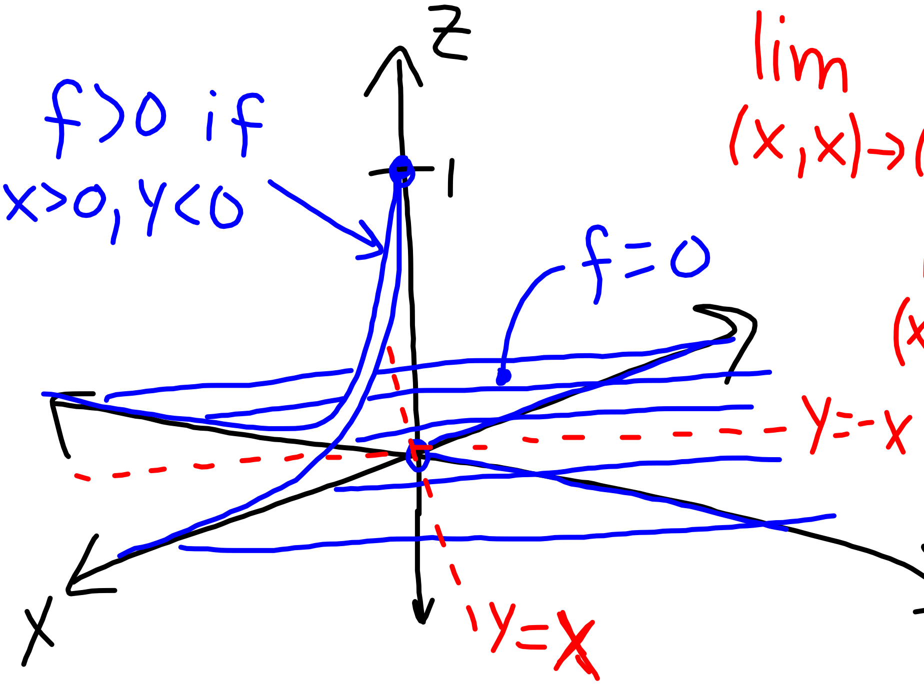
Review for Exam 2

Math 2110Q

In 2D (or higher) the limit may only exist along specific paths.



$$\lim_{(x,x)\to(0,0)} f(x,x)=0$$

$$\lim_{(x,-x)\to(0,0)} f(x,-x)=DNE$$

Find the limit or else show that it does not exist:

$$(a)\ \lim_{(x,y)\to(1,2)} \frac{x+y-2}{x+y-3} \begin{matrix}\leftarrow \text{get "1"} \\ \leftarrow \text{looks like "0"}\end{matrix}$$

This "blows up" somehow. Take $x=1$

$$\Rightarrow \frac{1+y-2}{1+y-3} = \frac{y-1}{y-2} \left\} \begin{matrix}\text{positive as } y\to 2^+ \\ \text{negative as } y\to 2^-\end{matrix}\right.$$

So this goes to $\pm\infty$ depending on the way we approach $\Rightarrow$ D.N.E.

(b) $\lim_{(x,y)\to(-1,1)} \dfrac{x^2+y^3}{x-y+\sin(x+y)}$.

So $(-1,1)$ is in the domain here...

- cite continuity
- plug in point to evaluate

$$\lim_{(x,y)\to(-1,1)} \frac{x^2+y^3}{x-y+\sin(x+y)} = \frac{1+1}{-1-1+0} = \frac{2}{-2} = \boxed{-1}$$

(c) $$\lim_{(x,y)\to(0,0)} \frac{\sin(x)}{\sin(y)\cos(y)} \quad \Big\} \quad \text{"}\frac{0}{0}\text{"}$$

- Try $x=0,\ y\neq 0 \Rightarrow \dfrac{\sin(0)}{\sin(y)\cos(y)} = 0$

- Look for a way to get a different result; $y=x \Rightarrow \dfrac{\cancel{\sin(x)}}{\cancel{\sin(x)}\cos(x)}$

$$= \frac{1}{\cos(x)} \xrightarrow[x\to 0]{} 1 \neq 0 \ (\text{DNE})$$

Implicit differentiation

Let $z = z(x,y)$ satisfy

$$x^3+y^2+z^4+xyz=1.$$

Then find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ by differentiating through the whole equation...

