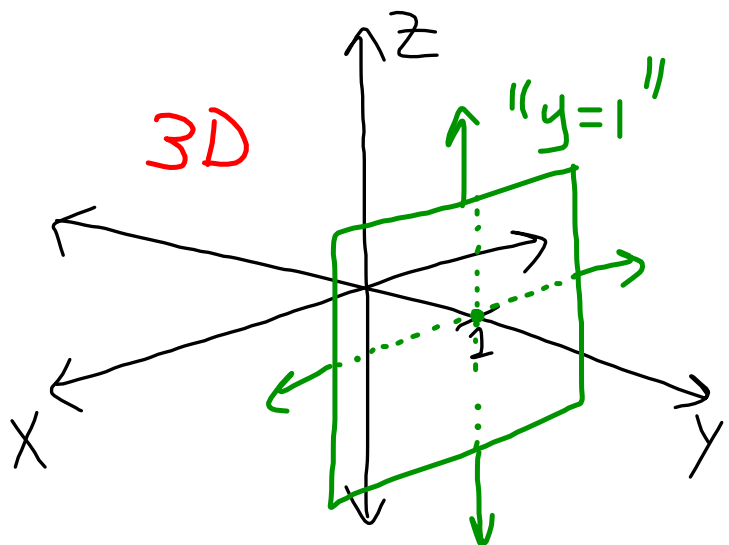
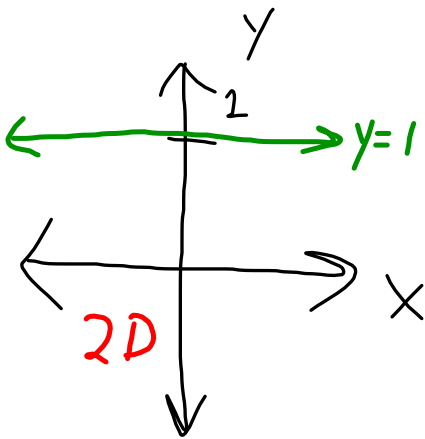
SURFACES. 2D lines become surfaces in 3D.



There is no restriction on z in 3D for the relationship $y = -x$, so ALL z -values are included:

$$\{(x, y, z) \mid y = -x\}.$$

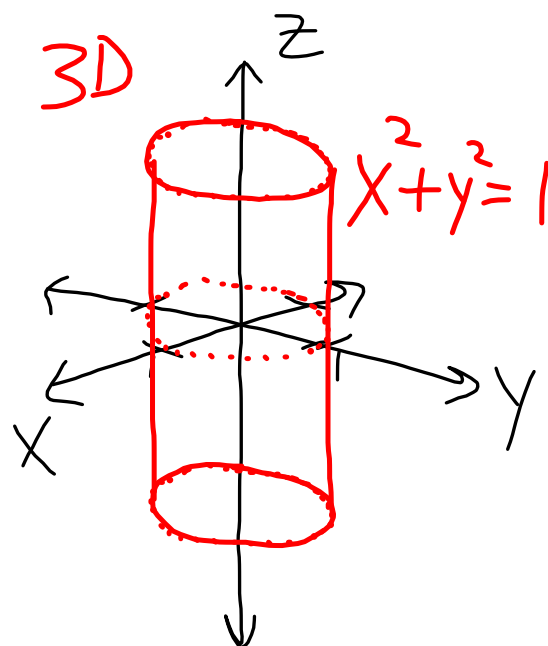
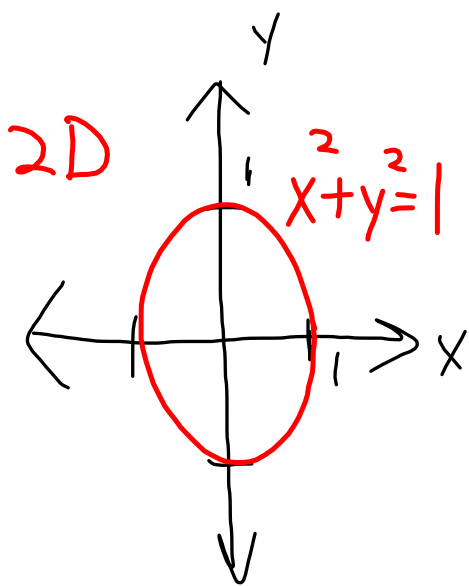
Another common example would be $y=1$.



