

MATH 2110 - Review for Exam 4

Line integrals

$$\int_C f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt$$

$$\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

$$\vec{F} = \langle P, Q, R \rangle \Rightarrow \int_C \vec{F} \cdot d\vec{s} = \int_C P dx + Q dy + R dz .$$

Ex: Let C be the curve $y = x^{5/5}$,
with initial point $(0,0)$ and end point $(1, 1/5)$.
Calculate $\int_C x^2 y ds$.

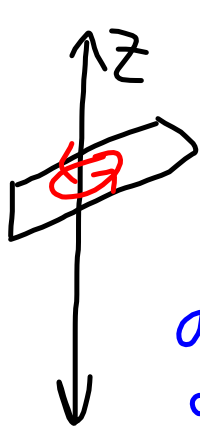
Use the natural parameterization $y = y(x)$
with $0 \leq x \leq 1$. Our formula is

$$\int_C x^2 y ds = \int_0^1 x^2 \cdot \frac{x^5}{5} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \frac{1}{5} \int_0^1 x^7 \sqrt{1 + x^8} dx$$

$$= \frac{1}{5} \int_0^1 \frac{d}{dx} \left\{ \frac{1}{12} (1+x^8)^{3/2} \right\} dx$$

$$= \frac{1}{5} \cdot \frac{1}{12} \cdot (2\sqrt{2}-1) = \boxed{\frac{2\sqrt{2}-1}{60}}$$

EX: Find $\int 2dx + ydy - dz$, if C is the intersection c of the cylinder $x^2 + y^2 = 4$ and the plane $z = 1 + x + y$, oriented counter-clockwise when viewed from "above" (+z-axis).



$$x = 2 \cos(t)$$

$$y = 2 \sin(t)$$

$$z = 1 + x + y = 1 + 2(\cos(t) + \sin(t))$$

$$dx = -2 \sin(t) dt$$

$$dy = 2 \cos(t) dt$$

$$dz = 2(\cos(t) - \sin(t)) dt$$

$$\int_C 2dx + ydy - dz$$

$$= \int_0^{2\pi} \left[-4\cancel{\sin(t)} + \underbrace{4\cos(t)\sin(t)}_{u=\sin(t)} - 2\cancel{\cos(t)} + 2\cancel{\sin(t)} \right] dt$$

$$du = \cos(t)dt$$

$$= 4 \int_{u=0}^{u=\sin(2\pi)=0} u du$$

$$= 0$$

Well, this was a conservative field and a closed curve...

Conservative vector fields

* Check if $\vec{F} = \nabla f$ by verifying
 $\nabla \times \vec{F} = 0$ (3D) or $\vec{F} = \langle P, Q \rangle \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

* Know the procedure to find f .

* "Fundamental Theorem"

$$\vec{F} = \nabla f \Rightarrow \int_C \vec{F} \cdot d\vec{s} = f(B) - f(A).$$

end-point on C starting point on C.

EX: Let $\vec{F} = \left\langle ye^{yx} - 1 + \frac{3}{2}y^2\sqrt{x}, y + xe^{yx} + 2yx^{3/2} \right\rangle$.

Is \vec{F} conservative? If so, find the potential function $f(x, y)$.

$$P = ye^{yx} - 1 + \frac{3}{2}y^2\sqrt{x}, \quad Q = y + xe^{yx} + 2yx^{3/2}$$

$$Q_x = e^{yx} + xy e^{yx} + 3y\sqrt{x}$$

$$P_y = e^{yx} + xy e^{yx} + 3y\sqrt{x}$$

$$\left. \begin{array}{l} Q_x = e^{yx} + xy e^{yx} + 3y\sqrt{x} \\ P_y = e^{yx} + xy e^{yx} + 3y\sqrt{x} \end{array} \right\} \begin{array}{l} Q_x = P_y \\ \Rightarrow \text{conservative.} \end{array}$$