$$\frac{\partial}{\partial x}\left(x^3+y^2+z^4+xyz\right)=\frac{\partial}{\partial x}(1)=0$$

$$\Rightarrow 3x^2+4z^3\frac{\partial z}{\partial x}+yz+xy\frac{\partial z}{\partial x}=0.$$

Now group $\frac{\partial z}{\partial x}$ terms and factor it out:

$$3x^2+yz+\frac{\partial z}{\partial x}\left\{4z^3+xy\right\}=0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x}=\frac{-3x^2-yz}{4z^3+xy}}.$$

$$\frac{\partial}{\partial y}\left(x^3+y^2+z^4+xyz\right)=\frac{\partial}{\partial y}(1)=0$$

$$\Rightarrow 2y+4z^3\frac{\partial z}{\partial y}+xz+xy\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{\partial z}{\partial y}(4z^3+xy) = -xz-2y$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-xz-2y}{4z^3+xy}}.$$

$*$ If $(x,y,z(x,y))$ solve the original equation then you can plug them in to get the partial derivatives.

Given that $xy+xz+e^z=2$

find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ at $(1,1,0)$.

$$\frac{\partial}{\partial x}\left(xy+xz+e^z\right)=\frac{\partial}{\partial x}(2)=0$$

$$\Rightarrow y+z+x\frac{\partial z}{\partial x}+e^z\frac{\partial z}{\partial x}=0$$

(plug in $(1,1,0)$...)

$$\Rightarrow 1+0+\frac{\partial z}{\partial x}+\underbrace{e^0}_{1}\frac{\partial z}{\partial x}=0=1+2\frac{\partial z}{\partial x}$$

$$\Rightarrow \underline{\frac{\partial z}{\partial x} = -\frac{1}{2}}.$$

$$\frac{\partial}{\partial y}\left(xy + xz + e^{z}\right) = x + x\frac{\partial z}{\partial y} + e^{z}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow 1 + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \underline{\frac{\partial z}{\partial y} = -\frac{1}{2}}.$$

The Chain Rule revisited

Recall $y = y(x)$ & $x = x(t)$

$$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{(Chain Rule)}$$

But now we can have $f = f(x, y)$

with $x = x(t)$ & $y = y(t)$

$$\Rightarrow \frac{df}{dt} = \underset{\uparrow}{\frac{\partial f}{\partial x}} \frac{dx}{dt} + \underset{\uparrow}{\frac{\partial f}{\partial y}} \frac{dy}{dt} .$$

note "partials"

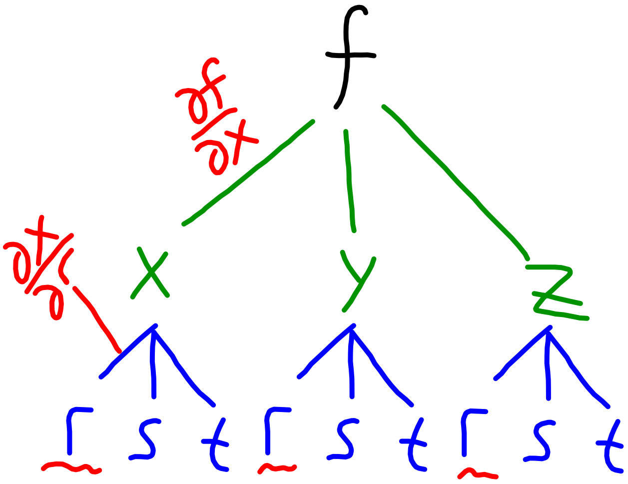
Tree diagrams

f=f(x,y,z)

x=x(r,s,t)

y=y(r,s,t)

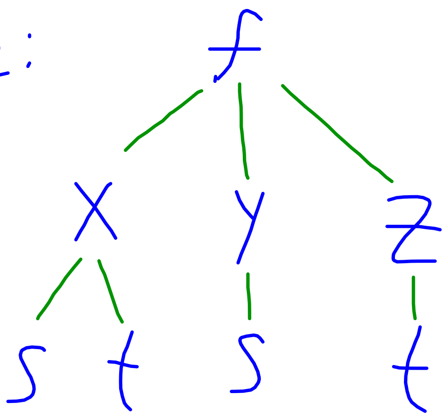
z=z(r,s,t)