In 3D, the surface $y=1$ is a plane parallel to the x-z plane but intersecting the y-axis at $y=1$;

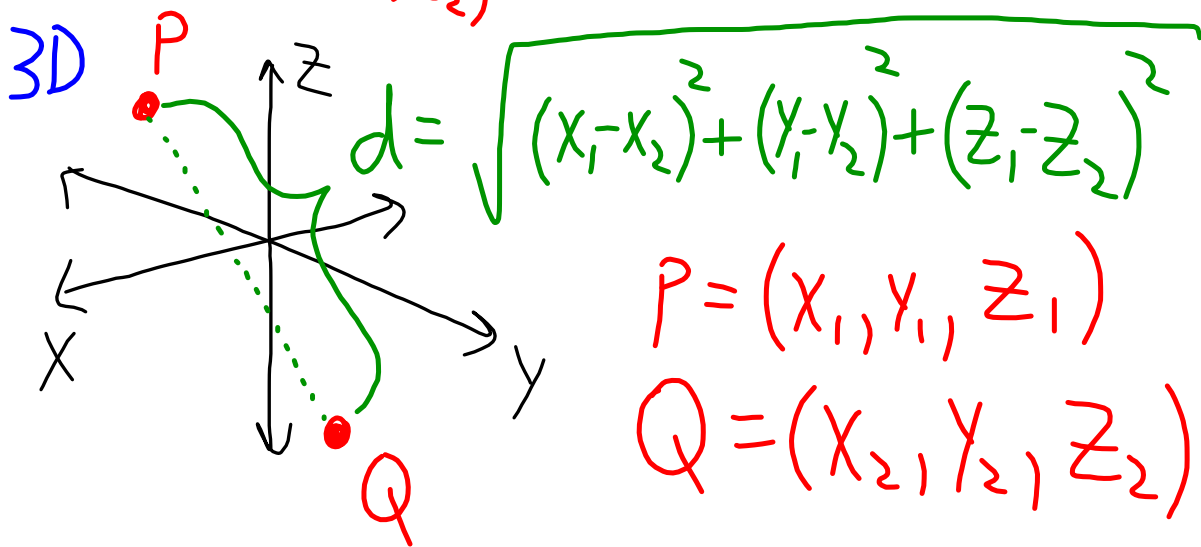
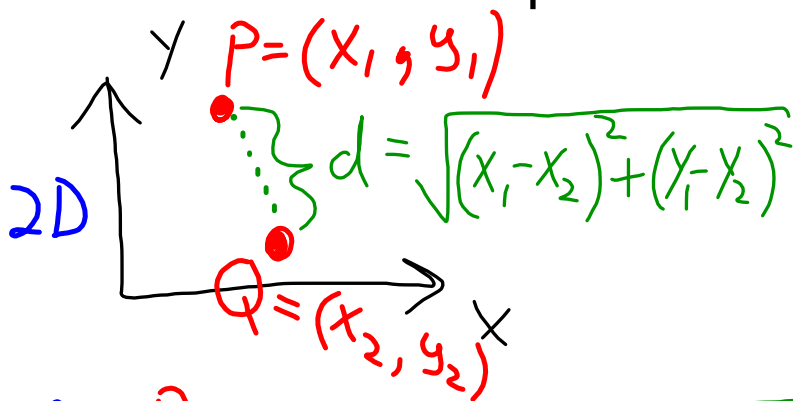
$$\{(x, y, z) \mid y=1\}.$$

A more complicated example is a CYLINDER.



$$\{(x, y, z) \mid x^2 + y^2 = 1\}.$$

Distance between points.



Note: sometimes we say $d = |PQ|$.

Ex. 1: Find $|PQ|$ if $P = (1, 2, 3)$ and
 $Q = (-1, 3, 4)$.

$$\begin{aligned} |PQ| &= \sqrt{(1 - (-1))^2 + (2 - 3)^2 + (3 - 4)^2} \\ &= \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6} \end{aligned}$$

Ex. 2: Find $|PQ|$ if $P=(1,2,3)$ & $Q=(1,2,0)$.

$$\begin{aligned} |PQ| &= \sqrt{(1-1)^2 + (2-2)^2 + (3-0)^2} \\ &= \sqrt{3^2} = 3. \end{aligned}$$

Note Q is the projection of P onto the x - y plane here. The distance is just the size of the z -coordinate of P .

