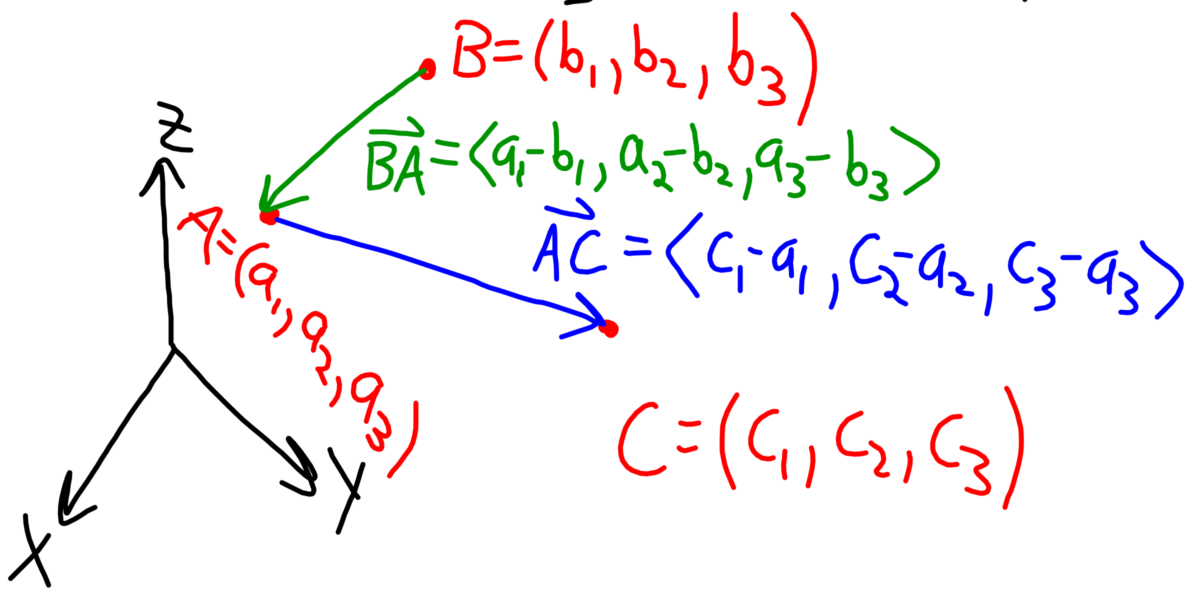
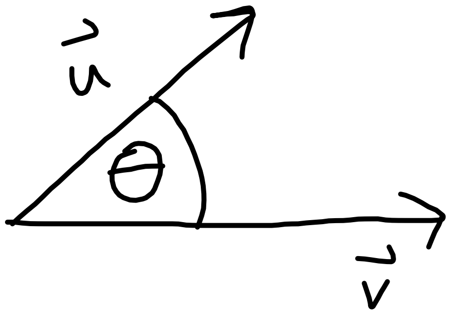
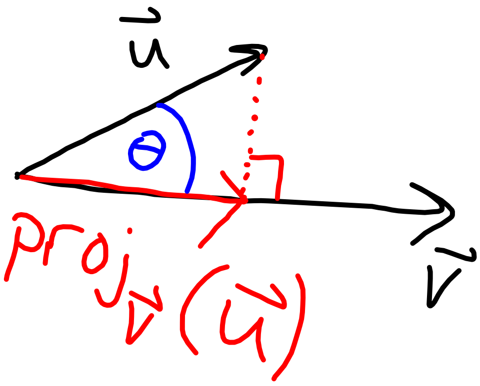
Recall constructing vectors to denote moving between 2 points:





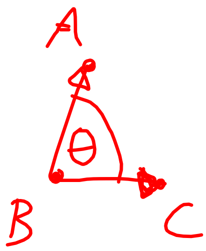
$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right).$$



$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

Ex: Let $A=(2,-1,1)$, $B=(3,1,1)$, $C=(-1,0,4)$. Find $\angle ABC$.



Use vectors $\overrightarrow{BA}$, $\overrightarrow{BC}$

$$\overrightarrow{BA}=\langle 2-3,-1-1,1-1\rangle=\langle -1,-2,0\rangle$$

$$\overrightarrow{BC}=\langle -1-3,0-1,4-1\rangle=\langle -4,-1,3\rangle$$

$$\theta=\cos^{-1}\left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}\right)=\cos^{-1}\left(\frac{4+2}{\sqrt{5}\cdot\sqrt{26}}\right)=\cos^{-1}\left(\frac{6}{\sqrt{130}}\right)$$

