Math 2110Q Worksheet 14 Solutions

1. Find the volume of the region bounded by the surfaces $z = -1$, $z = 2$ and the hyperboloid $x^2 + y^2 - z^2 = 1$. *Hint: think about the curves on the surface of the hyperboloid for fixed z-values.*

Solution. Recall the one-sheet hyperboloid has a vase-like shape, which has rotational symmetry about the z-axis. The planes $z = -1$ and $z = 2$ then cut horizontally through this and form flat bottom and top surfaces, respectively. It is ideal to describe in cylindrical coordinates. Inside the given region, one can rotate around the z-axis freely, so that the bounds on the angle of rotation are always just $0 \leq \theta \leq 2\pi$. Meanwhile, for any z between -1 and 2, we can bound the distance r from the z-axis for a point in the region as follows. We can decrease r (keeping z and θ fixed) until we are on the axis, so $0 \leq r$, regardless of where we start inside the region. On the other hand, if we try to increase r, we can proceed until we hit the "sides" of the vase. This bound depends on the z coordinate; for a fixed z, the equation for the bounding hyperboloid represents a circle with a fixed r:

$$r^2 = x^2 + y^2 = 1 + z^2 \text{ (fixed)},$$

so then $r = \sqrt{1+z^2}$ gives the bound on r. We can now set up our integral for the volume of the region:

$$\int_0^{2\pi} \int_{-1}^{2} \int_0^{\sqrt{1+z^2}} r\,dr\,dz\,d\theta = \frac{1}{2} \int_0^{2\pi} \int_{-1}^{2} 1 + z^2 dz\,d\theta = \frac{1}{2} 2\pi \left[z + \frac{1}{3} z^3 \right]_{-1}^{2} = 6\pi.$$

2. Find the volume of the region bounded between the surfaces $z = 4 + x^2 + y^2$ and $z = 1 + 4x^2 + 4y^2$.

Solution. These are both elliptic paraboloids, opening "upward" (positive z direction) and have rotational symmetry about the z axis. The rotational symmetry makes cylindrical coordinates ideal to calculate the volume. However, the order of integration is (arguably) best handled differently than the previous problem, above. In the current situation, the most intuitive approach is to bound the z coordinate using the given surface equations. More precisely, if we transform the equations into cylindrical coordinates, we find $z = 4 + r^2$ and $z = 1 + 4r^2$. Note that we have a way to express bounds in z in terms of r from these equations. One surface is "lower" on the z-axis; when $r = 0$ (on the z-axis), we have $z = 1$ for the lower surface and $z = 4$ for the upper surface.
What does this volume look like? The two paraboloids intersect because the one that is lower on the z axis has a higher growth rate as r increases: $4r^2$ is greater than r^2. So, as r increases, the lower surface catches up with the upper surface and they intersect where they achieve the same z-value:

$$1 + 4r^2 = z = 4 + r^2 \Rightarrow 3r^2 = 3 \Rightarrow r = 1.$$

That is, the projection of the volume down into the xy-plane looks like a circle of radius 1. This makes bounding the r and θ variables standard, and we calculate the volume to be

$$\int_0^{2\pi} \int_0^1 \int_{1+4r^2}^{4+r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 r(3 - 3r^2)\,dr\,d\theta$$

$$= 3 \left(\int_0^{2\pi} d\theta \right) \left(\int_0^1 (r - r^3)\,dr \right) = 6\pi \left[\frac{1}{2} r^2 - \frac{1}{4} r^4 \right]_0^1 = \frac{3\pi}{2}.$$