One useful idea is the set of all points that are EQUIDISTANT from a central point.

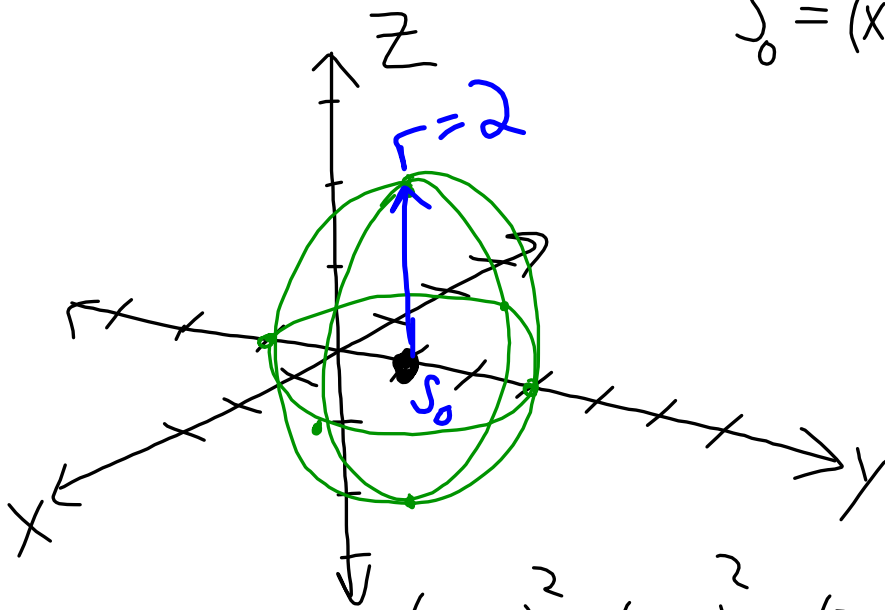
"Sphere of radius r "

S : sphere $S_0 = (x_0, y_0, z_0)$: center

$$\text{Then } S = \left\{ (x, y, z) \mid \underbrace{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}_{\text{"STANDARD FORM"}} = r^2 \right\}$$

*Note also $S = \left\{ P = (x, y, z) \mid |PS_0| = r \right\}$.

$$S_0 = (x_0, y_0, z_0) = (0, 1, 0)$$



$$(x-0)^2 + (y-1)^2 + (z-0)^2 = 2^2 = 4$$

$$\text{OR } x^2 + (y-1)^2 + z^2 = 4$$

Ex. 3: Put $x^2 + y^2 + z^2 + 3x + 2z - 4y = 10$
into standard form.

Remember: "complete the square"...

$$\begin{aligned}\left(x + \frac{3}{2}\right)^2 + (y - 2)^2 + (z + 1)^2 &= 10 + \left(\frac{3}{2}\right)^2 + (-2)^2 + 1^2 \\ &= 10 + \frac{9}{4} + 4 + 1 \\ &= \frac{40 + 9 + 16 + 4}{4} = \frac{69}{4}.\end{aligned}$$