$$\int P dx = \int y e^{yx} - 1 + \frac{3}{2} y^2 \sqrt{x} dx$$

$$= e^{yx} - x + y^2 x^{3/2} + g(y) = f(x, y).$$

Set y-derivative = \vec{j} -component, Q

$$Q = y + x e^{yx} + 2y x^{3/2} = \frac{\partial}{\partial y} (e^{yx} - x + y^2 x^{3/2} + g(y))$$

$$= x e^{yx} + 2y x^{3/2} + g'(y)$$

So $y = g'(y) \Rightarrow \frac{1}{2} y^2 = g(y)$. Plug in above to get

$$f(x, y) = e^{yx} - x + y^2 x^{3/2} + \frac{1}{2} y^2.$$

EX: Given $f(x, y, z) = x - y^2 + zy$, find

$$\int_C \nabla f \cdot d\vec{r}, \text{ where } \vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

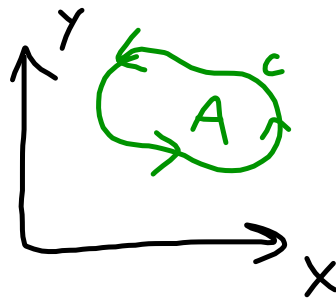
C for $0 \leq t \leq \pi/2$ parameterizes C .

$$\text{Note } \vec{r}(0) = \langle 1, 0, 0 \rangle \text{ \& } \vec{r}(\pi/2) = \langle 0, 1, \pi/2 \rangle.$$

Thus (Fundamental Theorem)

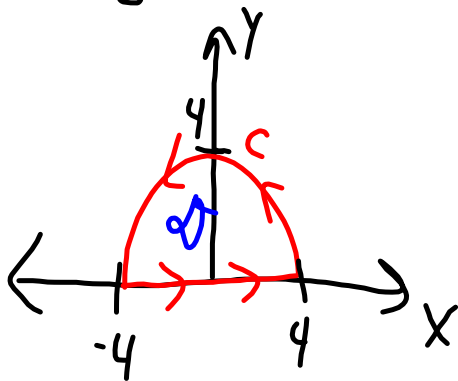
$$\int_C \nabla f \cdot d\vec{r} = f(0, 1, \pi/2) - f(1, 0, 0) = -1 + \frac{\pi}{2} - 1 = \frac{\pi}{2} - 2.$$

Green's Theorem



$$\int_c P dx + Q dy = \int_c \vec{F} \cdot d\vec{s}$$
$$= \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

EX: Let C be the boundary of the region $\{(x, y) \mid x^2 + y^2 \leq 16, y \geq 0\}$, with positive orientation. Find $\int_C y dx + xy dy$, using Green's Theorem.



$$\int_C P dx + Q dy = \iint_D Q_x - P_y dA$$

$$= \iint_D y - 1 dA = \int_0^\pi \int_0^4 (r \sin \theta - 1) r dr d\theta$$

$$= \int_0^{\pi} \int_0^4 r^2 \sin \theta - r \, dr \, d\theta = \int_0^{\pi} \left[\frac{1}{3} \cdot 4^3 \sin \theta - \frac{1}{2} \cdot 4^2 \right] d\theta$$

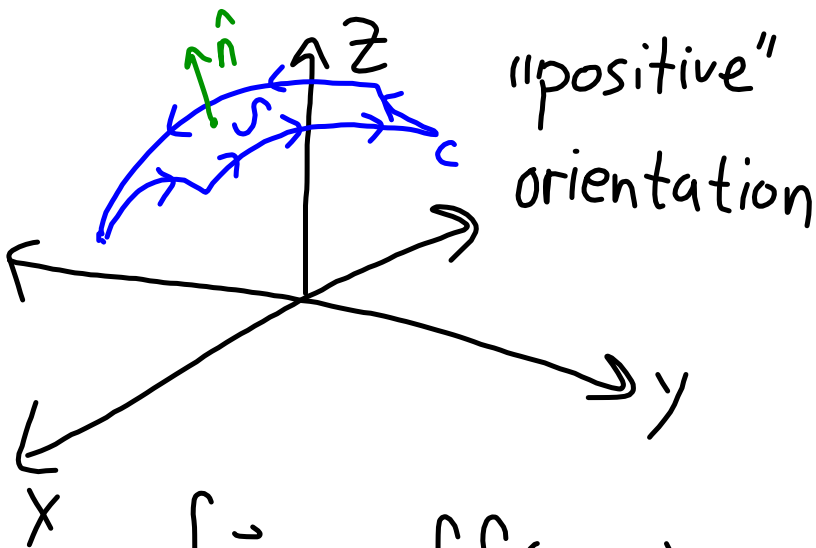
$$= \frac{64}{3} (-\cos \theta) \Big|_0^{\pi} - 8\pi$$