Ex: If $\vec{v} = 2\vec{i} - \vec{j} + 6\vec{k}$, $\vec{u} = \vec{j} - \vec{k}$,

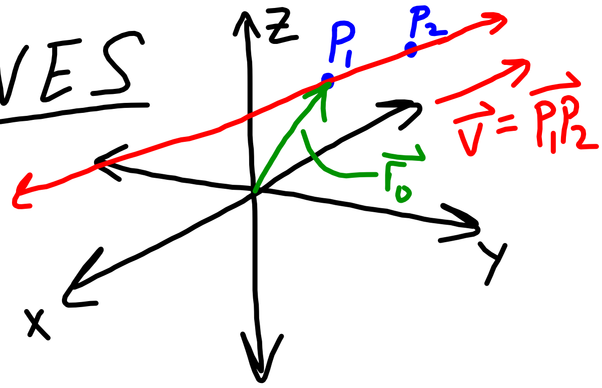
what is $\text{proj}_{\vec{v}} \vec{u}$?

$$\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \frac{0-1-6}{4+1+36}\vec{v} = \frac{-7}{41}\vec{v}$$

$$= \frac{-14}{41}\vec{i} + \frac{7}{41}\vec{j} - \frac{42}{41}\vec{k}.$$

in the direction of $\vec{v}$, or "onto" $\vec{v}$

LINES



Red line can be created by adding multiples of $\vec{v}$ to $\vec{r}_0$ ...

$$\langle x, y, z\rangle = \vec{r}_0 + t\,\vec{v}$$ vector equation for the line

$$\vec{r}_0 = \langle x_0, y_0, z_0\rangle$$

$$\vec{v} = \langle v_1, v_2, v_3\rangle$$

Then

$$\langle x, y, z \rangle = \langle x_0 + t v_1, y_0 + t v_2, z_0 + t v_3 \rangle$$

"Parametric equations"

$$\boxed{\begin{aligned} x &= x_0 + t v_1 \\ y &= y_0 + t v_2 \\ z &= z_0 + t v_3 . \end{aligned}}$$

"Symmetric" equations...

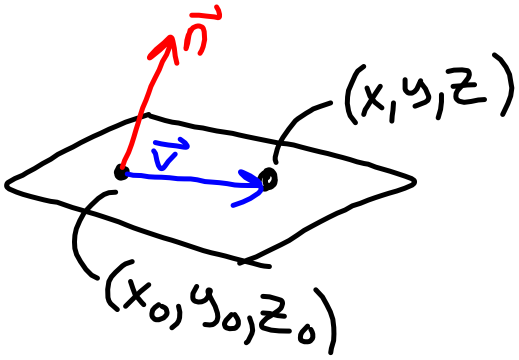
Solve for t: $x = x_0 + t v_1 \Rightarrow t = \dfrac{x - x_0}{v_1}$

Also, $t = \boxed{\dfrac{y - y_0}{v_2} = \dfrac{z - z_0}{v_3} = \dfrac{x - x_0}{v_1}}$

symmetric equations

If, e.g. $v_3 = 0$, then $z = z_0 + \overset{0}{\cancel{t v_3}} = z_0$

$\Rightarrow \boxed{\dfrac{x - x_0}{v_1} = \dfrac{y - y_0}{v_2} \;\&\; z = z_0.}$



$$\vec{n} = \langle n_1, n_2, n_3 \rangle$$

$$\vec{v} = \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$\vec{n} \cdot \vec{v} = \underbrace{n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0}_{\text{Scalar equation for plane}}$$

Note how we can algebraically manipulate the equation of the plane...

$$n_1x + n_2y + n_3z - n_1x_0 - n_2y_0 - n_3z_0 = 0.$$

looks like $\underbrace{ax + by + cz + d = 0}$.

So if you see this, then you know $\vec{n} = \langle a, b, c \rangle$. ☺

Ex: One line is given by $\frac{x-1}{4}=\frac{z+2}{3}$, $y=-2$

and another by $\vec{r}(t)=\langle t+1, 2t-2, t-2\rangle$.

They lie in a common plane; find the equation of the plane.

Direction vectors are $\vec{v}_1=\langle 4,0,3\rangle$

$\vec{v}_2=\langle 1,2,1\rangle$

$\Rightarrow \vec{n}=\vec{v}_1\times\vec{v}_2=\langle 0\cdot1-3\cdot2, 1\cdot3-4\cdot1, 4\cdot2-0\cdot1\rangle=\langle -6,-1,8\rangle$

We need a point (x_0, y_0, z_0) in the plane...
<u>any</u> point on either line works, e.g. $(1,-2,-2)$

Plane equation: $\vec{n} \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

$$\Rightarrow \langle -6, -1, 8 \rangle \cdot \langle x-1, y+2, z+2 \rangle = 0$$

$$\Rightarrow -6(x-1) - (y+2) + 8(z+2) = 0.$$

Ex: Find the equation (vector form) of the line passing through $(0,-1,1)$ perpendicular to the plane $\underbrace{2y-z+5x-1=0}_{\vec{n}=\langle 5,2,-1\rangle}$.