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}$$

Ex: If $f(x,y,z) = xy + z^2$ and we have $x = x(s,t)$, $y = y(s)$, $z = z(t)$, find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Tree:



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Let $F(x,y,z)=e^{x}(2y+4z)$, with $x=t+s$, $y=2r-3t$, $z=r-t+s$. Use the Chain Rule to find $\frac{\partial F}{\partial t}$ with $(r,s,t)=(1,0,1)$.

$$F_t = F_x x_t + F_y y_t + F_z z_t$$

$$\left.\begin{array}{ll} F_x = e^{x}(2y+4z) & x_t = 1 \\ F_y = 2e^{x} & y_t = -3 \\ F_z = 4e^{x} & z_t = -1 \end{array}\right\} \text{still need } (x,y,z)$$

$x = 1 + 0 = 1$
$y = 2 - 3 = -1$
$z = 1 - 1 + 0 = 0$

$\Rightarrow F_x = e(2(-1) + 4(0))$
$= -2e$
$F_y = 2e, \; F_z = 4e$

Thus, $F_t = -2e + 2e(-3) + 4e(-1)$
$= (-2 - 6 - 4)e = \underline{-12e}.$

Directional derivatives

Recall f_x : rate of change, x-direction

f_y : rate of change, y-direction

$\hat{u} = \langle a, b \rangle$: unit vector

Rate of change in direction $\hat{u}$ is

$$D_{\hat{u}} f = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}.$$

$\underline{\text{DEFINE}}$: $\underset{\text{"gradient"}}{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

$$\Longrightarrow \boxed{D_{\hat{u}} f = \nabla f \cdot \hat{u}}$$

$$\overset{\nabla f}{\langle f_x, f_y \rangle} \cdot \overset{\hat{u}}{\langle a, b \rangle} = a f_x + b f_y$$

Let $f(x,y)=4xy+x^2-2y^2$

(a) Find the derivative of f in the direction $\langle 4,3\rangle$ at $(-1,0)$.

$$|\langle 4,3\rangle| = \sqrt{16+9} = \sqrt{25} = 5 \Rightarrow \hat{u} = \frac{1}{5}\langle 4,3\rangle.$$

$$\nabla f = \langle 4y+2x, 4x-4y\rangle \left.\right\} \text{plug in } \begin{matrix} x=-1 \\ y=0 \end{matrix}$$

$$\nabla f(-1,0) = \langle -2,-4\rangle$$

$$\Rightarrow D_{\hat{u}} f = \frac{1}{5}\langle -2,-4\rangle \cdot \langle 4,3\rangle = \frac{-8-12}{5} = \boxed{-4.}$$

(b) Find $D_{\hat{u}} f$ at $(2,-1)$ if $\hat{u}$ points in the direction of fastest increase.

Then $\hat{u} = \frac{\nabla f}{|\nabla f|}$ and $D_{\hat{u}} f = |\nabla f|$

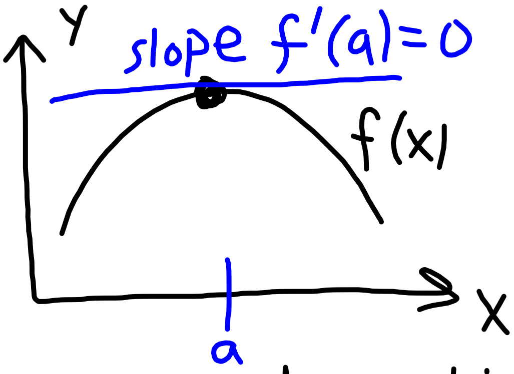
$$\nabla f\Big|_{(2,-1)} = \langle 4y+2x, 4x-4y \rangle\Big|_{(2,-1)}$$

$$= \langle -4+4, 8+4 \rangle = \langle 0, 12 \rangle$$

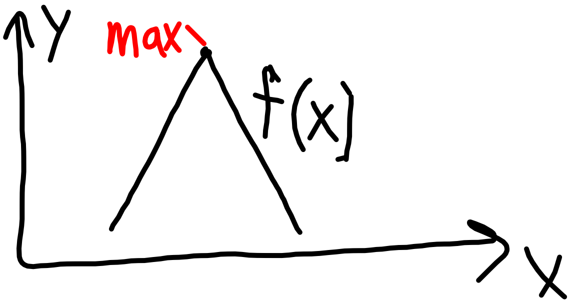
$$\Rightarrow |\nabla f| = D_{\hat{u}} f = \underline{12}.$$

When derivatives exist at a max/min, they must be zero.