$$= \boxed{\frac{128}{3} - 8\pi}$$

$$\int_S f(x, y, z) dS = \int_D f(\vec{r}(u, v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv$$

$$\int_S \vec{F} \cdot \hat{n} dS = \int_D \vec{F}(\vec{r}(u, v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du dv$$

Stoke's Theorem



$$\int_{c=\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS .$$

Ex: Let S be the section of the surface $z=1+y^2$ that lies over $D = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$.

Given $\vec{F} = \langle y, yz^2, z^4 y^2 \rangle$, find $\int \vec{F} \cdot d\vec{s}$ if ∂S has positive orientation with ∂S respect to the upward-pointing normal on S .



Line integral requires 4 pieces,
so let's apply Stoke's.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & yz^2 & yz^4 \end{vmatrix} = \langle 2yz^4 - 2yz, 0, -1 \rangle$$

Recall $\vec{n} = \langle -z_x, -z_y, 1 \rangle = \langle 0, -2y, 1 \rangle = \vec{r}_x \times \vec{r}_y$

if we choose $\vec{r}(x,y) = \langle x, y, 1+y^2 \rangle$.

We get

$$\int_C \vec{F} \cdot d\vec{s} = \int_{-1}^1 \int_{-1}^1 (\text{Div} \vec{F}) (\vec{r}_x \times \vec{r}_y) dy dx$$

$$= \int_{-1}^1 \int_{-1}^1 \langle 2yz^4 - 2yz, 0, -1 \rangle \cdot \langle 0, -2y, 1 \rangle dy dx$$
$$= \int_{-1}^1 \int_{-1}^1 -1 dy dx = \boxed{-4}.$$

Divergence Theorem

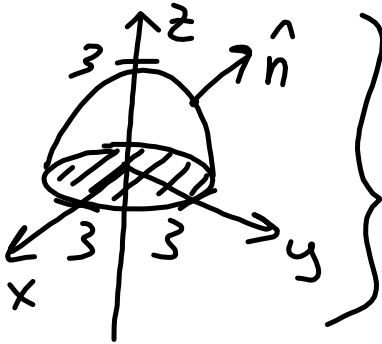
$$\int_{\partial V} \vec{F} \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV$$

where \hat{n} points "outward".

EX: Let S be the surface of the region

$V = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq a, z \geq 0\}$, \hat{n} outward,

$\vec{F} = \langle x, y, z^2 \rangle$. Find $\int_{\partial V} \vec{F} \cdot \hat{n} \, dS$.



Surface integral requires two pieces and is tedious.
Use Divergence Theorem:

$$\begin{aligned}
 \iint_{\partial V} \vec{F} \cdot \hat{n} \, dS &= \iiint_V \left(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2) \right) dV \\
 &= \iiint_V 2(1+z) \, dV = \underbrace{2|V|}_{\frac{4}{3}\pi \cdot 3^3} + 2 \underbrace{\iiint_V z \, dV}_{\text{use spherical coordinates}}.
 \end{aligned}$$

$$= 36\pi + 2 \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 \rho \cos\phi \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 36\pi + 2 \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi \cos\phi \, d\phi \int_0^3 \rho^3 \, d\rho$$

$$= 36\pi + 4\pi \left[\frac{1}{2} \sin^2\phi \right]_0^{\pi/2} \left[\frac{1}{4} \rho^4 \right]_0^3$$

$$= 36\pi + 4\pi \frac{1}{2} \cdot \frac{1}{4} \cdot 3^4 = 36\pi + \frac{81\pi}{2}$$

$$= \boxed{\frac{153\pi}{2}}$$