$\vec{n}=\langle 5,2,-1\rangle$: direction of line

$$\vec{r}(t)=\langle 0,-1,1\rangle+t\langle 5,2,-1\rangle$$
$$=\langle 5t,2t-1,1-t\rangle.$$

Review of other surfaces

(a) one-sheet hyperboloid

(b) elliptic cylinder

(c) ellipsoid

(d) hyperbolic paraboloid

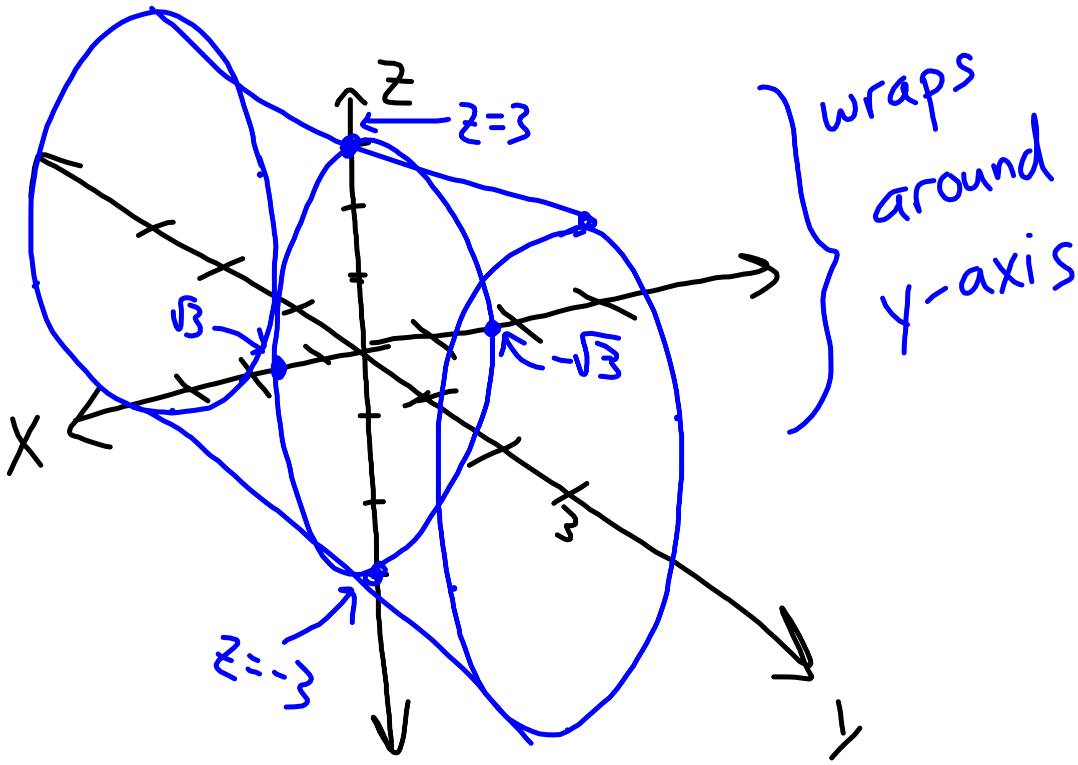
(e) two-sheet hyperboloid

(f) elliptic paraboloid

