Math 2110Q Worksheet 6 - Solutions
September 21, 2016

1. Given a curve with parameterization $\vec{r}(t) =< \sqrt{2}\cos(t), 1+t, \sqrt{2}\sin(t) >$, find the TNB-frame vectors AND the curvature at the point $(-\sqrt{2}, 1+\pi, 0)$.

Note that for the given point, the corresponding parameter value is $t = \pi$.

$$\begin{aligned}
\vec{r}\,'(t) &=< -\sqrt{2}\sin(t), 1, \sqrt{2}\cos(t) >, \ \ \left|\vec{r}\,'(t)\right| = \sqrt{2(\sin^2(t)+\cos^2(t))+1} = \sqrt{3} \\
\Rightarrow \vec{T}(t) &= \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} =< -\sqrt{2/3}\sin(t), \sqrt{3}/3, \sqrt{2/3}\cos(t) > \\
\vec{T}\,'(t) &=< -\sqrt{2/3}\cos(t), 0, -\sqrt{2/3}\sin(t) >, \ \ \left|\vec{T}\,'(t)\right| = \sqrt{2/3} \\
\Rightarrow \vec{N}(t) &= \frac{\vec{T}\,'(t)}{\left|\vec{T}\,'(t)\right|} =< -\cos(t), 0, -\sin(t) > .
\end{aligned}$$

Now, plug in $t = \pi \ldots$

$$\begin{aligned}
\vec{T}(t=\pi) &=< 0, \sqrt{3}/3, -\sqrt{2/3} > \\
\vec{N}(t=\pi) &=< 1, 0, 0 > \\
\vec{B}(t=\pi) &= \vec{T} \times \vec{N} =< 0, -\sqrt{2/3}, -\sqrt{3}/3 >
\end{aligned}$$

Two approaches are possible for the curvature... one is to find $\vec{r}\,''(t) =< -\sqrt{2}\cos(t), 0, -\sqrt{2}\sin(t) >$ and then

$$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{\left| < 0, 1, -\sqrt{2} > \times < \sqrt{2}, 0, 0 > \right|}{3\sqrt{3}} = \frac{\left| < 0, -2, -\sqrt{2} > \right|}{3\sqrt{3}} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}.$$

The other is the alternative formula $\kappa = |\vec{T}\,'(\pi)|/|\vec{r}\,'(\pi)|$, plugging values in from the above formulas. This is much faster for this problem; the rule of thumb is that when $\vec{T}'(t)$ is not too cumbersome to find, then the second approach is faster. However, in practice it will often be cumbersome.

2. Let $\vec{a}(t) =< 4e^{2t}, 6t, 0 >$ describe the acceleration of a particle (neglecting units). Find the position of the particle at time t if the initial position is $(1,0,1)$ and the initial velocity is $< 2, 0, 0 >$.

This is slightly more challenging than the examples from class, since the constant vectors from integration are not just the initial velocity and initial position; no credit is removed for missing this detail, since we did not explicitly cover it. Note that

$$\vec{v}(t) = \int \vec{a}(t)\,dt =< 2e^{2t}, 3t^2, 0 > + \vec{C}_1,$$

where $\vec{v}(0) =< 2, 0, 0 >$ works to match our given initial velocity with $\vec{C}_1 =< 0, 0, 0 >$. Thus, we have position vector

$$\vec{r}(t) = \int \vec{v}(t)\,dt =< e^{2t}, t^3, 0 > + \vec{C}_2,$$

where $\vec{C}_2$ is found from the equation

$$\vec{r}(0) =< 1, 0, 0 > + \vec{C}_2 =< 1, 0, 1 > \Rightarrow \vec{C}_2 =< 0, 0, 1 > .$$

Thus, $\vec{r}(t) =< e^{2t}, t^3, 1 >$.