Case 1 : derivative = 0



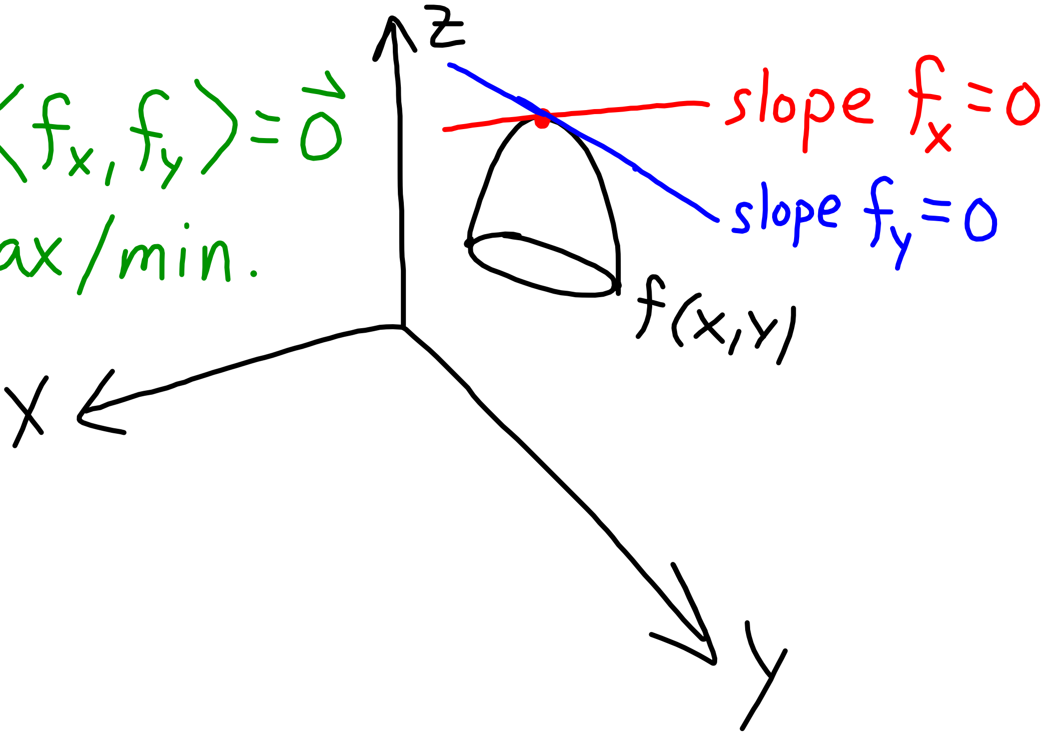
Case 2: no derivative



With multiple variables, the partial derivatives will be zero when they exist at max/min points.

$\nabla f = \langle f_x, f_y \rangle = \vec{0}$

at max/min.