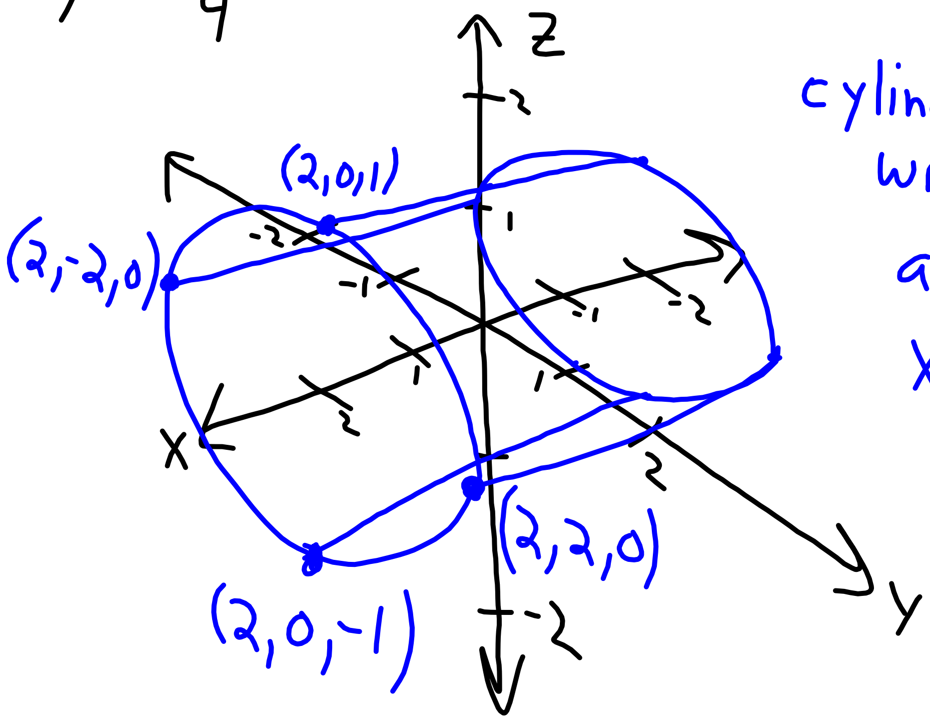
(g) cone

(h) non-elliptic cylinder

(a) $\frac{x^2}{3} - \frac{y^2}{9} + \frac{z^2}{9} = 1$

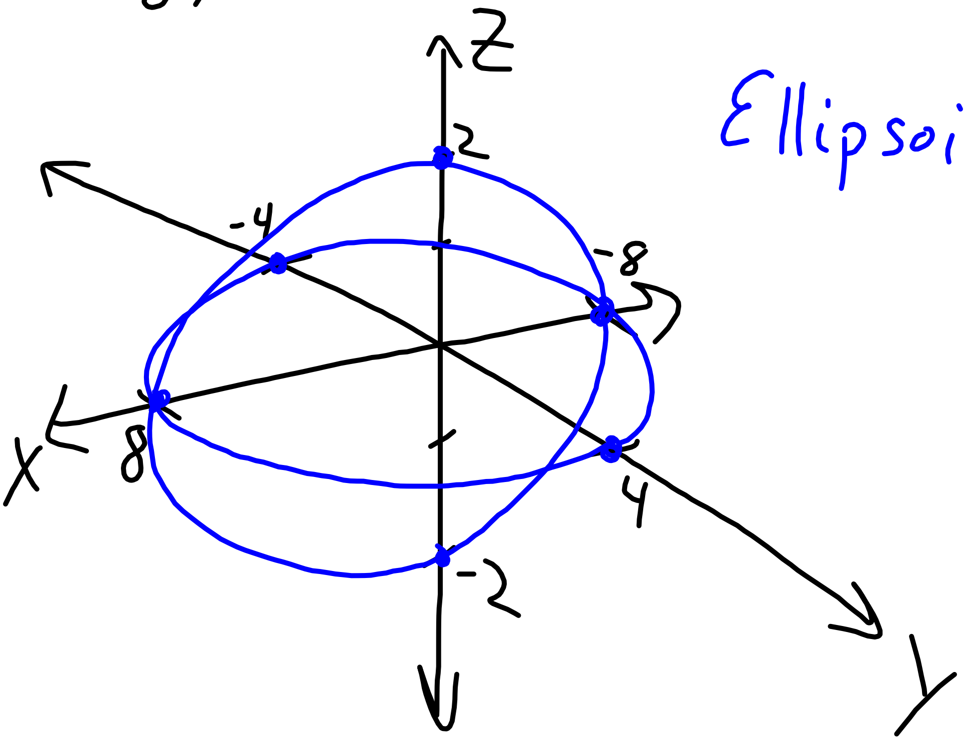


(b) $\dfrac{y^2}{4} + z^2 = 1$



cylinder wraps around x-axis

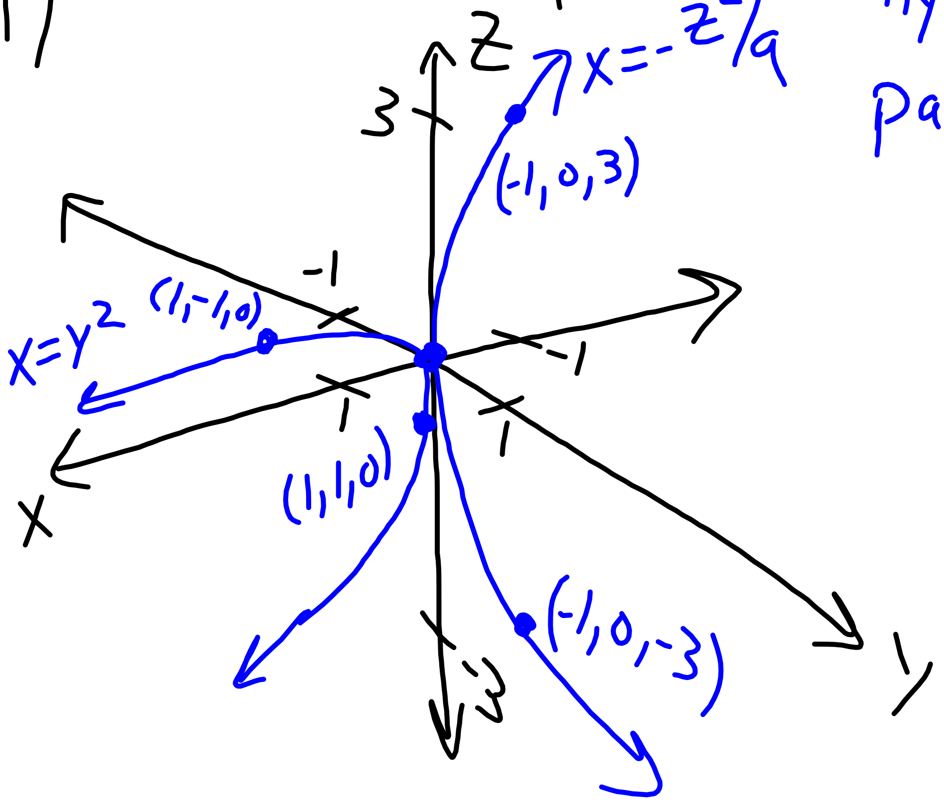
(c) $\left(\frac{x}{8}\right)^2+\left(y/4\right)^2+\left(\frac{z}{2}\right)^2=1$



