Math 2110Q Worksheet 20 Solutions

Trigonometric integrals:

$$\int_0^{2\pi} \cos^2(t)\,dt = \int_0^{2\pi} \sin^2(t)\,dt = \pi.$$

1. Let S denote the portion of the sphere $x^2+y^2+z^2=1$ with $z \geq 0$ and let $\hat{n}$ be the unit normal pointing upward on S. Given $\vec{F}(x,y,z) =< z,x,xy >$, calculate

$$\int\int_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\,dS.$$

Solution: It is possible to do this directly. The parameterization requires a bit of work to do so. Instead, apply Stoke's Theorem with the boundary curve parameterized as $\vec{r}(t) =< \cos(t),\, \sin(t),\, 0 >$ for $0 \leq t \leq 2\pi$:

$$\begin{aligned}\int\int_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\,dS &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}^{\,'}(t)\,dt \\ &= \int_0^{2\pi} < 0,\, \cos(t),\, \sin(t)\cos(t) > \cdot < -\sin(t),\, \cos(t),\, 0 > \,dt = \int_0^{2\pi} \cos^2(t)dt = \pi.\end{aligned}$$

2. Let S denote the surface of the box

$$V = \{(x,y,z) \mid -1 \leq x \leq 1,\, -1 \leq y \leq 1,\, -1 \leq z \leq 1\},$$

and let $\hat{n}$ be the unit normal pointing outward on S. Given $\vec{F}(x,y,z) =< yz, xz, z^3 >$, calculate

$$\int\int_S \vec{F} \cdot \hat{n}\,dS.$$

Solution: Apply the Divergence Theorem;

$$\begin{aligned}\int\int_S \vec{F} \cdot \hat{n}\,dS &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \nabla \cdot \vec{F}\,dx\,dy\,dz = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 0+0+3z^2\,dx\,dy\,dz \\ &= 12\int_{-1}^1 z^2\,dz = 4z^3|_{-1}^1 = 8.\end{aligned}$$