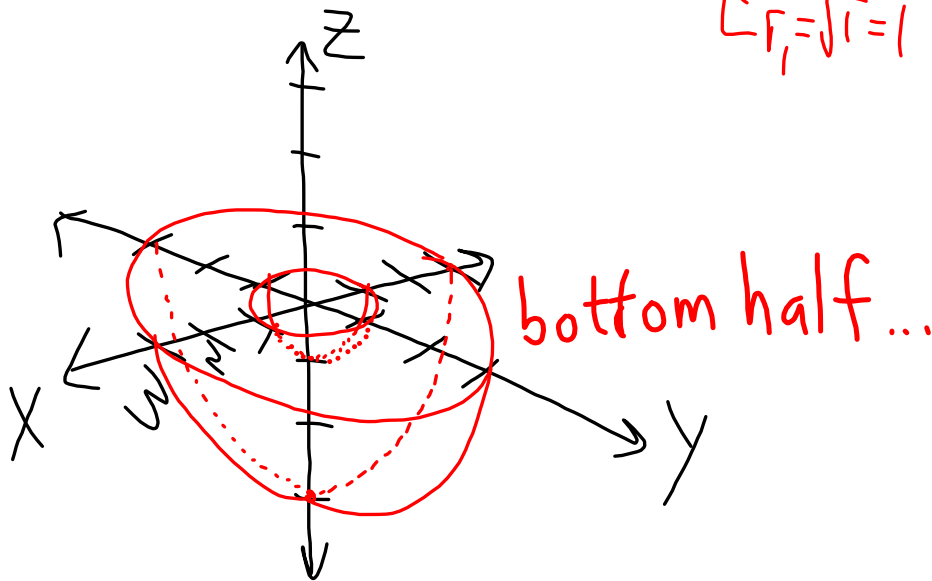
We can also envision things like spherical
VOLUMES;

$$r_1 \leq \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \leq r_2$$

Ex. 4: Sketch $V = \{(x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 9\}$.

$$\uparrow r_1 = \sqrt{1} = 1$$

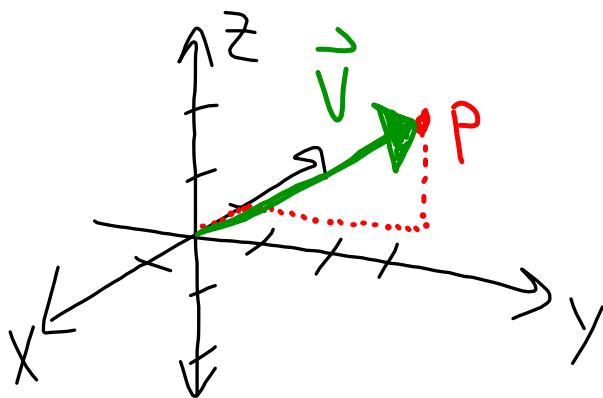
$$\uparrow r_2 = \sqrt{9} = 3$$



VECTORS. These are defined by precisely two characteristics: (1) direction and (2) magnitude.