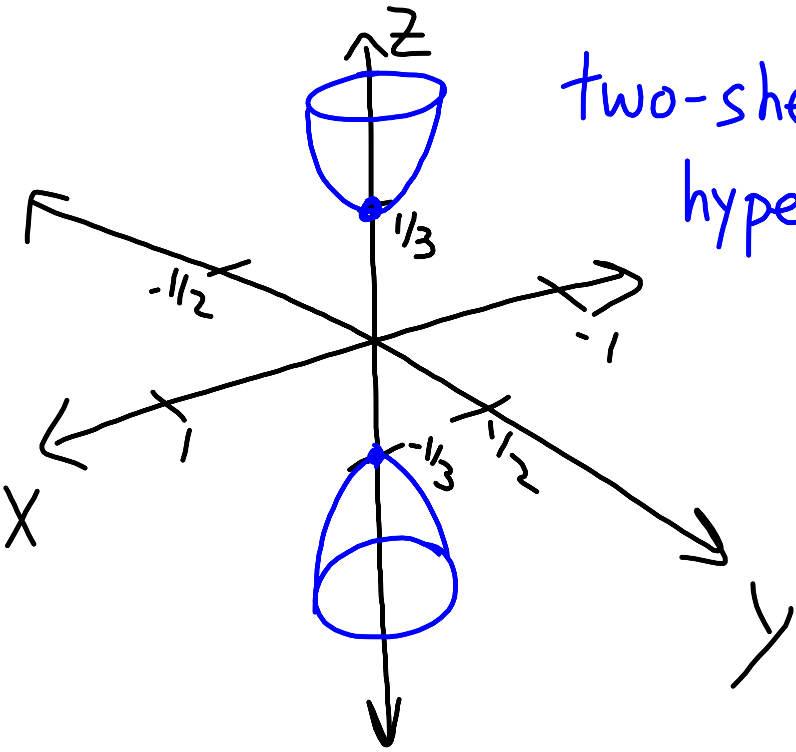
Ellipsoid

(d) $x = y^2 - z^2/9$

Hyperbolic paraboloid



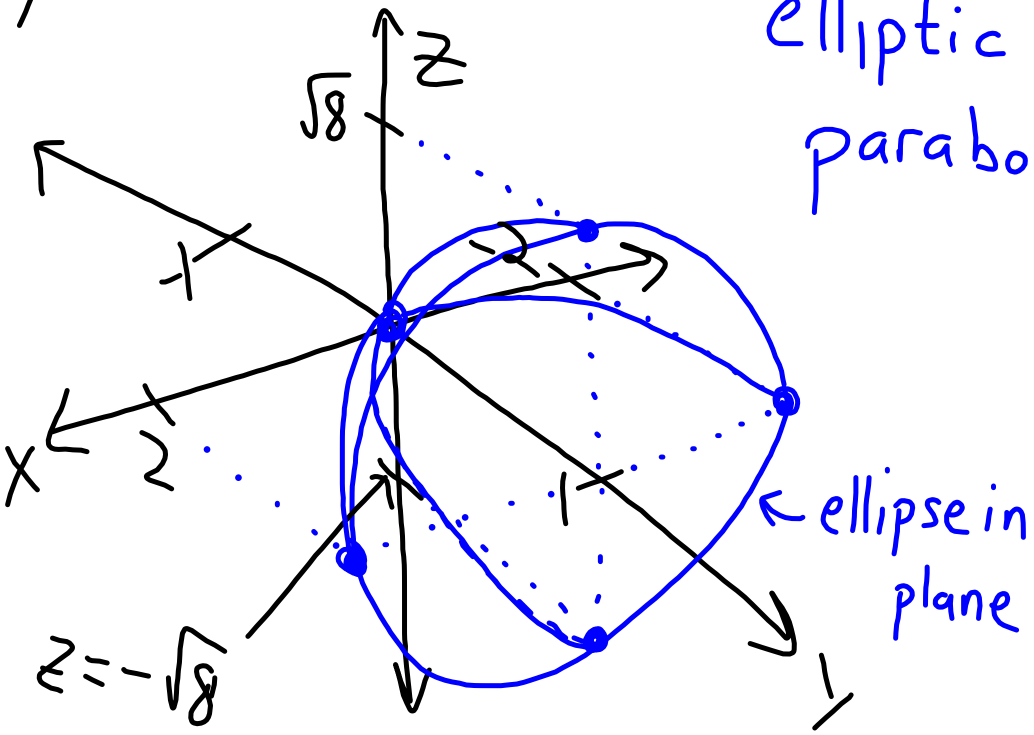
(e) $\left(\frac{z}{1/3}\right)^2 - \left(\frac{y}{1/2}\right)^2 - x^2 = 1$



