Math 2110Q Worksheet 10 Solutions

Let $T \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$ (including the boundary of the triangle). Find the global maximum and minimum values of $f(x,y)$ on the domain T.

$$f(x,y) = \frac{y}{1+x^2+y^2}$$

Solution. We search first for critical points inside T. Solve

$$\begin{aligned} & \frac{\partial f}{\partial x} = \frac{-2xy}{(1+x^2+y^2)^2} &= 0 \\ \text{and} \quad & \frac{\partial f}{\partial y} = \frac{1+x^2-y^2}{(1+x^2+y^2)^2} &= 0. \end{aligned}$$

The numerators must be zero, so either $x = 0$ or $y = 0$ from the first equation. Substitute into $1 + x^2 - y^2 = 0$ to get $y = -1$ or $y = 1$ when $x = 0$, and no solution if $y = 0$. However, the points $(0, \pm 1)$ are both *outside* the region T, so there are no critical points to check inside T. One proceeds to check along the boundary. It will be necessary to check the value of f at all vertices, so we list

$$f(0,0) = 0, \quad f(1,0) = 0, \quad f(1,1) = \frac{1}{3}.$$

We then reduce $f(x,y)$ to a function of a single variable along each of the three edges, then check the extrema of the resulting function along each respective edge (away from the end points, since we already checked the vertices of the triangle). Along the edge $y = 0$, $0 \le x \le 1$ we have $f = f(x,0) = 0$. There is nothing to do. Along the edge $x = 1$, $0 \le y \le 1$ we have

$$f = f(1,y) = \frac{y}{2+y^2} \Rightarrow \frac{d}{dy}\left(\frac{y}{2+y^2}\right) = \frac{2-y^2}{(2+y^2)^2} = 0$$

holds for $y = \pm\sqrt{2}$. However, we discard these possibilities since we need $0 \le y \le 1$ on this edge.

The third edge lies along the line $y = x$, for $0 \le x \le 1$. In this case,

$$f = f(x, y = x) = \frac{x}{1+2x^2} \Rightarrow \frac{d}{dx}\left(\frac{x}{1+2x^2}\right) = \frac{1-2x^2}{(1+2x^2)^2} = 0$$

holds for $x = \pm\sqrt{2}/2$. Since $x = \sqrt{2}/2$ lies in the range $0 \le x \le 1$, we check the value of f:

$$f(x, y = x)|_{x=\sqrt{2}/2} = \left.\frac{x}{1+2x^2}\right|_{x=\sqrt{2}/2} = \frac{\sqrt{2}}{4}.$$

In summary, our candidate extreme values for f come from this last result plus the values at the vertices of T. If follows that the global minimum is $f = 0$ and the global maximum is $f = \sqrt{2}/4$.