$$\vec{V} = \langle -1, 3, 2 \rangle \quad (\text{vector})$$

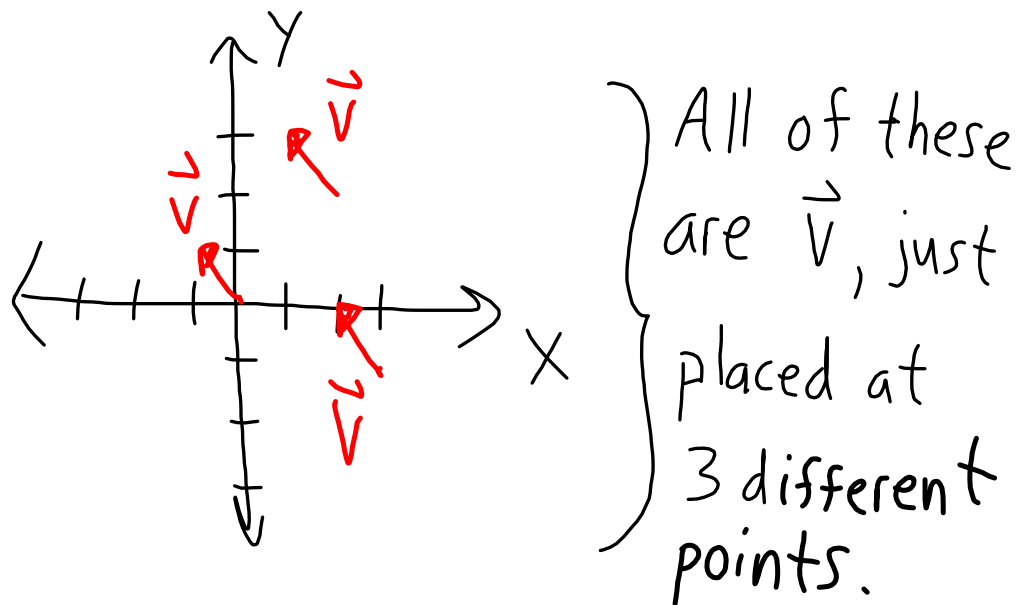
$$P = (-1, 3, 2) \quad (\text{point})$$



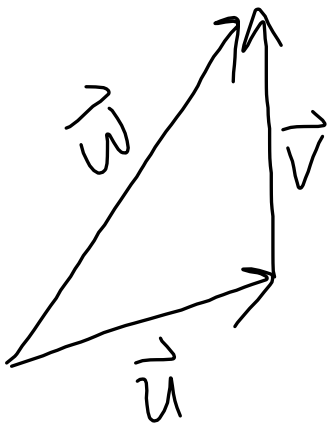
} One way to visualize \vec{V} .

A vector can be VISUALIZED with the tail at any desired point; it is still the same vector.

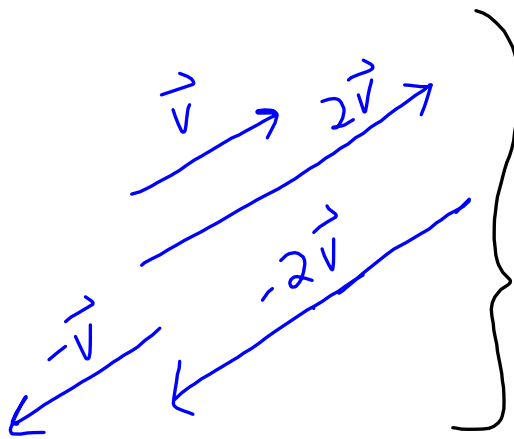
2D example $\vec{v} = \langle -1, 1 \rangle$



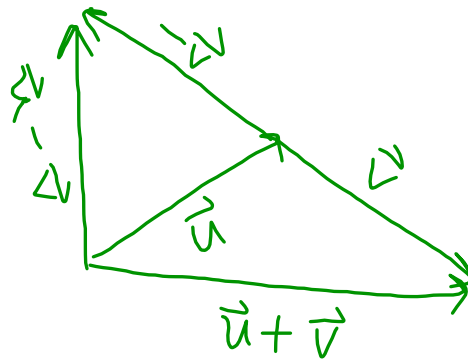
Vector addition. Put tail of \vec{v} at head of \vec{u} to make $\vec{w} = \vec{u} + \vec{v}$:



Multiplication by a scalar. Let $c \in \mathbb{R}$ and \vec{v} a vector. Then $\vec{w} = c\vec{v}$ is also a vector. If c is negative then the direction flips.



This also gives a way to think of vector subtraction:



Here are the algebraic rules; things are simply done "component-wise".

$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

Addition/subtraction:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

$$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

Scalar multiplication:

$$c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle.$$

Ex. 5 : $\vec{v} = \langle 1, 0, -5 \rangle$. Find $4\vec{v}$.

$$4\vec{v} = \langle 4 \cdot 1, 4 \cdot 0, 4 \cdot (-5) \rangle = \langle 4, 0, -20 \rangle.$$