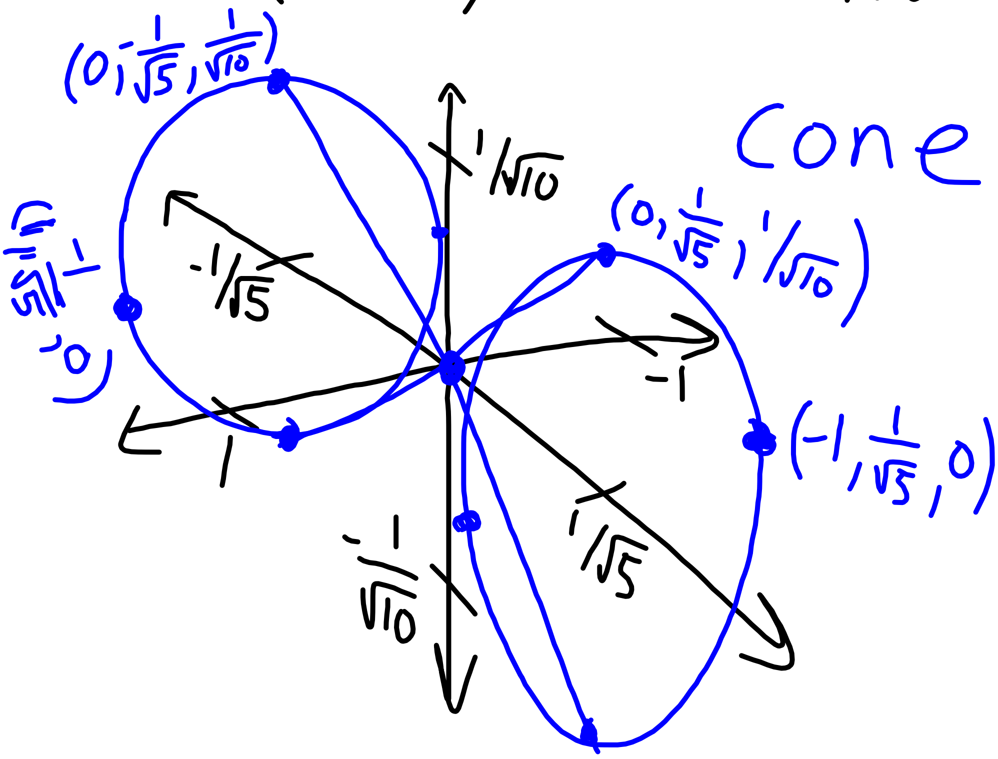
two-sheet hyperboloid

(f) $y = \frac{x^2}{4} + \frac{z^2}{8}$

elliptic paraboloid

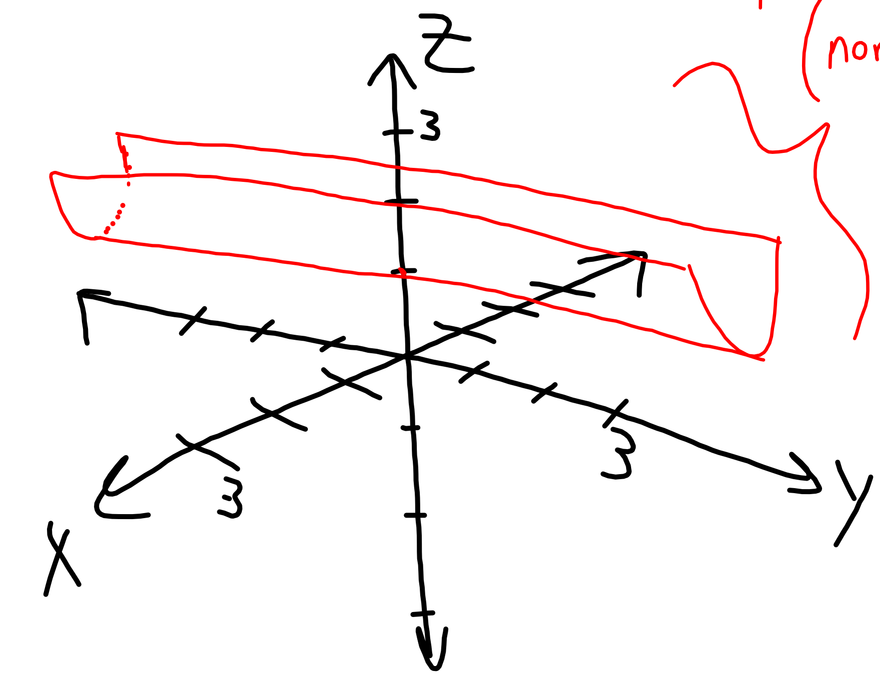


(9) $\left(\frac{y}{1/\sqrt{5}}\right)^2 = x^2 + \left(\frac{z}{1/\sqrt{10}}\right)^2$