... similarly for more variables

$f(x,y,z)$ has MAX/MIN

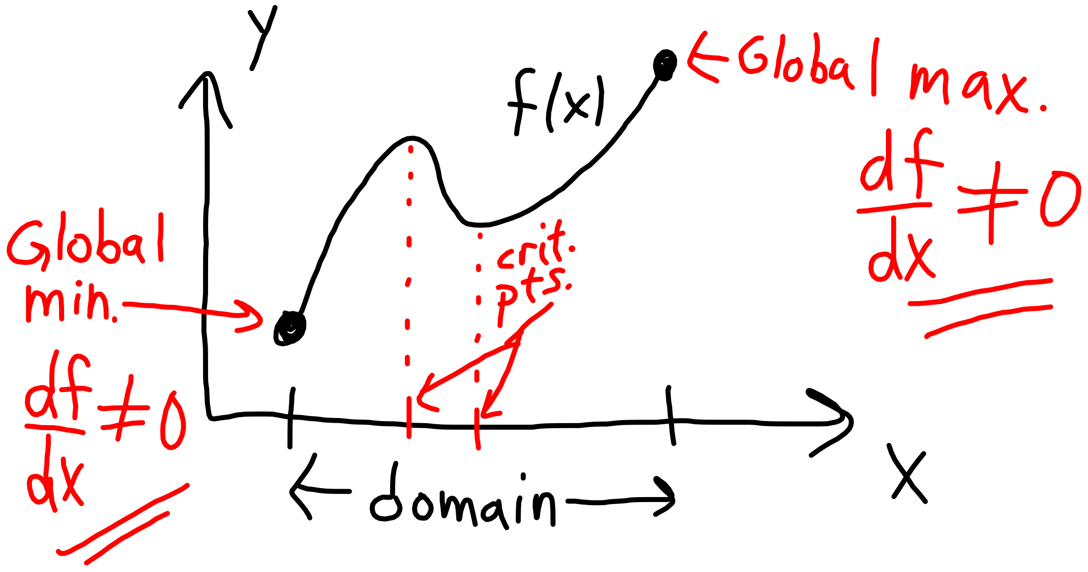
at $(a,b,c) \Rightarrow \nabla f = \langle f_x, f_y, f_z \rangle$

$= \langle 0,0,0 \rangle$ at (a,b,c).

Since ∇f may not exist at MAX/MIN points we check all CRITICAL POINTS, i.e.

where $\nabla f = \vec{0}$ <u>or</u> $\nabla f =$ D.N.E.

Looking at critical points gives us local extrema and sometimes global extrema; not generally on a closed and bounded domain.



Identifying GLOBAL extrema takes work:

(1) Find f-values at all critical points inside the domain.

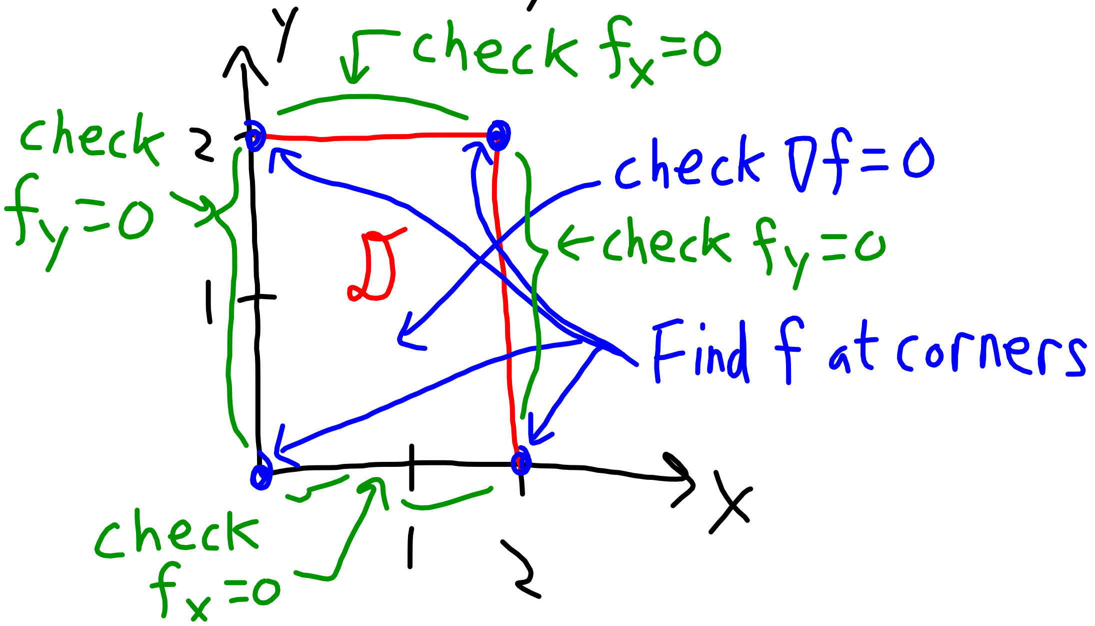
(2) Find max/min f-values along the domain boundary.