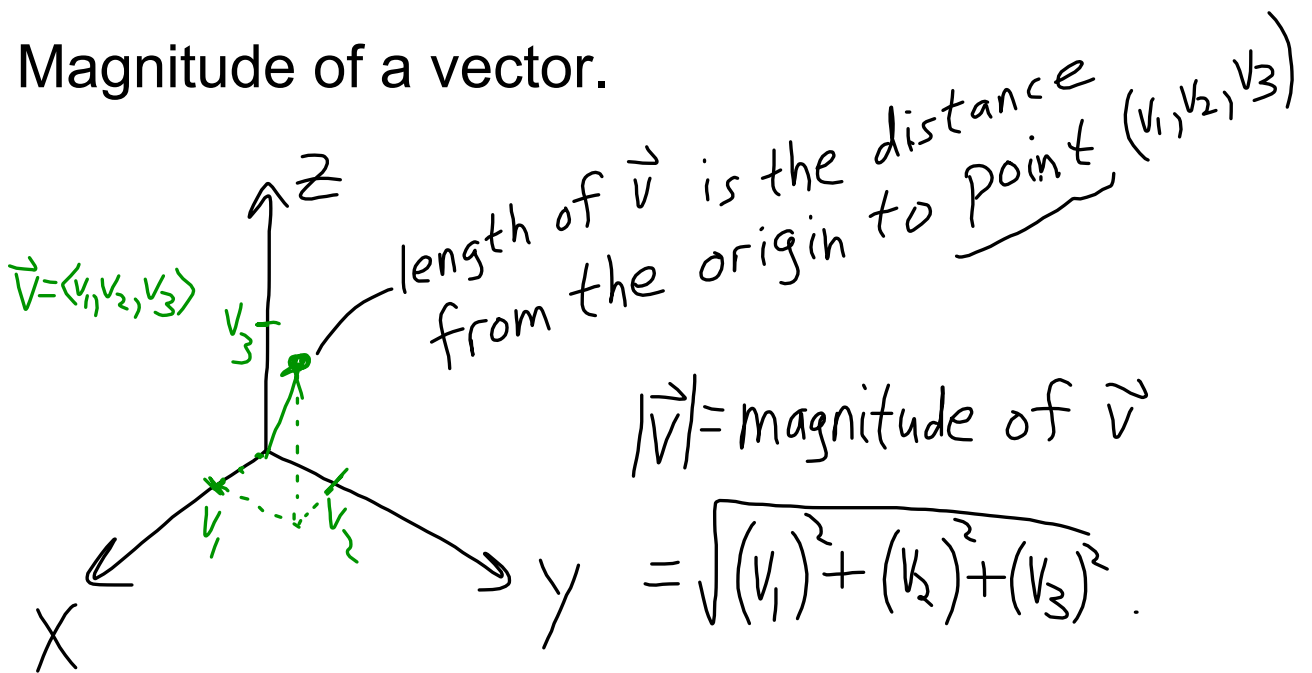
Ex. 6 : $\vec{u} = \langle -2, 7, -3 \rangle$, $\vec{v} = \langle 1, 2, 0 \rangle$.

Find $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$.

$$\vec{u} + \vec{v} = \langle -2 + 1, 7 + 2, -3 + 0 \rangle = \langle -1, 9, -3 \rangle.$$

$$\vec{u} - \vec{v} = \langle -2 - 1, 7 - 2, -3 - 0 \rangle = \langle -3, 5, -3 \rangle.$$

Magnitude of a vector.



Ex. 7: Find $|\vec{v}|$ if $\vec{v} = \langle -1, 1, 2 \rangle$.

$$|\vec{v}| = \sqrt{(-1)^2 + (1)^2 + (2)^2} = \sqrt{6}$$

Practice #1

Which of the following points is closest to the origin? $P = (1, -2, 2)$ or $Q = (3, 0, 1)$?

$$\text{For } P, d = \sqrt{1 + 4 + 4} = \sqrt{9}$$

$$\text{For } Q, d = \sqrt{9 + 0 + 1} = \sqrt{10}$$

} P is closer.

Practice #2

Find the distance from $P=(5, -7, 4)$ to the y - z plane.

projection is $(0, -7, 4)$

$$d = \sqrt{(5-0)^2 + (-7-(-7))^2 + (4-4)^2} = 5$$

Practice #3.

Find an equation for a sphere with center $(0,-1,2)$ and radius 7.

$$(x-0)^2 + (y-(-1))^2 + (z-2)^2 = 7^2$$

$$\Rightarrow x^2 + (y+1)^2 + (z-2)^2 = 49.$$

Practice #4.

Find the center and radius of the sphere

$$x^2 - 2x + y^2 + 2y + z^2 - 6z = -10.$$

$$\begin{aligned}(x-1)^2 + (y+1)^2 + (z-3)^2 &= -10 + 3^2 + 1^2 + 1^2 \\ &= -10 + 9 + 2 = 1 = r^2\end{aligned}$$

So $r=1$, center $(1, -1, 3)$.

Practice #5.

If $\vec{u} = \langle 5, -1 \rangle$ and $\vec{v} = \langle 3, 4 \rangle$, find $|\vec{u} - \vec{v}|$.

$$\vec{u} - \vec{v} = \langle 2, -5 \rangle$$

$$\Rightarrow |\vec{u} - \vec{v}| = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}.$$