cone

(h) $z = x^2 + 1$



parabolic (non-elliptic) cylinder

Intersections of surfaces

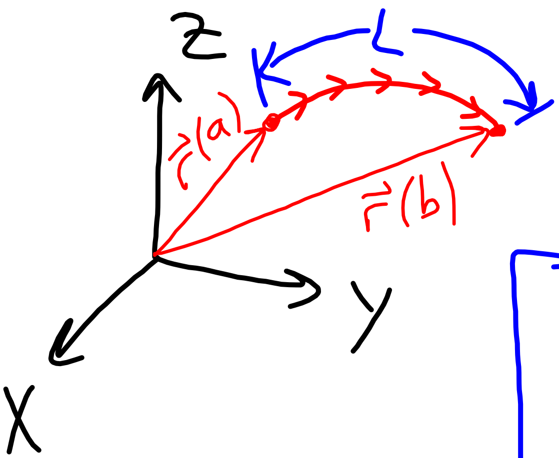
For example, $z = 1 + y^2 + \frac{1}{2}x^2$

& $y = 3x + 5$

Look for an easy way to get 2 variables in terms of a third...

$$\left.\begin{aligned} z &= 1 + (3x+5)^2 + \tfrac{1}{2}x^2 \\ y &= 3x+5 \end{aligned}\right\}$$ Let "x" be the parameter

Arc length formula



$$\boxed{L = \int_a^b |\vec{r}\,'(t)|\,dt}$$

Ex: Let a curve have parameterization

$$\vec{r}(t) = \left\langle t, \frac{2\sqrt{2}}{3} t^{3/2}, \frac{1}{2} t^2 \right\rangle, \quad 0 \le t \le 2.$$

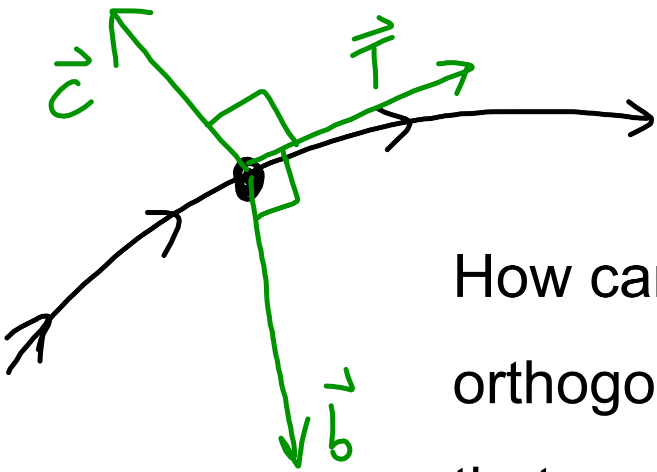
Find the length of the arc segment.

$$\vec{r}\,'(t) = \left\langle 1, \frac{2\sqrt{2}}{3} \cdot \frac{3}{2} \cdot \sqrt{t}, t \right\rangle = \langle 1, \sqrt{2t}, t \rangle.$$

$$\int_0^2 |\vec{r}\,'(t)|\,dt = \int_0^2 \sqrt{1+2t+t^2}\,dt = \int_0^2 \sqrt{(1+t)^2}\,dt$$

$$= \int_0^2 1+t\,dt = 2 + \frac{1}{2}t^2 \Big|_0^2 = 4.$$

The normal and binormal vectors



How can we find three orthogonal unit vectors that move with a particle through space?

$\vec{T}$ points in the direction of motion, so that will be the first vector. Note that $|\vec{T}| = 1 \Rightarrow |\vec{T}|^2 = \vec{T} \cdot \vec{T} = 1$

$$\Rightarrow \frac{d}{dt}(\vec{T} \cdot \vec{T}) = \frac{d}{dt}(1) = 0$$

$$\Rightarrow 2\vec{T} \cdot \vec{T}' = 0$$

$$\Rightarrow \underline{\vec{T} \perp \vec{T}'}$$

so normalize this...

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}.$$

$\vec{N} = \dfrac{\vec{T}'}{|\vec{T}'|}$: NORMAL vector
for a curve (not a plane).

Bi-normal : $\vec{B} = \vec{T} \times \vec{N}$

$|\vec{B}| = |\vec{T}||\vec{N}| \sin(90^\circ) = 1 \cdot 1 \cdot 1 = \underline{1}$.

$\{\vec{T}, \vec{N}, \vec{B}\}$: the "TNB frame".

