Math 2110Q Worksheet 19 Solutions
December 5, 2016

1. Let S denote the portion of the surface $z = 1 + x\sqrt{3} + y^2$ for $-1 \leq x \leq 2$ and $0 \leq y \leq 1$. Calculate the value of the surface integral

$$\int\int_S xy\,dS.$$

Solution: The natural parameterization should be applied here. Use x and y as the parameters for the surface and take $z = f(x,y) = 1 + x\sqrt{3} + y^2$. Then the integral is

$$\int_{-1}^{2}\int_{0}^{1} xy\sqrt{1+(f_x)^2+(f_y)^2}\,dy\,dx = \int_{-1}^{2}\int_{0}^{1} xy\sqrt{4+4y^2}\,dy\,dx = \int_{-1}^{2}\int_{0}^{1} x\left(2y\sqrt{1+y^2}\right)\,dy\,dx$$
$$= \int_{-1}^{2} x\,dx \int_{1}^{2}\sqrt{u}\,du = 2^{3/2}-1.$$

2. Let S denote the portion of the surface $x+y+z=0$ that lies inside the cylinder $x^2+y^2=1$. Given the vector field $\vec{F}(x,y,z) = < x^2, y^2, z^2 >$, calculate

$$\int\int_S \vec{F}\cdot\hat{n}\,dS,$$

where $\hat{n}$ is the upward-oriented, unit normal vector for S.
Solution: There are some different ways to think about setting this up. One way is to apply the natural parameterization again for the planar surface; $z = -x-y$, but the parameters x and y lie inside a circle of radius 1. Let us call this region $\mathscr{D}$. It follows that

$$\int\int_S \vec{F}\cdot\hat{n}\,dS = \int\int_{\mathscr{D}} \vec{F}(x,y)\cdot < -z_x, -z_y, 1 > \,dx\,dy = \int\int_{\mathscr{D}} < x^2, y^2, (x+y)^2 > \cdot < 1,1,1 > \,dx\,dy$$
$$= \int\int_{\mathscr{D}} x^2+y^2+(x+y)^2\,dx\,dy.$$

Now we convert into polar coordinates to finish the job:

$$\int\int_{\mathscr{D}} x^2+y^2+(x+y)^2\,dx\,dy = \int_0^{2\pi}\int_0^1 2r^3(1+\sin(\theta)\cos(\theta))\,dr\,d\theta$$
$$\frac{1}{2}\int_0^{2\pi} 1+\sin(\theta)\cos(\theta)\,d\theta = \frac{1}{2}\left(1+\frac{1}{2}\sin^2(\theta)\right)\Big|_0^{2\pi} = \pi.$$

Another way one might think about it is to directly parameterize the surface using

$$\vec{r}(r,\theta) = < r\cos(\theta), r\sin(\theta), -r(\sin(\theta)+\cos(\theta)) >,$$

and apply the appropriate integration formula. The downside of this approach is that you would calculate $|\vec{r}_r \times \vec{r}_\theta|$ directly, which is perhaps a bit more tedious than applying the short-cut formulas above for the natural parameterization.