Math 2110Q Worksheet 20 Solutions

Trigonometric integrals:

$$\int_0^{2\pi} \cos^2(t)\,dt = \int_0^{2\pi} \sin^2(t)\,dt = \pi.$$

1. Let S denote the portion of the sphere $x^2+y^2+z^2=1$ with $z \geq 0$ and let $\hat{n}$ be the unit normal pointing upward on S. Given $\vec{F}(x,y,z) = < z, x, xy >$, calculate

$$\int\int_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\,dS.$$

Solution: It is possible to do this directly. The parameterization requires a bit of work to do so. Instead, apply Stoke's Theorem with the boundary curve parameterized as $\vec{r}(t) = < \cos(t), \sin(t), 0 >$ for $0 \leq t \leq 2\pi$:

$$\begin{aligned}\int\int_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\,dS &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}^{\,'}(t)\,dt \\ &= \int_0^{2\pi} < 0, \cos(t), \sin(t)\cos(t) > \cdot < -\sin(t), \cos(t), 0 > \,dt = \int_0^{2\pi} \cos^2(t)dt = \pi.\end{aligned}$$

Now, for the purpose of illustration, let us do the calculation the long way. That is, do the surface integral of the curl directly. One should find

$$\nabla \times \vec{F} = < x, 1-y, 1 > .$$

The surface should be parameterized using spherical coordinates with the constant radius of $\rho = 1$. That is,

$$\vec{r}(\phi,\theta) = < \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) >,$$

with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$. It is shown in the lecture slides that

$$\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = < \cos(\theta)\sin^2(\phi), \sin(\theta)\sin^2(\phi), \sin(\phi)\cos(\phi) > .$$

One inserts the parameterization into the above formula for the curl to get

$$\nabla \times \vec{F} = < x, 1-y, 1 > = < \cos(\theta)\sin(\phi), 1-\sin(\theta)\sin(\phi), 1 > .$$

The last two equations are used now to evaluate

$$\begin{aligned}&\int\int_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\,dS = \int_0^{2\pi}\int_0^{\pi/2} < \cos(\theta)\sin(\phi), 1-\sin(\theta)\sin(\phi), 1 > \cdot \left(\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}\right) d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2} \left(\cos^2(\theta)\sin^3(\phi) + \sin(\theta)\sin^2(\phi) - \sin^2(\theta)\sin^3(\phi) + \sin(\phi)\cos(\phi)\right) d\phi\,d\theta\end{aligned}$$

Now apply the following:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/2} \left(\cos^2(\theta)\sin^3(\phi) - \sin^2(\theta)\sin^3(\phi)\right) d\phi\,d\theta &= \int_0^{2\pi} \left(\cos^2(\theta) - \sin^2(\theta)\right) d\theta \int_0^{\pi/2} \sin^3(\phi)\,d\phi \\ &= (\pi - \pi)\int_0^{\pi/2} \sin^3(\phi)\,d\phi = 0\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/2} \sin(\theta)\sin^2(\phi)\,d\phi\,d\theta &= \int_0^{2\pi} \sin(\theta)\,d\theta \int_0^{\pi/2} \sin^2(\phi)\,d\phi \\ &= 0\int_0^{\pi/2} \sin^2(\phi)\,d\phi = 0\end{aligned}$$

$$\int_0^{2\pi}\int_0^{\pi/2} \sin(\phi)\cos(\phi)\,d\phi\,d\theta = 2\pi\left(\frac{1}{2}\sin^2(\phi)\right)\Big|_0^{\pi/2} = \pi.$$

This yields the same answer as before.

2. Let S denote the surface of the box

$$V = \{(x,y,z) \mid -1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1\},$$

and let $\hat{n}$ be the unit normal pointing outward on S. Given $\vec{F}(x,y,z) = < yz, xz, z^3 >$, calculate

$$\int\int_S \vec{F} \cdot \hat{n}\, dS.$$

Solution: Apply the Divergence Theorem;

$$\begin{aligned}\int\int_S \vec{F} \cdot \hat{n}\, dS &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \nabla \cdot \vec{F}\, dx\, dy\, dz = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 0 + 0 + 3z^2\, dx\, dy\, dz \\ &= 12\int_{-1}^{1} z^2\, dz = 4z^3|_{-1}^{1} = 8.\end{aligned}$$

As an exercise, you could calculate the six flux integrals corresponding to the six sides of the box directly and add them together to see that you get the same answer.