(3) Take MAX/MIN of all f-values from steps (1) & (2).

Given $f(x,y) = e^{-x^2+2x-2y^2+4y-3}$

find the absolute max/min values for f

on $0 \leq x \leq 2$, $0 \leq y \leq 2$.



$$f_x = 0 = (-2x+2)\underbrace{e^{-x^2+2x-2y^2+4y-3}}_{\text{never} = 0}$$

$$\Rightarrow x = 1$$

Candidate

$$f_y = 0 = (-4y+4)e^{(\dots)} \Rightarrow y = 1$$

So check $f(1,1) = e^{-1+2-2+4-3} = e^{0} = 1.$

Check corners of $\mathcal{D}$...

$f(0,0) = e^{-3}$ $\quad f(2,0) = e^{-4+4-3} = e^{-3}$

$f(0,2) = e^{-8+8-3} = e^{-3}$ $\quad f(2,2) = e^{-3}$

So far, max $=1$, min $= e^{-3}$. Now we check $f_x = 0$ along $y=0$, $y=2$...

$f_x = 0 \Rightarrow x = 1$ (already solved this)

So now check f at $(1,0)$ & $(1,2)$

$$f(1,0) = e^{-1+2-3} = e^{-2}$$

$$f(1,2) = e^{-1+2-8+8-3} = e^{-2}$$

Similarly, $f_y = 0 \Rightarrow y = 1$, so check $f(0,1)$, $f(2,1)$...

$$f(0,1) = e^{-2+4-3} = e^{-1} \quad \& \quad f(2,1) = e^{-1}$$

MAX $=1$, MIN $= e^{-3}$.

Lagrange Multipliers: used to maximize or minimize a function under constraints.

$f(x,y)$: A function to minimize or maximize

$g(x,y)=K$: "Constraint equation" restricts (x,y) allowed (think "design parameters")

*First step: identify these for your problem.

Let $f=f(x,y,z)$... $g(x,y,z)=K$ constraint.
To find the MAX or MIN of f, solve

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} \\ g(x,y,z) &= K \end{aligned}\right\}$$

Solving for x, y, z, λ

4 variables

4 equations

The volume of a cylinder in terms of radius r and height h is given by $V = \pi r^2 h$. The surface area is

$$S = 2\pi r^2 + 2\pi r h.$$

Maximize the volume if $S=4$, fixed.

Solve $\begin{cases} \nabla V = \lambda \nabla S \\ S = 4 \end{cases}$

for r, h, λ.

1) $V_r = 2\pi r h = \lambda S_r = \lambda(4\pi r + 2\pi h)$

2) $V_h = \pi r^2 = \lambda S_h = \lambda(2\pi r)$

3) $S = 2\pi r^2 + 2\pi r h = 4$

$(r \neq 0)$

$(2) \Rightarrow r^2 = 2\lambda r \Rightarrow r(r-2\lambda) = 0 \Rightarrow r = 2\lambda$

Insert in (1)... $4\pi\lambda h = \lambda(8\pi\lambda + 2\pi h)$

$\Rightarrow 2\lambda h = 4\lambda^2 + \lambda h$

$\Rightarrow 0 = 4\lambda^2 - \lambda h = \lambda(4\lambda - h).$

$\Rightarrow \cancel{\lambda = 0}$ or $4\lambda = h$

Put $r=2\lambda$, $h=4\lambda$ into $S=4$:

$$2\pi(2\lambda)^2+2\pi(2\lambda)(4\lambda)=4$$

$$8\pi\lambda^2+16\pi\lambda^2=4$$

$$24\pi\lambda^2=4$$

$$\lambda^2=\frac{4}{24\pi}=\frac{1}{6\pi}\Rightarrow \overbrace{\lambda=\frac{1}{\sqrt{6\pi}}}^{\text{positive root}}$$

$$\Rightarrow r=\frac{2}{\sqrt{6\pi}},\ h=\frac{4}{\sqrt{6\pi}}$$

$$\Rightarrow V=\pi r^2 h=\pi\frac{4}{6\pi}\cdot\frac{4}{\sqrt{6\pi}}=\boxed{\begin{array}{c}\text{MAX } V\\ \dfrac{8}{3\sqrt{6\pi}}\end{array}}$$