$\underline{\text{Ex}}$: Find the TNB-frame vectors for $\vec{r}(t) = \langle \sqrt{2}\cos(t), 4t, \sqrt{2}\sin(t) \rangle$ at the point $\underbrace{(0, 2\pi, \sqrt{2})}_{\text{so } t = \pi/2}$.

$$\vec{r}\,'(t) = \langle -\sqrt{2}\sin(t), 4, \sqrt{2}\cos(t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{2\left(\sin^2(t) + \cos^2(t)\right) + 16} = \sqrt{18} = 3\sqrt{2}$$

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \left\langle -\frac{1}{3}\sin(t), \frac{2\sqrt{2}}{3}, \frac{1}{3}\cos(t) \right\rangle$$

$$\vec{T}\,' = \left\langle -\frac{1}{3}\cos(t), 0, -\frac{1}{3}\sin(t) \right\rangle$$

$$\vec{N} = \left\langle -\cos(t), 0, -\sin(t) \right\rangle$$

$$\vec{T}\left(\frac{\pi}{2}\right) = \left\langle -\frac{1}{3}, \frac{2\sqrt{2}}{3}, 0 \right\rangle$$

$$\vec{N}\left(\frac{\pi}{2}\right) = \left\langle 0, 0, -1 \right\rangle$$

$$\vec{B} = \left\langle -\frac{2\sqrt{2}}{3}, -\frac{1}{3}, 0 \right\rangle.$$

Velocity and acceleration

$\vec{r}(t)$: position at time t.

$\vec{v}(t) = \vec{r}\,'(t)$; velocity at time t.

$$\vec{v} = |\vec{v}|\,\hat{v}$$

speed direction

$\vec{a}(t) = \vec{v}\,'(t)$

$= \vec{r}\,''(t)$; acceleration

Given acceleration, we can also find velocity and position by integrating...

$$\vec{v} = \int \vec{a}(t)\,dt$$

$$\vec{r} = \int \vec{v}(t)\,dt$$

We require unknown constants to be added at each step; their values are often determined later from "initial conditions".

Newton's Second Law

$$\underbrace{\vec{F}}_{\text{force}} = \underbrace{m}_{\text{mass}} \vec{a}$$

Often we just want the acceleration:

$$\vec{a} = \frac{1}{m}\vec{F}.$$

Ex: A particle with mass 4 kg is initially moving with velocity <0 , 1 , 1/2> m/s and is acted on by a force $\vec{F}(t) = \langle -\sin(\pi t), t^2, 1 \rangle$ N, until 2 seconds later. Find the distance the particle has moved during those 2 seconds.

$$\vec{F} = m\vec{a}$$

$$\vec{a} = \frac{1}{m}\vec{F} = \frac{1}{4}\langle -\sin(\pi t), t^2, 1 \rangle$$

$$\int \vec{a} = \vec{v} = \frac{1}{4}\left\langle \frac{1}{\pi}\cos(\pi t), \frac{1}{3}t^3, t \right\rangle + \vec{C}_1$$

$$\vec{v}(0) = \left\langle \frac{1}{4\pi}, 0, 0 \right\rangle + \vec{C}_1 = \underbrace{\left\langle 0, 1, \tfrac{1}{2} \right\rangle}_{\text{given}}$$

$$\text{so } \vec{C}_1 = \left\langle \frac{-1}{4\pi}, 1, \tfrac{1}{2} \right\rangle.$$

$$\vec{r} = \int \vec{v} = \frac{1}{4}\left\langle \frac{1}{\pi^2}\sin(\pi t), \frac{1}{12}t^4, t^2/2 \right\rangle + t\vec{C}_1 + \vec{C}_2$$

$\vec{r}(0) = \vec{C}_2$ $\quad (\sin(0) = 0)$

so distance travelled is

$$d = \Big|\underbrace{\vec{r}(2)}_{= \ldots + \vec{C}_2} - \underbrace{\vec{r}(0)}_{\vec{C}_2}\Big| \Big\} \ \vec{C}_2 \text{ cancels out}$$

$$= \Big| \frac{1}{4}\Big\langle \frac{1}{\pi^2} \underbrace{\cancel{\sin(2\pi)}}_{0}, \underbrace{\frac{2^4}{12}}_{4/3}, \underbrace{\frac{2^2}{2}}_{2} \Big\rangle + 2\vec{C}_1 \Big|$$

$$d = \left| \left\langle 0, \frac{1}{3}, \frac{1}{2} \right\rangle + 2\left\langle \frac{-1}{4\pi}, 1, 1/2 \right\rangle \right|$$

$$= \left| \left\langle \frac{-1}{2\pi}, 7/3, 3/2 \right\rangle \right|$$

$$= \sqrt{\frac{1}{4\pi^2} + \frac{49}{9} + \frac{9}{4}} \quad \Bigg\} \text{ close enough (in meters)}$$